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An approximation scheme for optimal stochastic control problems*

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1 Introduction

We consider the optimal stochastic control problems over a finite horizon $T \in (0, \infty)$ with value function

$$v(t,x) = \inf_{\alpha \in \mathcal{A}} \mathbb{E}\left[f(X_T^{t,x,\alpha}) + \int_t^T g(s, X_s^{t,x,\alpha}, \alpha_s) ds \right], \quad (t,x) \in [0,T] \times \mathbb{R}^m, \tag{1.1}$$

where the controlled process $\{X_s^{t,x,\alpha}\}$ is governed by

$$\begin{cases} dX_s^{t,x,\alpha} = \mu(s, X_s^{t,x,\alpha}, \alpha_s) ds + \sigma(s, X_s^{t,x,\alpha}, \alpha_s) dW_s, \quad s \in [t,T], \\ X_t^{t,x,\alpha} = x \in \mathbb{R}^m. \end{cases} \tag{1.2}$$

Here, $\mu : [0,T] \times \mathbb{R}^m \times A \to \mathbb{R}^m$, $\sigma : [0,T] \times \mathbb{R}^m \times A \to \mathbb{R}^{m \times d}$, $f : \mathbb{R}^m \to \mathbb{R}$, $g : [0,T] \times \mathbb{R}^m \times A \to \mathbb{R}$ and $A \subset \mathbb{R}^k$. The conditions imposed on these functions are described in Section 2 below. The process $\{W_s\}_{0 \le t \le T}$ is a d-dimensional standard Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \le t \le T}, \mathbb{P})$ satisfying the usual conditions. The class $\mathcal{A}$ of controls is the collection of A-valued $\{\mathcal{F}_t\}_{0 \le t \le T}$-progressively measurable processes $\{\alpha_t\}_{0 \le t \le T}$.

One of the main approaches for optimal stochastic control problems is to solve the Hamilton-Jacobi-Bellman (HJB) equation that the value function should satisfy, and then to construct a control strategy via the verification argument. For our problem (1.1), the corresponding HJB equation is given by

$$-\partial_t v + F(t,x,Dv,D^2v) = 0, \quad (t,x) \in [0,T) \times \mathbb{R}^m, \tag{1.3}$$

with the terminal condition $v(T,x) = f(x)$, $x \in \mathbb{R}^m$, where

$$F(t,x,p,M) = \sup_{a \in A} \left\{ -\mu^*(t,x,a)p - \frac{1}{2}\mathrm{tr}((\sigma\sigma^*)(t,x,a)M) - g(t,x,a) \right\}$$

for $x \in \mathbb{R}^m$, $p \in \mathbb{R}^m$, and M, a symmetric and nonnegative definite $m \times m$ matrix. Here we have denoted by ∂_t the partial differential operator with respect to the time variable t, by D^j the j-th order partial differential operator with respect to the space variable x, and by c^* the transpose of a matrix c.

Since analytical solutions for HJB equations are rarely available, a number of numerical methods have been proposed in the literature. However, for practical applications there still remain several challenging problems. First one is that in some schemes the coefficients of the

*This paper is an abbreviated version of Nakano [16]. All proofs are omitted.

control problems have to be very restrictive for ensuring its convergence. For examples, in the finite-difference scheme (see, e.g., Kushner and Dupuis [13]), the diffusion matrix $\sigma\sigma^*$ in our notation should basically be diagonally dominant. Although this restriction can be weakened by considering the generalized finite-difference that involves a nontrivial adjustment of the diffusion matrix, we may need a large size of the stencil depending on a problem, as well as the further computational efforts for the implementation. See Bonnans and Zidani [3], Bonnans et al. [2] and the references therein. The optimal quantization approach taken by Pagès et al. [17] works under mild conditions on the coefficients, in particular, under quadratic growth conditions on f and g. However, their scheme requires most components of the process $\{X_s^{x,\pi}\}$ being actually control-free. In Fahim et al. [8], they extract an uncontrolled generator from the nonlinearity F, and then use a probabilistic representation of the HJB equation based on the process associated with such generator. Then the partial derivatives of the value function in that representation are computed by the expectations of random variables involving the value function itself via the integration-by-parts. These tricks cost some strong non-degeneracy conditions on F.

