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Existence of positive solution for the Cauchy problem for an ordinary differential equation

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Abstract

In this paper we consider the existence of positive solution for the Cauchy problem of the second order differential equation $u''(t)=f(t,u(t))$.

1 Introduction

The following ordinary differential equations arise in many different areas of applied mathematics and physics; see [2, 4]. In [3] Knežević-Miljanović considered the Cauchy problem

$$\begin{cases} u''(t)=P(t)t^{a}u(t)^{\sigma},\ t\in(0,1],\\ u(0)=0,\ u'(0)=\lambda, \end{cases} \tag{1}$$

where $a,\sigma,\lambda\in\mathbf{R}$ with $\sigma<0$ and $\lambda>0$, and P is a continuous mapping of $[0,1]$ such that $\int_{0}^{1}|P(t)|t^{a+\sigma}dt<\infty$. On the other hand in [1] Erbe and Wang considered the equation

$$u''(t)=f(t,u(t)),\ t\in(0,1]. \tag{2}$$

In this paper we consider the second order Cauchy problem

$$\begin{cases} u''(t)=f(t,u(t)),\ \text{for almost every } t\in[0,1],\\ u(0)=0,\ u'(0)=\lambda, \end{cases} \tag{3}$$

where f is a mapping from $[0,1]\times(0,\infty)$ into $\mathbf{R}$ satisfying the Carathéodory condition and $\lambda\in\mathbf{R}$ with $\lambda>0$.

2 Main results

Theorem 2.1. *Suppose that a mapping f from $[0,1] \times (0, \infty)$ into $\mathbf{R}$ satisfies the following.*

(a) *The mapping f satisfies the Carathéodory condition, that is, the mapping $t \longmapsto f(t,u)$ is measurable for any $u \in (0,\infty)$ and the mapping $u \longmapsto f(t,u)$ is continuous for almost every $t \in [0,1]$.*

(b) $|f(t,u_1)| \geq |f(t,u_2)|$ *for almost every $t \in [0,1]$ and for any $u_1, u_2 \in (0,\infty)$ with $u_1 \leq u_2$.*

(c) *There exists $\alpha \in \mathbf{R}$ with $0 < \alpha < \lambda$ such that*

$$\int_0^1 |f(t,\alpha t)|dt < \infty.$$

(d) *There exists $\beta \in \mathbf{R}$ with $\beta > 0$ such that*

$$\left|\frac{\partial f}{\partial u}(t,u)\right| \leq \frac{\beta |f(t,u)|}{u}$$

for almost every $t \in [0,1]$ and for any $u \in (0,\infty)$.

Then there exist $h \in \mathbf{R}$ with $0 < h \leq 1$ such that the Cauchy problem (3) has a unique solution in X, where X is a subset

$$X = \left\{ u \;\middle|\; \begin{array}{l} u \in C[0,h], u(0) = 0, u'(0) = \lambda \\ \textit{and } \alpha t \leq u(t) \textit{ for any } t \in [0,h] \end{array} \right\}$$

of $C[0,h]$, which is the class of continuous mappings from $[0,h]$ into $\mathbf{R}$.

Proof. It is noted that $C[0,h]$ is a Banach space by the maximum norm

$$\|u\| = \max\{|u(t)| \mid t \in [0,h]\}.$$

Instead of the Cauchy problem (3) we consider the integral equation

$$u(t) = \lambda t + \int_0^t (t-s) f(s,u(s))ds.$$

By the condition (c) there exists $h \in \mathbf{R}$ with $0 < h \leq 1$ such that

$$\int_0^h |f(t,\alpha t)|dt < \min\left\{\lambda - \alpha, \frac{\alpha}{\beta}\right\}.$$

