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CHOQUET INTEGRAL REPRESENTATION OF COMONOTONICALLY ADDITIVE FUNCTIONALS

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ABSTRACT. The Daniell-Stone type representation theorem of Greco leads us to an improvement of the Riesz type representation theorem of Sugeno, Narukawa, and Murofushi for comonotonically additive, monotone functionals.

1. INTRODUCTION

Let X be a locally compact Hausdorff space. Let $C_{00}^{+}(X)$ denote the space of all nonnegative, continuous functions on X with compact support and let $C_{0}^{+}(X)$ denote the space of all nonnegative, continuous functions on X vanishing at infinity. In [8], Sugeno et al. succeeded in proving an analogue of the Riesz type integral representation theorem in nonadditive measure theory. More precisely, they gave a direct proof of the assertion that every comonotonically additive, monotone functional on $C_{00}^{+}(X)$ can be represented as the Choquet integral with respect to a nonadditive measure on X with some regularity properties. Their theorem gives a functional analytic characterization of the Choquet integrals and is inevitable in order to develop nonadditive measure theory based on the topology of the underlying spaces on which measures are defined.

In this paper, we give an improvement of the above theorem with the help of the Greco theorem [4], which is the most general Daniell-Stone type integral representation theorem for comonotonically additive, monotone functionals on function spaces. By using the same approach, we also give a Riesz type integral representation theorem for a bounded functional on $C_{0}^{+}(X)$.

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2. Notation and preliminaries

Let X be a non-empty set and let 2^X denote the family of all subsets of X. For each $A \subset X$, let χ_A denote the characteristic function of A. Let $\mathbb{R}$ and $\mathbb{R}^+$ denote the set of all real numbers and the set of all nonnegative real numbers, respectively. Also let $\overline{\mathbb{R}}$ and $\overline{\mathbb{R}}^+$ denote the set of all extended real numbers and the set of all nonnegative extended real numbers, respectively. Let $\mathbb{N}$ denote the set of all natural numbers. For any functions $f, g : X \to \overline{\mathbb{R}}$, let $f \vee g := \max(f, g)$ and $f \wedge g := \min(f, g)$. For any bounded f, let $\|f\|_\infty := \sup_{x \in X} |f(x)|$.

Definition 1. A set function $\mu : 2^X \to \overline{\mathbb{R}}^+$ is called a *nonadditive measure* on X if $\mu(\emptyset) = 0$ and $\mu(A) \leq \mu(B)$ whenever $A \subset B$.

Let μ be a nonadditive measure on X and let $f : X \to \overline{\mathbb{R}}^+$ be a function. Since the function $t \in \mathbb{R}^+ \mapsto \mu(\{f > t\})$ is non-increasing, it is Lebesgue integrable on $\mathbb{R}^+$. Therefore, the following formalization is well-defined; see [2] and [7].

Definition 2. Let μ be a nonadditive measure on X. The *Choquet integral* of a nonnegative function $f : X \to \overline{\mathbb{R}}^+$ with respect to μ is defined by

$$(\mathrm{C})\int_X f d\mu := \int_0^\infty \mu(\{f > t\}) dt,$$

where the right hand side of the above equation is the usual Lebesgue integral.

Remark 1. For any nonadditive measure μ on X and any function $f : X \to \overline{\mathbb{R}}^+$, the two Lebesgue integrals $\int_0^\infty \mu(\{f > t\})dt$ and $\int_0^\infty \mu(\{f \geq t\})dt$ are equal, since $\mu(\{f \geq t\}) \geq \mu(\{f > t\}) \geq \mu(f \geq t + \varepsilon\})$ for every $\varepsilon > 0$ and $0 \leq t < \infty$. This fact will be used implicitly in this paper.

See [3], [6], and [9] for more information on nonadditive measures and Choquet integrals.

For the reader's convenience, we introduce the Greco theorem [4, Proposition 2.2], which is the most general Choquet integral representation theorem for comonotonically additive, monotone, extended real-valued functionals. Recall that two functions $f, g : X \to \overline{\mathbb{R}}$ are *comonotonic* and is written by $f \sim g$ if, for every $x, x' \in X$, $f(x) < f(x')$ implies $g(x) \leq g(x')$.

Theorem 1 (The Greco theorem). *Let $\mathcal{F}$ be a non-empty family of functions $f : X \to \overline{\mathbb{R}}$. Assume that $\mathcal{F}$ satisfies*

(i) $0 \in \mathcal{F}$,

(ii) $f \geq 0$ *for every* $f \in \mathcal{F}$ *(nonnegativity), and*

(iii) *if* $f \in \mathcal{F}$ *and* $c \in \mathbb{R}^+$, *then* $cf,\ f \wedge c,\ f - f \wedge c = (f-c)^+ \in \mathcal{F}$ *(the Stone condition).*

Assume that a functional $I : \mathcal{F} \to \overline{\mathbb{R}}$ *satisfies*

(iv) $I(0) = 0$,

(v) *if* $f, g \in \mathcal{F}$ *and* $f \leq g$, *then* $I(f) \leq I(g)$ *(monotonicity),*

(vi) *if* $f, g \in \mathcal{F}, f + g \in \mathcal{F}$, *and* $f \sim g$, *then* $I(f+g) = I(f) + I(g)$ *(comonotonic additivity),*

(vii) $\lim_{a \to +0} I(f - f \wedge a) = I(f)$ *for every* $f \in \mathcal{F}$, *and*

(viii) $\lim_{b \to \infty} I(f \wedge b) = I(f)$ *for every* $f \in \mathcal{F}$.