Second problem is computational difficulties in high-dimensional cases. In general, the finite-difference scheme needs a spatial grid with its size growing exponentially as the dimension m becomes large. As for the method by [17], the quantization error with respect to an ℓ-dimensional random variable is known to be $O(N^{1/\ell})$ if we denote by N the number of discretizing points for the random variable. The scheme by [8] can be applied to high-dimensional problems because it is based on Monte Carlo simulation for computing the conditional expectation, but the kernel density estimation used in this procedure is in general suffered from the curse-of-dimensionality. The finite-element like schemes studied by e.g., Camilli and Falcone [5] and Debrabant and Jakobsen [7], often called Semi-Lagrangian schemes, solve the first problem. That is, their scheme converge to the original value functions under no special assumption on the diffusion matrix $\sigma\sigma^*$. The difficulty in their scheme is that they require the interpolation of the value functions in the state space, and need involved computational procedures for the implementation in high-dimensional problems (see Carlini et al. [6]).

In this paper, we propose a new time-discretization scheme for the problem (1.1). It is based on a probabilistic representation for the convolution of v by a probability density function. In Section 2 below, we first give a rough derivation of the semi-discrete scheme, and then prove its convergence results by the viscosity solution method presented in Barles and Souganidis [1]. With the choices of the kernel and with the estimation methods for the conditional expectations, various numerical methods can be generated. Resulting numerical methods allow us to use uncontrolled Markov processes to estimate the conditional expectations in the dynamic programming procedure. Moreover, they can be implemented without the interpolation of the value function or the adjustment of the diffusion matrix. We present one of possible methods with Gaussian distributions in Section 3. A numerical experiment is also performed there. We focus on an artificial problem having two-dimensional state space, where an analytical solution for (1.3) is easily obtained.

Throughout this paper, we write $|a| = (\sum_{i,j} a_{i,j}^2)^{1/2}$ for a matrix $a = (a_{i,j})$. By C we denote positive constants that may not be necessarily equal with each other.

2 Approximation of the value function

We discuss the general stochastic control problem (1.1) under the following assumptions on the coefficients:

Assumption 2.1 *1. The functions μ, σ, f, g are Borel measurable with respect to (t,x,a) and continuous with respect to (t,x) uniformly over a.*

2. There exists a positive constant K such that, for every $s,t \in [0,T]$, $x,y \in \mathbb{R}^m$ and $a \in A$,

$$|\mu(t,x,a)| + |\sigma(t,x,a)| \leq K(1+|x|),$$
$$|\mu(s,x,a) - \mu(t,y,a)| + |\sigma(s,x,a) - \sigma(t,y,a)| \leq K|s-t| + K|x-y|,$$
$$|f(x)| + |g(t,x,a)| \leq K.$$

With these conditions, the controlled stochastic differential equation (1.2) has a unique strong solution for each control $\alpha \in \mathcal{A}$ (see, e.g., Fleming and Soner [9] or Krylov [12]) and the value function v in (1.1) becomes bounded. Moreover, it is known that v satisfies the viscosity solution property. To be precise, recall that an $\mathbb{R}$-valued, upper-semicontinuous function u on $[0,T] \times \mathbb{R}^m$ is said to be a viscosity subsolution of (1.3) if, for any $(t,x) \in [0,T) \times \mathbb{R}^m$ and any smooth function φ such that $u - \varphi$ has a local maximum at (t,x) we have

$$-\partial_t \varphi(t,x) + F(t,x,D\varphi(t,x), D^2\varphi(t,x)) \leq 0.$$

Similarly, an $\mathbb{R}$-valued, lower-semicontinuous function u on $[0,T] \times \mathbb{R}^m$ is said to be a viscosity supersolution of (1.3) if, for any $(t,x) \in [0,T) \times \mathbb{R}^m$ and any smooth function φ such that $u - \varphi$ has a local minimum at (t,x) we have

