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The Arcsine Law and "Quantum-Classical Correspondence"

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1 Introduction

We will discuss an aspect of a famous classical probability distribution known as the Arcsine law from the view point of "quantum-classical correspondence".

The (normalized) Arcsine law μ_{As} is the probability distribution on $\mathbb{R}$ with support $[-\sqrt{2}, \sqrt{2}]$ defined as

$$\mu_{As}(dx) = \frac{1}{\pi}\frac{dx}{\sqrt{2-x^2}},$$

whose n-th moment $M_n := \int_{\mathbb{R}} x^n \mu_{As}(dx)$ is given by

$$M_{2m+1} = 0, \quad M_{2m} = \frac{1}{2^m}\binom{2m}{m}.$$

In this case, the moment sequence $\{M_n\}$ characterize μ_{As} ("the moment problem is deterministic").

2 Noncommutative Probability

To describe the relation between the Arcsine law and "quantum-classical correspondence", we need some basic notions in "Noncommutative probability".

Noncommmutative probability is a generalization of probability theory, partly motivated to include quantum thoery in itself. The basic notion is "a state defined on a $*-$ algebra".

Let $\mathcal{A}$ be a $*$-algebra ("observable algebra"). We call a linear map $\varphi : \mathcal{A} \to \mathbb{C}$ a state on $\mathcal{A}$ if it satisfies

$$\varphi(1) = 1, \quad \varphi(a^* a) \geq 0.$$

A pair $(\mathcal{A}, \varphi)$ of a $*$-algebra and a state on it is called an algebraic probability space. When the algebra $\mathcal{A}$ is commutative, the theory can be reduced to classical probability theory. On the other hand, noncommutativity of the algebra leads to "Unceretainty principle" or "Bell's inequality", which characterize the limitation of classical theory, under the positivity condition in the definition of state.

Here we adopt a notation for a state $\varphi : \mathcal{A} \to \mathbb{C}$, an element $X \in \mathcal{A}$ and a probability distribution μ on $\mathbb{R}$.

Notation 2.1. $X \sim_{\varphi} \mu \Longleftrightarrow \quad \varphi(X^m) = \int_{\mathbb{R}} x^m \mu(dx)$ for all $m \in \mathbb{N}$.

Remark 2.2. Existence of μ for X which satisfies $X \sim_{\varphi} \mu$ always holds. The uniqueness of such μ holds if the moment problem is deterministic.

3 Quantum Harmonic Oscillator

A classical Harmonic oscillator is a movement which occurs under "Hooke's law" (Force is proportional to displacement). As long as force is sufficiently small, Hooke's law holds universally. So a harmonic oscillator is one of the most important model case of mechanics. However, in the quantum region, we need more subtle treatment of harmonic oscillators using a kind of algebraic probability space defined as follows.

Definition 3.1 (Quantum Harmonic Oscillator). A quantum harmonic oscillator is a quadruple $(\Gamma(\mathbb{C}), \{\Phi_n\}_{n=0}^{\infty}, a, a^*)$ where $\Gamma(\mathbb{C})$ is a Hilbert space $\Gamma(\mathbb{C}) := \oplus_{n=0}^{\infty} \mathbb{C}\Phi_n$ with inner product given by $< \Phi_n, \Phi_m >= \delta_{n,m}$, and a, a^* are operators defined as follows:

$$a\Phi_0 = 0, \quad a\Phi_n = \sqrt{n}\Phi_{n-1} (n \geq 1)$$

$$a^*\Phi_n = \sqrt{n+1}\Phi_{n+1}$$

Let $\mathcal{A}$ be the $*$-algebra generated by a, a^* and $\varphi_n(\cdot)$ be the state defined as $\varphi(\cdot) :=< \Phi_n, (\cdot)\Phi_n >$. Then $(\mathcal{A}, \varphi_n(\cdot))$ becomes an algebraic probability space. It is well known that $\frac{1}{\sqrt{2}}(a + a^*)$ represents the "position" observable and that

$$\frac{1}{\sqrt{2}}(a + a^*) \sim_{\varphi_0} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx.$$

That is, in $n = 0$ case, the distribution of position operator is Gaussian ("zero point motion").