Let A be an operator from X into $C[0,h]$ defined by

$$Au(t) = \lambda t + \int_0^t (t-s) f(s,u(s))ds.$$

Since a mapping $t \longmapsto \lambda t$ belongs to X, $X \neq \emptyset$. Moreover $A(X) \subset X$. Indeed by the condition (a) $Au \in C[0,h]$, $Au(0) = 0$,

$$(Au)'(0) = \left[\lambda + \int_0^t f(s,u(s))ds\right]_{t=0} = \lambda$$

and by the condition (b)

$$\begin{aligned} Au(t) &= \lambda t + \int_0^t (t-s) f(s,u(s))ds \\ &\geq \lambda t - t\int_0^h |f(s,u(s))|ds \\ &\geq \lambda t - t\int_0^h |f(s,\alpha s)|ds \\ &\geq \alpha t \end{aligned}$$

for any $t \in [0,h]$. We will find a fixed point of A. Let φ be an operator from X into $C[0,h]$ defined by

$$\varphi[u](t) = \begin{cases} \frac{u(t)}{t}, & \text{if } t \in (0,h], \\ \lambda, & \text{if } t = 0, \end{cases}$$

and

$$\begin{aligned} \varphi[X] &= \{\varphi[u] \mid u \in X\} \\ &= \{v \mid v \in C[0,h], v(0) = \lambda \text{ and } \alpha \leq v(t) \text{ for any } t \in [0,h]\}. \end{aligned}$$

Then $\varphi[X]$ is a closed subset of $C[0,h]$ and hence it is a complete metric space. Let Φ be an operator from $\varphi[X]$ into $\varphi[X]$ defined by

$$\Phi\varphi[u] = \varphi[Au].$$

By the mean value theorem for any $u_1, u_2 \in X$ there exists a mapping ξ such that

$$\frac{f(t,u_1(t)) - f(t,u_2(t))}{u_1(t) - u_2(t)} = \frac{\partial f}{\partial u}(t,\xi(t))$$

and

$$\min\{u_1(t), u_2(t)\} \leq \xi(t) \leq \max\{u_1(t), u_2(t)\}$$

for any $t \in [0,h]$. By the conditions (b) and (d)

$$\begin{aligned} |f(t,u_1(t)) - f(t,u_2(t))| &= \left|\frac{\partial f}{\partial u}(t,\xi(t))(u_1(t) - u_2(t))\right| \\ &\leq \left|\frac{\beta f(t,\xi(t))}{\xi(t)}\right| |u_1(t) - u_2(t)| \\ &\leq \left|\frac{\beta f(t,\alpha t)}{\alpha t}\right| |u_1(t) - u_2(t)| \end{aligned}$$

for almost every $t \in [0, h]$. Therefore

$$\begin{aligned}
|\Phi\varphi[u_1](t) - \Phi\varphi[u_2](t)| &= \left|\frac{1}{t}\int_0^t (t-s)(f(s,u_1(s)) - f(s,u_2(s)))ds\right| \\
&\leq \int_0^h \left|\frac{\beta f(s,\alpha s)}{\alpha s}\right| |u_1(s) - u_2(s)|ds \\
&\leq \frac{\beta}{\alpha}\int_0^h |f(s,\alpha s)|ds\|\varphi[u_1] - \varphi[u_2]\|
\end{aligned}$$

for any $t \in [0, h]$. Therefore

$$\|\Phi\varphi[u_1] - \Phi\varphi[u_2]\| \leq \frac{\beta}{\alpha}\int_0^h |f(s,\alpha s)|ds\|\varphi[u_1] - \varphi[u_2]\|.$$

By the Banach fixed point theorem there exists a unique mapping $\varphi[u] \in \varphi[X]$ such that $\Phi\varphi[u] = \varphi[u]$. Then $Au = u$. □

Theorem 2.2. *Suppose that a mapping f from $[0,1] \times (0,\infty)$ into $\mathbf{R}$ satisfies the following.*

(a) *The mapping f satisfies the Carathéodory condition, that is, the mapping $t \longmapsto f(t,u)$ is measurable for any $u \in (0,\infty)$ and the mapping $u \longmapsto f(t,u)$ is continuous for almost every $t \in [0,1]$.*