$$-\partial_t \varphi(t,x) + F(t,x,D\varphi(t,x), D^2\varphi(t,x)) \geq 0.$$

We say that u is a viscosity solution of (1.3) if it is both a viscosity subsolution and a viscosity supersolution of (1.3). Then, Theorem 4.3.1 in Pham [19] tells us that our value function v is indeed a viscosity solution of (1.3). In addition, by Theorem 4.4.5 in [19] the following comparison principle holds: for every bounded, upper-semicontinuous viscosity subsolution u of (1.3) and bounded lower-semicontinuous viscosity supersolution w of (1.3) such that $u(T,x) \leq w(T,x)$, $x \in \mathbb{R}^m$, we have

$$u(t,x) \leq w(t,x), \quad (t,x) \in [0,T] \times \mathbb{R}^m.$$

Let $\{t_i\}_{i=0}^n$ be a fixed set of time indices such that $0 = t_0 < t_1 < \cdots < t_n = T$ with $h = t_i - t_{i-1} = T/n$, $i = 1, \ldots, n$. We consider n sufficiently large so that $h \in (0,1]$. To find a time discretization scheme for the value function, first we write down the dynamic programming principle as follows:

$$v(t_i,x) = \inf_{\alpha \in \mathcal{A}} \mathbb{E}\left[v(t_{i+1}, X_{t_{i+1}}^{t_i,x,\alpha}) + \int_{t_i}^{t_{i+1}} g(s, X_s^{t_i,x,\alpha}, a_s) ds \right].$$

Replacing $X_{t_{i+1}}^{t_i,x,\alpha}$ by its Euler-Maruyama approximation $\hat{X}_{t_{i+1}}^{t_i,x,a}$ defined by

$$\hat{X}_s^{t,x,a} = x + \mu(t,x,a)(s-t) + \sigma(t,x,a)\sqrt{s-t}G, \quad t \le s \le t+h, \ \ a \in A,$$

formally we have

$$\mathbb{E}\left[v(t_{i+1}, X_{t_{i+1}}^{t_i,x,\alpha}) + \int_{t_i}^{t_{i+1}} g(s, X_s^{t_i,x,\alpha}, \alpha_s)ds\right] \approx \mathbb{E}\left[v(t_{i+1}, \hat{X}_{t_{i+1}}^{t_i,x,\alpha_{t_i}})\right] + hg(t_i,x,\alpha_{t_i}).$$

Here, $G = (G_1, \dots, G_d)^*$ is a random variable such that G_i's are mutually independent and that

$$\mathbb{E}[G_i] = 0, \ \ \mathbb{E}[G_iG_j] = \delta_{ij}, \ \ \mathbb{E}[|G_i|^3] < \infty, \quad i,j = 1, \dots, d, \tag{2.1}$$

where δ_{ij} denote the Kronecker's delta.

Now we consider the convolution $\phi^h * v(t_{i+1}, \cdot)$ by some kernel ϕ^h to approximate $v(t_{i+1}, \cdot)$. To this end, let ϕ be a probability density function on $\mathbb{R}^m$ with full support, i.e., $\phi(y) > 0$ for all $y \in \mathbb{R}^m$, and then define

$$\phi^h(x) := \frac{\phi(x/\lambda(h))}{\lambda(h)^m}, \quad x \in \mathbb{R}^m,$$

where λ is a positive function on $(0,1]$. Then, for any bounded function u, the convolution $\phi^h * u$ can be represented as

$$\begin{aligned}(\phi^h * u)(x+\xi) &= \int_{\mathbb{R}^m} u(z)\frac{\phi\left((x+\xi-z)/\lambda(h)\right)}{\lambda(h)^m}dz \\ &= \int_{\mathbb{R}^m} u(x+\lambda(h)z)\phi\left(\frac{\xi}{\lambda(h)} - z\right)dz, \quad x, \xi \in \mathbb{R}^m.\end{aligned}$$