For each $A \subset X$, *define the set functions* $\alpha, \beta : 2^X \to \overline{\mathbb{R}}^+$ *by*

$$\alpha(A) := \sup\{I(f) : f \in \mathcal{F},\ f \leq \chi_A\},$$
$$\beta(A) := \inf\{I(f) : f \in \mathcal{F},\ \chi_A \leq f\},$$

where let $\inf \emptyset := \infty$.

(1) *The set functions* α *and* β *are nonadditive measures on* X *with* $\alpha \leq \beta$.

(2) *For any nonadditive measure* λ *on* X, *the following two conditions are equivalent:*

(a) $\alpha \leq \lambda \leq \beta$.

(b) $I(f) = (\mathrm{C})\int_X f d\lambda$ *for every* $f \in \mathcal{F}$.

Remark 2. The functional I given in Theorem 1 is nonnegative, that is, $I(f) \geq 0$ for every $f \in \mathcal{F}$, and positively homogeneous, that is, $I(cf) = cI(f)$ for every $f \in \mathcal{F}$ and $c \in \overline{\mathbb{R}}^+$. See, for instance, [3, page 159] and [5, Proposition 4.2].

3. Riesz type integral representation theorems

In this section, we give an improvement of the Sugeno-Narukawa-Murofushi theorem [8, Theorem 3.7]. This can be done by the effective use of the Greco theorem and the following technical lemma.

Lemma 1. *Let* $\mathcal{F}$ *and* I *satisfy the same hypotheses as Theorem* 1.

(1) *Assume that, for any* $f \in \mathcal{F}$, *there is a* $g \in \mathcal{F}$ *such that* $\chi_{\{f>0\}} \leq g$ *and* $I(g) < \infty$ *(in particular,* $1 \in \mathcal{F}$ *and* $I(1) < \infty$*). Then, condition* (vii) *of Theorem* 1 *holds.*

(2) *Assume that every* $f \in \mathcal{F}$ *is bounded. Then, condition* (viii) *of Theorem* 1 *holds.*

(3) *Assume that every* $f \in \mathcal{F}$ *is bounded. Also assume that* I *is bounded, that is, there is a constant* $M > 0$ *such that* $I(f) \leq M\|f\|_\infty$ *for every* $f \in \mathcal{F}$. *Then, conditions* (vii) *and* (viii) *of Theorem* 1 *hold.*

From this point forwards, X is a locally compact Hausdorff space. For any real-valued function f on X, let $S(f)$ denote the support of f, which is defined by the closure of $\{f \neq 0\}$.

The following regularity properties give a tool to approximate general sets by more tractable sets such as open and compact sets. They are still important in nonadditive measure theory.

Definition 3. Let μ be a nonadditive measure on X.

(1) μ is said to be *outer regular* if, for every subset A of X, $\mu(A) = \inf\{\mu(G) : A \subset G, G \text{ is open}\}$.

(2) μ is said to be *quasi outer regular* if, for every compact subset K of X, $\mu(K) = \inf\{\mu(G) : K \subset G, G \text{ is open}\}$.

(3) μ is said to be *inner Radon* if, for every subset A of X, $\mu(A) = \sup\{\mu(K) : K \subset A, K \text{ is compact}\}$.

(4) μ is said to be *quasi inner Radon* if, for every open subset G of X, $\mu(G) = \sup\{\mu(K) : K \subset G, K \text{ is compact}\}$.

The following theorem is an improvement of [8, Theorem 3.7] and it has essentially been derived from the Greco theorem.

Theorem 2. *Let a functional* $I : C_{00}^+(X) \to \mathbb{R}$ *satisfy the following conditions:*

(i) *if* $f, g \in C_{00}^+(X)$ *and* $f \leq g$, *then* $I(f) \leq I(g)$ (*monotonicity*), *and*

(ii) *if* $f, g \in C_{00}^+(X)$ *and* $f \sim g$, *then* $I(f+g) = I(f) + I(g)$ (*comonotonic additivity*).