On the other hand, the asymptotic behavior of the distribution of position operator as n tends to infinity is quite nontrivial.

4 Emergence of the Arcsine Law

Theorem 4.1. *Let μ_N be a probability distribution on $\mathbb{R}$ such that*

$$\frac{a+a^*}{\sqrt{2(N+1)}} \sim_{\varphi_n} \mu_N.$$

Then μ_N weakly converge to μ_{As}.

Proof. We only have to prove moment convergece.

First we can easily prove that

$$\varphi_N((\frac{a+a^*}{\sqrt{2(N+1)}})^{2m+1}) =< \Phi_N, (\frac{a+a^*}{\sqrt{2(N+1)}})^{2m+1}\Phi_N >= 0$$

since $< \Phi_N, \Phi_M >= 0$ when $N \neq M$. To consider the moments of even degrees, we introduce the following notations:

- $\Lambda^{2m} := \{\text{maps from } \{1, 2, ..., 2m\} \text{ to } \{1, *\}\}$,
- $\Lambda^{2m}_m := \{\lambda \in \Lambda^{2m}; |\lambda^{-1}(1)| = |\lambda^{-1}(*)| = m\}$.

Then

$$\begin{aligned}
\varphi_N((\frac{a+a^*}{\sqrt{2(N+1)}})^{2m}) &= \tfrac{1}{2^m} \textstyle\sum_{\lambda\in\Lambda^{2m}} \tfrac{1}{(N+1)^m} < \Phi_N, a^{\lambda_1}a^{\lambda_2}\cdots a^{\lambda_{2m}}\Phi_N > \\
&= \tfrac{1}{2^m} \textstyle\sum_{\lambda\in\Lambda^{2m}_m} \tfrac{1}{(N+1)^m} < \Phi_N, a^{\lambda_1}a^{\lambda_2}\cdots a^{\lambda_{2m}}\Phi_N > \\
&\to \quad \tfrac{1}{2^m}|\Lambda^{2m}_m| = \tfrac{1}{2^m}\tbinom{2m}{m} \quad (N \to \infty)
\end{aligned}$$

because

$$N(N-1)\cdots(N-m+1) \leq< \Phi_N, a^{\lambda_1}a^{\lambda_2}\cdots a^{\lambda_{2m}}\Phi_N >\leq (N+1)(N+2)\cdots(N+m)$$

for sufficiently large N and then

$$\frac{1}{(N+1)^m} < \Phi_N, a^{\lambda_1}a^{\lambda_2}\cdots a^{\lambda_{2m}}\Phi_N > \to 1 \quad (N \to \infty).$$

This completes the proof. $\square$

5 The Physical Meaning of the Result

Here we discuss a physical implication of the result above: Let us see the relationship between the Arcsine law and the Classical harmonic oscillator.

Let $x(t) = A\sin(t)$ be a harmonic oscillator with amplitude A. Then it is easy to see that "the probability distribution $\mu$of x at random time t" has a form

$$\mu(dx) = C\frac{dx}{\sqrt{A^2 - x^2}}$$

where C denotes the normalizing constant. In $A = \sqrt{2}$ case, $\mu = \mu_{As}$.

Therefore, the meaning of the result above is, roughly, "the (time averaged) behavior of quantum harmonic oscillator tends to that of classical harmonic oscillator when the quantum number (energy level) tends to infinity", which is nothing but a typical example of "quantum-classical correspondence". It is related to very fundamental aspects of Quantum theory and asymptotic analysis [1]. We have analyzed it from the viewpoint of noncommutative algebraic probability with quite a simple combinatorial argument.

References

[1] H. Ezawa "Asymptotic analysis" (in Japanese) Iwanami-Shoten.