Let $\mathcal{D}^h$ be a subset of $\mathbb{R}^m$ satisfying $\int_{\mathbb{R}^m \setminus \mathcal{D}^h} \phi(\xi/\lambda(h) - z)dz \to 0$, $h \searrow 0$, and let ψ^h be an another probability density function having $\mathcal{D}^h$ as the support set. Then, roughly speaking, $(\phi^h * u)(x+\xi)$ is approximated by $\int_{\mathcal{D}^h} u(x+\lambda(h)z)\phi(\xi/\lambda(h) - z)dz$ and this can be written as

$$\begin{aligned}\int_{\mathcal{D}^h} u(x+\lambda(h)z)\phi\left(\frac{\xi}{\lambda(h)} - z\right)dz &= \int_{\mathcal{D}^h} u(x+\lambda(h)z)\phi\left(\frac{\xi}{\lambda(h)} - z\right)\frac{\psi^h(z)}{\psi^h(z)}dz \\ &= \mathbb{E}\left[u(x+\lambda(h)Z^h)\phi\left(\frac{\xi}{\lambda(h)} - Z^h\right)\frac{1}{\psi^h(Z^h)}\right],\end{aligned}$$

where Z^h is a random variable with probability density ψ^h, which is assumed to be independent of G. Thus, denoting $H_h^{t,x,a} = \mu(t,x,a)h + \sigma(t,x,a)\sqrt{h}G$, we have

$$\begin{aligned}\mathbb{E}\left[(\phi^h * u)(\hat{X}_{t+h}^{t,x,a})\right] &\approx \mathbb{E}\left[\mathbb{E}\left[u(x+\lambda(h)Z^h)\phi\left(\frac{H_h^{t,x,a}}{\lambda(h)} - Z^h\right)\frac{1}{\psi^h(Z^h)}\,\middle|\, H_h^{t,x,a}\right]\right] \\ &= \mathbb{E}\left[u(x+\lambda(h)Z^h)\phi\left(\frac{H_h^{t,x,a}}{\lambda(h)} - Z^h\right)\frac{1}{\psi^h(Z^h)}\right].\end{aligned}$$

The last expression might be useful since the argument of u is control-free. Then, as $\lambda(h) \searrow 0$, the quantity $\mathbb{E}[(\phi^h * u)(\hat{X}_{t+h}^{t,x,a})]$ converges to $\mathbb{E}[u(\hat{X}_{t+h}^{t,x,a})]$ under mild conditions on u, so we expect that the scheme defined by

$$\begin{cases} v^h(T,x) = f(x), \\ v^h(t_i,x) = \Phi^h[v^h(t_{i+1},\cdot)](t_i,x) \end{cases}$$

with

$$\Phi^h[u](t,x) = \inf_{a\in A}\left\{\mathbb{E}\left[u(x+\lambda(h)Z^h)\phi\left(\frac{H_h^{t,x,a}}{\lambda(h)} - Z^h\right)\frac{1}{\psi^h(Z^h)}\right] + hg(t,x,a)\right\}$$

gives an approximation for the value function v.

Let β be a function on $(0,1]$ defined by

$$\beta(h) = \sup_{\substack{t\in[0,T],\ \xi\in\mathbb{R}^m \\ a\in A}} \mathbb{E}\int_{\mathbb{R}^m\setminus\mathcal{D}^h} \phi\left(\frac{H_h^{t,\xi,a}}{\lambda(h)} - z\right)dz. \tag{2.2}$$

Then we are able to state the following convergence result.

Theorem 2.2 *Let $v : [0,T]\times\mathbb{R}^m \to \mathbb{R}$ be defined by (1.1). Suppose that Assumption 2.1 is satisfied. Suppose also that*

$$\lim_{h\searrow 0}\frac{\lambda(h)+\beta(h)}{h} = 0.$$

Then,

$$v^h(t_i,y) \longrightarrow v(t,x)$$

as $h\searrow 0$, $t_i\to t$, and $y\to x$, uniformly on any compact subset of $\mathbb{R}^m$.