For each $A \subset X$, *define the set functions* $\alpha, \beta, \gamma : 2^X \to \overline{\mathbb{R}}^+$ *by*

$$\alpha(A) := \sup\{I(f) : f \in C_{00}^+(X), f \leq \chi_A\},$$

$$\beta(A) := \inf\{I(f) : f \in C_{00}^+(X), \chi_A \leq f\},$$

$$\gamma(A) := \sup\{I(f) : f \in C_{00}^+(X), 0 \leq f \leq 1, S(f) \subset A\},$$

where let $\inf \emptyset := \infty$, *and their regularizations* $\alpha^*, \beta^*, \gamma^* : 2^X \to \overline{\mathbb{R}}^+$ *by*

$$\alpha^*(A) := \inf\{\alpha(G) : A \subset G, G \text{ is open}\},$$

$$\beta^{**}(A) := \sup\{\beta(K) : K \subset A, K \text{ is compact}\},$$

$$\gamma^*(A) := \inf\{\gamma(G) : A \subset G, G \text{ is open}\}.$$

(1) *The set functions* $\alpha, \beta, \gamma, \alpha^*, \beta^{**}$, *and* γ^* *are nonadditive measures on* X.

(2) *For any nonadditive measure* λ *on* X, *the following two conditions are equivalent:*

   (a) $\alpha \leq \lambda \leq \beta$.

   (b) $I(f) = (\mathrm{C})\int_X f d\lambda$ *for every* $f \in C_{00}^+(X)$.

(3) $\gamma^*(K) = \beta(K) < \infty$ *for every compact subset* K *of* X.

(4) γ^* *is quasi inner Radon and outer regular.*

(5) β^{**} *is inner Radon and quasi outer regular.*

(6) $\beta^{**}(G) = \gamma(G)$ *for every open subset* G *of* X.

(7) *The defined nonadditive measures are comparable, that is,* $\alpha = \gamma \leq \beta^{**} \leq \alpha^* = \gamma^* \leq \beta$, *so that any of them is a representing measure of* I.

Remark 3. Define the functional $I : C_{00}^+(\mathbb{R}) \to \mathbb{R}$ by $I(f) := \int_{-\infty}^{\infty} f(t)dt$ for every $f \in C_{00}^+(\mathbb{R})$. Then I satisfies (i) and (ii) of Theorem 2, but it is not bounded. So, Theorem 2 does not follow from (3) of Lemma 1.

From Theorem 2 and Lemma 1, we can derive a representation theorem for *bounded*, comonotonically additive, monotone functionals on $C_0^+(X)$.

Theorem 3. *Let a functional* $I : C_0^+(X) \to \mathbb{R}$ *satisfy*

(i) *if* $f, g \in C_0^+(X)$ *and* $f \leq g$, *then* $I(f) \leq I(g)$ (*monotonicity*),

(ii) *if* $f, g \in C_0^+(X)$ *and* $f \sim g$, *then* $I(f+g) = I(f) + I(g)$ (*comonotonic additivity*), *and*

(iii) *there is a constant* $M > 0$ *such that* $I(f) \leq M\|f\|_\infty$ *for every* $f \in C_0^+(X)$ (*boundedness*).

For each $A \subset X$, *define the set functions* $\alpha, \beta, \gamma : 2^X \to \overline{\mathbb{R}}^+$ *by*

$$\alpha(A) := \sup\{I(f) : f \in C_0^+(X), f \leq \chi_A\},$$
$$\beta(A) := \inf\{I(f) : f \in C_0^+(X), \chi_A \leq f\},$$
$$\gamma(A) := \sup\{I(f) : f \in C_0^+(X), 0 \leq f \leq 1, S(f) \subset A\},$$

where let $\inf \emptyset := \infty$, *and their regularizations* $\alpha^*, \beta^{**}, \gamma^* : 2^X \to \overline{\mathbb{R}}^+$ *by*

$$\alpha^*(A) := \inf\{\alpha(G) : A \subset G, G \text{ is open}\},$$
$$\beta^{**}(A) := \sup\{\beta(K) : K \subset A, K \text{ is compact}\},$$
$$\gamma^*(A) := \inf\{\gamma(G) : A \subset G, G \text{ is open}\}.$$

(1) *The set functions* α, β, γ, α^*, β^{**}, *and* γ^* *are nonadditive measures on* X *and they satisfy properties* (3)–(7) *in Theorem* 2.

(2) *For any nonadditive measure λ on X, the following two conditions are equivalent:*

(a) $\alpha \leq \lambda \leq \beta$.

(b) $I(f) = (\mathrm{C})\int_X f d\lambda$ *for every* $f \in C_0^+(X)$.

(3) $\alpha(X) = \gamma(X) = \alpha^*(X) = \gamma^*(X) < \infty$.

(4) *Let λ be a nonadditive measure on X with $\lambda(X) < \infty$. Define the functional $I : C_0^+(X) \to \mathbb{R}$ by $I(f) := (\mathrm{C})\int_X f d\lambda$ for every $f \in C_0^+(X)$. Then I satisfies conditions* (i)–(iii).

4. Conclusion

In this paper, we gave an improvement of the Riesz type integral representation theorem of Sugeno, Narukawa, and Murofushi by the help of the Daniell-Stone type integral representation theorem of Greco. By using the same approach, we also gave a Riesz type integral representation theorem for a bounded functional on $C_0^+(X)$. Our approach will lead us to various Riesz type integral representation theorems on a wide variety of function spaces and sequence spaces.

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