(e) *$|f(t,u_1)| \leq |f(t,u_2)|$ for almost every $t \in [0,1]$ and for any $u_1, u_2 \in (0,\infty)$ with $u_1 \leq u_2$.*

(f) *There exists $\alpha \in \mathbf{R}$ with $0 < \alpha < \lambda$ such that*

$$\int_0^1 |f(t,(2\lambda - \alpha)t)|dt < \infty.$$

(d) *There exists $\beta \in \mathbf{R}$ with $\beta > 0$ such that*

$$\left|\frac{\partial f}{\partial u}(t,u)\right| \leq \frac{\beta|f(t,u)|}{u}$$

for almost every $t \in [0,1]$ and for any $u \in (0,\infty)$.

Then there exist $h \in \mathbf{R}$ with $0 < h \leq 1$ such that the Cauchy problem (3) has a unique solution in X, where X is a subset

$$X = \left\{ u \;\middle|\; \begin{array}{l} u \in C[0,h], u(0) = 0, u'(0) = \lambda \\ \text{and } \alpha t \leq u(t) \leq (2\lambda - \alpha)t \text{ for any } t \in [0,h] \end{array} \right\}$$

of $C[0,h]$.

Proof. By the condition (f) there exists $h \in \mathbf{R}$ with $0 < h \leq 1$ such that

$$\int_0^h |f(t,(2\lambda-\alpha)t)|dt < \min\left\{\lambda-\alpha, \frac{\alpha}{\beta}\right\}$$

and let A be an operator from X into $C[0,h]$ defined by

$$Au(t) = \lambda t + \int_0^t (t-s)f(s,u(s))ds.$$

Since a mapping $t \longmapsto \lambda t$ belongs to X, $X \neq \emptyset$. Moreover $A(X) \subset X$. Indeed by the condition (a) $Au \in C[0,h]$, $Au(0) = 0$,

$$(Au)'(0) = \left[\lambda + \int_0^t f(s,u(s))ds\right]_{t=0} = \lambda$$

and by the condition (e)

$$\begin{aligned} Au(t) &= \lambda t + \int_0^t (t-s)f(s,u(s))ds \\ &\geq \lambda t - t\int_0^h |f(s,u(s))|ds \\ &\geq \lambda t - t\int_0^h |f(s,(2\lambda-\alpha)s)|ds \\ &\geq \alpha t \end{aligned}$$

and

$$\begin{aligned} Au(t) &= \lambda t + \int_0^t (t-s)f(s,u(s))ds \\ &\leq \lambda t + t\int_0^h |f(s,u(s))|ds \\ &\leq \lambda t + t\int_0^h |f(s,(2\lambda-\alpha)s)|ds \\ &\leq (2\lambda-\alpha)t \end{aligned}$$

for any $t \in [0,h]$. We will find a fixed point of A. Let φ be an operator from X into $C[0,h]$ defined by

$$\varphi[u](t) = \begin{cases} \frac{u(t)}{t}, & t \in (0,h], \\ \lambda, & t = 0, \end{cases}$$

and

$$\begin{aligned} \varphi[X] &= \{\varphi[u] \mid u \in X\} \\ &= \{v \mid v \in C[0,h], v(0) = \lambda \text{ and } \alpha \leq v(t) \leq (2\lambda-\alpha) \text{ for any } t \in [0,h]\}. \end{aligned}$$

Then $\varphi[X]$ is a closed subset of $C[0,h]$ and hence it is a complete metric space. Let Φ be an operator from $\varphi[X]$ into $\varphi[X]$ defined by

$$\Phi\varphi[u] = \varphi[Au].$$

Then we can show just like Theorem 2.1 that by the Banach fixed point theorem there exists a unique mapping $\varphi[u] \in \varphi[X]$ such that $\Phi\varphi[u] = \varphi[u]$ and hence $Au = u$. $\square$

3 Examples

In this section we give some examples to illustrrate the results above.