To obtain the rates of convergence of our schemes, we impose more regularity conditions on the coefficients.

Assumption 2.3 *There exist positive constants K' such that, for every $s,t\in[0,T]$, $x,y\in\mathbb{R}^m$ and $a\in A$,*

$$|f(x)-f(y)| + |g(s,x,a)-g(t,y,a)| \le K'|x-y| + K'|s-t|^{1/2},$$
$$|\mu(t,x,a)|\le K', \quad |\sigma(t,x,a)|\le K'.$$

Theorem 2.4 *Let $v : [0,T]\times\mathbb{R}^m\to\mathbb{R}$ be defined by (1.1). Suppose that Assumptions 2.1 and 2.3 are satisfied. Suppose also that $\mathbb{E}[G_i^3]=0$, $\mathbb{E}[G_i^4]<\infty$, $i=1,\dots,d$. If λ and β satisfy $\lambda(h)+\beta(h)\le Ch^{7/6}$, then we have*

$$\max_{0\le i\le n}|v^h(t_i,x)-v(t_i,x)|\le Ch^{1/6}, \quad x\in\mathbb{R}^m.$$

Furthermore, if $\lambda(h)+\beta(h)\le Ch^{3/2}$ then we obtain

$$-Ch^{1/6}\le v(t_i,x)-v^h(t_i,x)\le Ch^{1/4}, \quad x\in\mathbb{R}^m,\ i=1,\dots,n.$$

3 A fully numerical method

Throughout this section, we suppose that Assumptions 2.1 and 2.3 are satisfied. We take the $\mathbb{R}^d$-valued random variable $G = G^{M,q}$ as an optimally quantized random variable in the sense that $G^{M,q}$ is a minimizer of

$$\mathbb{E}\left[|X-G|^q\right]$$

over finite random variables G with M-supporting points, where $q \geq 1$ and X is a d-dimensional standard Gaussian random variable. We refer to Graf and Luschgy [10] for a detailed account of optimal quantization theory, and to Pagés and Printems [18] for numerical procedures for obtaining $G^{M,2}$. Set

$$G^{M,q}(\Omega) = \{\gamma_1, \ldots, \gamma_M\}, \qquad \mathbb{P}(G^{M,q} = \gamma_i) = p_i, \quad i = 1, \ldots, M.$$

In most of numerical realizations, the random variables $G^{M,q}$'s approximately satisfy the moment condition (2.1). A reason why we use the quantization is that we need sufficiently many points of $H_h^{t,x,a}/\lambda(h)$'s near its mean to be shot by Monte Carlo simulations of Gaussian random variables, with less number of computations. See the comment made just after (3.2) below.

We choose the function λ as

$$\lambda(h)/h \to 0, \quad h \searrow 0.$$

For example, we may take $\lambda(h) = h/(-\log h)$.

To describe the realizations of $H_h^{t,x,a}$, we put

$$\eta_i^h(t,x,a) = \mu(t,x,a)h + \sigma(t,x,a)\sqrt{h}\gamma_i, \quad i = 1, \ldots, M.$$

As the functions ϕ and ψ^h, we examine

$$\phi(y) = \frac{e^{-|\xi|^2/2}}{(2\pi)^{m/2}}, \qquad \psi^h(y) = \frac{e^{-|\xi|^2/(2\tau(h))}}{(2\pi\tau(h))^{m/2}}, \quad y \in \mathbb{R}^m,$$

where $\tau(h) > 0$.

