数理解析研究所講究録
第 1820 巻 2012 年 69-72

Integral representation of monotone functions

Motoya Machida
Tennessee Technological University, Cookeville, TN
mmachida@tntech.edu

Abstract

Integral representation of monotone functions has been studied by Choquet [1], Murofushi and Sugeno [4], Norberg [5], and many others, but not necessarily been their primal interest due to the lack of uniqueness in their representations. Here we present a brief overview of different approaches and generalizations, and show our own version of integral representation from the ongoing investigation.

1 Choquet theory of integral representation

In his treatise on theory of capacity, Choquet outlined a series of applications for integral representation on the set $\mathcal{E}$ of extreme points of a compact convex Hausdorff space $\mathcal{C}$ (Chapter VII of [1]). Let L be a partially ordered set (poset) with a maximum element e, and let $\mathcal{C}$ be the convex set of nonnegative monotone functions φ on L with $\varphi(e) \leq 1$. Assuming the topology of simple (i.e., pointwise) convergence on functions over L, we can show that $\mathcal{C}$ is compact, and the set $\mathcal{E}$ of extreme points of $\mathcal{C}$ consists of indicator functions of the form

$$\chi(x) = \begin{cases} 1 & \text{if } x \in U; \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

The monotonicity of χ implies that $y \in U$ whenever $x \in U$ and $x \leq y$, and such subset U is called an *upper* set. The set $\mathcal{E}$ is compact, and any element φ of $\mathcal{C}$ is represented in the integral form

$$\varphi(x) = \int \chi(x)\, d\mu(\chi), \quad x \in L, \tag{2}$$

with a Radon measure μ on $\mathcal{E}$ (Section 40 of [1]).

Let S be a compact Hausdorff space, and $\mathcal{K}$ be the class of compact subsets of S. Then a nonnegative monotone function φ on $\mathcal{K}$ is called a *capacity* if it is upper semicontinuous (i.e., $\varphi(E) \downarrow \varphi(F)$ whenever $E \downarrow F$) in the exponential (i.e., Vietoris) topology. Here the convex set $\mathcal{C}$ of capacities φ with $\varphi(S) \leq 1$ is considered similarly; however, the topology of simple convergence is not suitable for the space $\mathcal{C}$. Over the convex cone $\mathcal{Q}$ of nonnegative continuous functions on S, a capacity φ uniquely corresponds to the functional

$$\varphi(\xi) = \int_0^{\max \xi} \varphi(\{x \in E : \xi(x) \geq r\})\, dr, \quad \xi \in \mathcal{Q}. \tag{3}$$

Then we can introduce the topology of vague convergence on capacities in which a net $\{\varphi_\alpha\}$ converges to φ if and only if $\varphi_\alpha(\xi)$ converges to $\varphi(\xi)$ for any $\xi \in \mathcal{Q}$. Under this topology the convex set $\mathcal{C}$ is compact Hausdorff, and the indicator function χ in (1) corresponds to a closed upper set U in the exponential topology (Section 48 of [1]).

When S is a locally compact Hausdorff space, it is not necessary for $\mathcal{K}$ to contain S. Here we can introduce a partial ordering on $\mathcal{K}$ by the dual (i.e., the reverse order) of inclusion, and denote the poset by L with the maximum element $\varnothing$. Then we can set the convex set $\mathcal{C}^*$ of lower semicontinuous and nonnegative monotone functions φ on L with $\varphi(\varnothing) \leq 1$. Observe that a lower semicontinuous and nonnegative monotone functions φ on L uniquely corresponds to a bounded capacity ψ on $\mathcal{K}$ via

$$\varphi(E) = \sup_{F \in \mathcal{K}} \psi(F) - \psi(E) + \psi(\varnothing), \quad E \in \mathcal{K}.$$

The topology of vague convergence is introduced by (3) over the convex cone $\mathcal{Q}$ of nonnegative continuous functions with compact support, in which the convex set $\mathcal{C}^*$ becomes compact Hausdorff.

2 A framework of continuous semilattice

In the application of integral representation for capacities on a locally compact Hausdorff S, the Hausdorff assumption seems indispensable in order for $\mathcal{C}^*$ to be compact Hausdorff. Then the set $\mathcal{E}^*$ of extreme points of $\mathcal{C}^*$ is compact and homeomorphic to the family of open upper subsets U, and the integral representation (2) of $\varphi \in \mathcal{C}^*$ is equivalently formulated as

$$\varphi(x) = \mu(\mathcal{U}_x), \quad x \in L, \tag{4}$$

where $\mathcal{U}_x := \{U \in \mathcal{E}^* : x \in U\}$ is an open set in $\mathcal{E}^*$.