Example 3.1. In [3] the Cauchy problem (1) is considered. Since $f(t,u) = P(t)t^a u^\sigma$, $a, \sigma, \lambda \in \mathbf{R}$ with $\sigma < 0$ and $\lambda > 0$ and P is a continuous mapping such that $\int_0^1 |P(t)|t^{a+\sigma}dt < \infty$, the conditions (a), (b), (c) and (d) are satisfied. Indeed (a), (b) and (c) are clear and since

$$\begin{aligned} \left|\frac{\partial f}{\partial u}(t,u)\right| &= |P(t)t^a \sigma u^{\sigma-1}| \\ &= \frac{|\sigma||f(t,u)|}{u}, \end{aligned}$$

(d) holds. By Theorem 2.1 the Cauchy problem (1) has a unique solution in

$$X = \left\{ u \;\middle|\; \begin{array}{l} u \in C[0,h], u(0) = 0, u'(0) = \lambda \\ \text{and } \alpha t \le u(t) \text{ for any } t \in [0,h] \end{array} \right\}.$$

Example 3.2. We consider the Cauchy problem

$$\begin{cases} u''(t) = a(t) + u(t)^\sigma, \; t \in [0,1], \\ u(0) = 0, \; u'(0) = \lambda, \end{cases} \tag{4}$$

where a is positive and integrable, $\sigma \in \mathbf{R}$ with $\sigma > 0$ and $\lambda \in \mathbf{R}$ with $\lambda > 0$. Since $f(t,u) = a(t) + u^\sigma$, the conditions (a), (e), (f) and (d) are satisfied. Indeed (a), (e) and (f) are clear and since

$$\left|\frac{\partial f}{\partial u}(t,u)\right| = \sigma u^{\sigma-1} \le \frac{\max\{\sigma,1\}(a(t)+u^\sigma)}{u} = \frac{\max\{\sigma,1\}|f(t,u)|}{u},$$

(d) holds. By Theorem 2.2 the Cauchy problem (4) has a unique solution in

$$X = \left\{ u \;\middle|\; \begin{array}{l} u \in C[0,h], u(0) = 0, u'(0) = \lambda \\ \text{and } \alpha t \le u(t) \le (2\lambda - \alpha)t \text{ for any } t \in [0,h] \end{array} \right\}.$$

Example 3.3. We consider the Cauchy problem

$$\begin{cases} u''(t) = a(t)u(t)^\sigma, \; t \in [0,1], \\ u(0) = 0, \; u'(0) = \lambda, \end{cases} \tag{5}$$

where $\int_0^1 |a(t)|t^\sigma dt < \infty$ and $\sigma, \lambda \in \mathbf{R}$ with $\lambda > 0$. Since $f(t,u) = a(t)u^\sigma$, the conditions (a), (b), (c) and (d) are satisfied if $\sigma < 0$ and the conditions (a), (e), (f) and (d) are satisfied if $\sigma \geq 0$. Indeed (a) is clear, (b) and (c) are clear if $\sigma < 0$, (e) and (f) are clear if $\sigma \geq 0$, and since

$$\begin{aligned} \left|\frac{\partial f}{\partial u}(t,u)\right| &= \begin{cases} |a(t)\sigma u^{\sigma-1}|, & \text{if } \sigma \neq 0, \\ 0, & \text{if } \sigma = 0, \end{cases} \\ &= \frac{|\sigma||f(t,u)|}{u}, \end{aligned}$$

(d) holds. By Theorem 2.1 if $\sigma < 0$ and by Theorem 2.2 if $\sigma > 0$ the Cauchy problem (5) has a unique solution in

$$X = \left\{ u \;\middle|\; \begin{array}{l} u \in C[0,h], u(0) = 0, u'(0) = \lambda \\ \text{and } \alpha t \leq u(t) \text{ for any } t \in [0,h] \end{array} \right\}$$

and

$$X = \left\{ u \;\middle|\; \begin{array}{l} u \in C[0,h], u(0) = 0, u'(0) = \lambda \\ \text{and } \alpha t \leq u(t) \leq (2\lambda - \alpha)t \text{ for any } t \in [0,h] \end{array} \right\},$$

respectively.

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