Let Z be an m-dimensional standard Gaussian random variable. Then $\Phi^h[u]$ is described by

$$\Phi^h[u](t,x) = \inf_{a\in A}\left\{\sum_{j=1}^M p_j \mathbb{E}\left[u\left(x + \lambda(h)\sqrt{\tau(h)}Z\right)\phi\left(\frac{\eta_j^h(t,x,a)}{\lambda(h)} - \sqrt{\tau(h)}Z\right)\frac{1}{\psi^h\left(\sqrt{\tau(h)}Z\right)}\right] + hg(t,x,a)\right\}.$$

Furthermore, we introduce the uncontrolled Markov process $\{S_k\}_{k=0}^n$ defined by

$$S_{k+1} = S_k + \lambda(h)\sqrt{\tau(h)}Z_{k+1}, \quad k = 0, \ldots, n-1,$$

where $\{Z_k\}_{k=1}^n$ denote an i.i.d. sequence with $Z_1 \sim Z$. With this process, $\Phi^h[u]$ can be written as

$$\begin{aligned}&\Phi^h[u](t_k,x)\\ &= \inf_{a\in A}\left\{\tau(h)^{m/2}\sum_{i=1}^M p_i \mathbb{E}\left[u\left(S_{k+1}\right)\exp\left(-\frac{1}{2}\left|\frac{\eta_i^h(t_k,x,a)}{\lambda(h)} - \sqrt{\tau(h)}Z\right|^2 + \frac{1}{2}|Z|^2\right)\,\middle|\, S_k = x\right] + hg(t_k,x,a)\right\}.\end{aligned} \tag{3.1}$$

To compute the conditional expectation in (3.2), we use the kernel density estimators with samples from Monte Carlo simulation. We refer to Longstaff and Schwartz [15], a landmark study of this approach in American option pricing, and to Bouchard and Touzi [4] and Lemor et al. [14] in the context of backward stochastic differential equations. It should be mentioned that this regression method is also adopted in [8] for the numerical study of HJB equations.

In our framework, an estimator $\tilde{\mathbb{E}}^h[Y \mid S_k = x]$ for $\mathbb{E}[Y \mid S_k = x]$ can be taken as

$$\tilde{\mathbb{E}}^h[Y \mid S_k = x] = \frac{\sum_{j=1}^N Y^{(j)} \kappa\left(\frac{x - S_k^{(j)}}{\Delta}\right)}{\sum_{j=1}^N \kappa\left(\frac{x - S_k^{(j)}}{\Delta}\right)}$$

if the denominator is nonzero. Otherwise the estimator is defined as zero. Here,

$$Y = u\left(S_{k+1}^h\right) \exp\left(-\frac{1}{2}\left|\frac{\eta_j^h(t_k, x, a)}{\lambda(h)} - \sqrt{\tau(h)}Z\right|^2 + \frac{1}{2}|Z|^2\right),$$

$Y^{(1)}, \ldots, Y^{(N)}$, $Z^{(1)}, \ldots, Z^{(N)}$, and $S_k^{(1)}, \ldots, S_k^{(N)}$ are samples from Monte Carlo simulations for Y, Z, and S_k respectively, and Δ denotes the bandwidth for the kernel function κ. A typical example for κ is the naive kernel defined by

$$\kappa(z) = 1_{\{|z| \le 1\}}, \quad z \in \mathbb{R}^m.$$

We refer to Györfi et al. [11] for other estimators and its convergence analyses.

Consequently, our scheme can be implemented as follows: $V_n^{(\ell)} = f(S_n^{(\ell)})$, $\ell = 1, \ldots, N$, and

$$V_k^{(\ell)} = \inf_{a \in A} \left\{ \tau(h)^{m/2} \sum_{i=1}^M p_i \frac{\sum_{j=1}^N V_{k+1}^{(j)} \exp\left(-\frac{1}{2}\left|\frac{\eta_i^h(t_k, S_k^{(\ell)}, a)}{\lambda(h)} - \sqrt{\tau(h)}Z^{(j)}\right|^2 + \frac{1}{2}|Z^{(j)}|^2\right) \kappa\left(\frac{S_k^{(\ell)} - S_k^{(j)}}{\Delta}\right)}{\sum_{j=1}^N \kappa\left(\frac{S_k^{(\ell)} - S_k^{(j)}}{\Delta}\right)} \right.$$

$$\left. + hg(t_k, S_k^{(\ell)}, a) \right\}, \quad \ell = 1, \ldots, N, \quad k = 0, \ldots, n-1. \tag{3.2}$$

We note that in general we need to set $\tau(h)$ large so that the values

$$\exp\left(-\frac{1}{2}\left|\frac{\eta_i^h(t_k, S_k^{(\ell)}, a)}{\lambda(h)} - \sqrt{\tau(h)}Z^{(j)}\right|^2\right)$$

actually contribute to the computation of $V_k^{(\ell)}$.