In the framework of continuous posets (cf. Giertz et al. [3]), the compact Hausdorff set $\mathcal{E}^*$ is homeomorphic to the family of Scott open subsets of L. Here the topology of vague convergence corresponds to the Lawson topology, which comes solely from the fact that L is a continuous semilattice. Norberg [5] showed that it is entirely possible to construct a Borel measure μ on the family $\mathcal{E}^*$ of Scott open subsets satisfying (4) if L is a continuous semilattice and $\mathcal{E}^*$ is second countable. Thus, we can choose S to be a locally compact sober and second countable space, which is not necessarily Hausdorff. Note that the Borel measure μ is a Radon measure when $\mathcal{E}^*$ is second countable; see [2].

We claim that $\mathcal{E}^*$ is not necessarily second countable, and demonstrate it by a rather straightforward construction of a Radon measure μ satisfying (4) due to

Murofushi and Sugeno [4]. Let $\varphi \in \mathcal{C}^*$ be fixed, and let e denote the top element of the continuous semilattice L. Observe that

$$F(r) = \{x \in L : \varphi(x) > r\} \tag{5}$$

maps from $r \in [0, \varphi(e))$ to $\mathcal{E}^*$, and F is Borel-measurable. For a Borel measurable subset $\mathcal{V}$ of $\mathcal{E}^*$ we can define $\mu(\mathcal{V}) := m(F^{-1}(\mathcal{V}))$ with the Lebesgue measure m on $[0, \varphi(e))$. Then we can show that μ is a Radon measure, and it satisfies

$$\mu(\mathcal{U}_x) = m([0, \varphi(x))) = \varphi(x).$$

It should be noted that Norberg [5] has investigated a Borel measure μ on the family $\mathcal{L}^*$ of Scott open filters in L, and proved a bijection between Borel measures on $\mathcal{L}^*$ and lower semicontinuous and completely monotone nonnegative functions on L. The above construction immediately fails for this purpose since (5) does not map into $\mathcal{L}^*$ in general even if φ is completely monotone.

Finally we present our own version of construction without assuming the second countable $\mathcal{E}^*$. Let $C(\mathcal{E}^*)$ be the space of continuous functions on $\mathcal{E}^*$, and let δ_x be a point mass probability measure (i.e., Dirac delta) at $x \in L$. Here we will use the following proposition, but leave the proof for the future publication.

Proposition 1. *There exists a subspace $\mathcal{R}$ of $C(\mathcal{E}^*)$ such that (i) each $g \in \mathcal{R}$ is uniquely extended to a signed Radon measure R on L so that $g(U) = R(U)$ for any $U \in \mathcal{E}^*$, and (ii) for each $x \in L$ there is an increasing net $\{g_\alpha\}$ of $\mathcal{R}$ satisfying $\sup_\alpha g_\alpha(U) = \delta_x(U)$ for any $U \in \mathcal{E}^*$.*

For a fixed $\varphi \in \mathcal{C}^*$, we can introduce a nonnegative homogeneous and superadditive functional on $C(\mathcal{E}^*)$ by

$$M(g) = \sup\left\{\int \varphi\, dR : R \le g, R \in \mathcal{R}\right\}, \quad g \in C(\mathcal{E}^*).$$

By applying the Hahn-Banach theorem we obtain a linear functional Φ on $C(\mathcal{E}^*)$ satisfying (a) $M \le \Phi$ on $C(\mathcal{E}^*)$, and (b) $M = \Phi$ on $\mathcal{R}$. The condition (a) implies that Φ is positive, and that Φ uniquely corresponds to a Radon measure μ on $\mathcal{E}^*$ via the Riesz representation $\Phi(g) = \int g\, d\mu$. By applying Proposition 1 together with the condition (b), we can show that if an increasing net $\{R_\alpha\}$ of $\mathcal{R}$ satisfies $\sup_\alpha R_\alpha(U) = \delta_x(U)$ for $U \in \mathcal{E}^*$ then

$$\mu(\mathcal{U}_x) = \sup_\alpha \Phi(R_\alpha) = \sup_\alpha M(R_\alpha) = \sup_\alpha \int \varphi\, dR_\alpha = \varphi(x),$$

as desired. A variation of this construction can be used to show the existence of a Radon measure μ whose support lies on $\mathcal{L}^*$ when φ is completely monotone (which is a part of the ongoing investigation).

References

[1] Choquet, G. (1954). Theory of capacities. *Ann. Inst. Fourier* **5**, 131–295.

[2] Folland, G. B. (1984). *Real Analysis*. John Wiley & Sons, New York.

[3] Giertz, G., Hofmann, K. H., Keimel, K., Lawson, J. D., Mislove, M., and Scott, D. S. (2003). *Continuous Lattices and Domains*. Cambridge University Press, Cambridge.

[4] Murofushi, T. and Sugeno, M. (1991). A theory of fuzzy measures: Representations, the Choquet integral, and null sets. *J. Math. Anal. Appl.* **159**, 532–549.

[5] Norberg, T. (1989). Existence theorems for measures on continuous posets with applications to random set theory. *Math. Scand.* **64**, 15–51.