As for control strategies, a minimizer $a_k^{(\ell)}$ of the right-hand side in (3.2) is a numerically optimal control at $(t_k, S_k^{(\ell)})$. In case one needs the values of reasonable control at given grids, the average values of $a_k^{(\ell)}$ distributed around the each grid point can be used.

Example 3.1 *As an illustration of our scheme, we examine the test problem described by*

$$T = 0.1, \quad d = 1, \quad m = 2, \quad A = \left\{a = (a_1, a_2)^* \in \mathbb{R}^2 : a_1^2 + a_2^2 = 1\right\},$$
$$\mu(t,x) = 0, \quad \sigma(t,x,a) = \sqrt{2}(a_1, a_2)^*,$$
$$f(x) = (1+T)T \sin x_1 \sin x_2, \quad g(t,x,a) = -2(1+t)Ta_1a_2 \cos x_1 \cos x_2 + tT \sin x_1 \sin x_2$$

for $t \in [0,T]$, $x = (x_1, x_2)^*$, and $a = (a_1, a_2)^* \in A$. This example is adopted in [7]. It is straightforward to see that the value function v in this problem is explicitly given by

$$v(t,x) = (1+t)T \sin x_1 \sin x_2,$$

and that any point in A is an optimal control.

In implementing our numerical method (3.2), we take $h = 0.01$, and set $\lambda(h) = h/(-\log(h))$ and $\tau(h) = (\sqrt{2h}/\lambda(h))^2$. We use $N = 3 \times 10^6$ samples to estimate the kernel density with the naive kernel and $\Delta = 0.1$. The quadratic quantization with $M = 100$ points is adopted for $G^{(M,q)}$. The initial value of the controlled process is set to be $(\pi/2, \pi/2)^*$. The resulting average absolute error between $V_k^{(\ell)}$ and $v(t_k, S_k^{(\ell)})$, $\ell = 1, \ldots, N$, at $k = 9$ is 0.0064. Figure 1 compares $v(t_k, x)$ and an estimated value function using $V_k^{(\ell)}$'s at $k = 9$. We use the average values of $V_k^{(\ell)}$'s in suitable clusters to estimate the value function at uniform grid points. The figure indicates that our method on the whole underestimates the analytical solution. This may be because the number of Monte Carlo paths shooting $\eta_i^h(t_k, S_k^{(\ell)}, a)/\lambda(h)$'s is still insufficient.

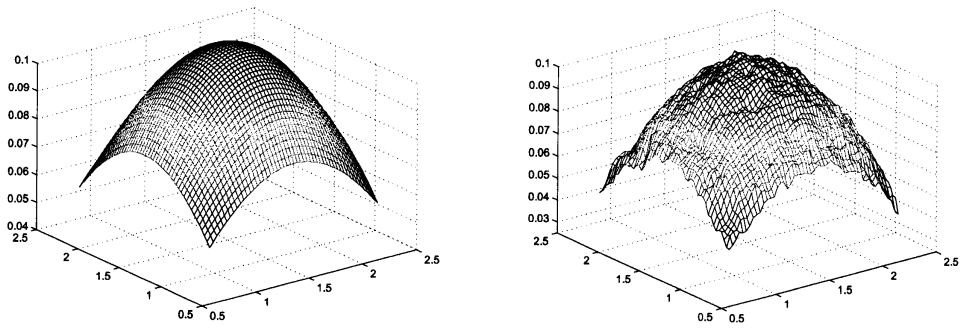

Figure 1: The analytical (left) and estimated (right) value functions at $t = 0.09$ around $x = (\pi/2, \pi/2)^*$.

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