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PREDICTABILITY AND UNPREDICTABILITY OF QUASI-PERIODIC DYNAMICAL SYSTEMS

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1. Introduction

According to the KAM theorem, if the irrational frequencies of integrable quasi-periodic Hamiltonian dynamical systems satisfy the Diophantine conditions (badly approximable by rationals), then the quasi-periodic tori are persistent (stable for small perturbations). On the other hand, in the converse KAM theorem, if the irrational frequencies are Liouville numbers (extremely well approximable numbers by rationals), then any small perturbations of the quasi-periodic systems contain the destructions of the q.p. tori. Here we consider the q.p. systems from some different view points; particularly, on their recurrent properties which are related to predictability or unpredictability of the systems.

In our previous papers [3], [4] we introduced the gaps between the upper and the lower recurrent dimensions as the index parameters, which measure unpredictability levels of the orbits. In [9], [10] we proved that the gaps of recurrent dimensions of q.p. orbits given by a rotation map with a single irrational frequency take positive values when this irrational frequency is a weak Liouville number; sufficiently well approximable number by rationals.

In this paper we consider q.p. systems with multiple irrational frequencies. For pairs of irrational numbers we have introduced Extended Common Multiples (ECM) conditions in [4] and we have shown some inequality relations between the parameters of Diophantine conditions and the ECM conditions. Using this relations, we have given some various examples of quasi peridic orbits, which have positive gap values of recurrent dimensions ([6], [7]). Here we investigate these conditions for a pair of irrational frequencies of q.p. systems and show the unpredictability of q.p. orbits by estimating gaps of their recurrent dimensions.

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2. Single frequency q.p. case

In [5] we classify irrational numbers according to badly or well approximable by rational numbers as follows:

We say that τ is an α-order Roth number if there exists $\alpha \geq 0$ such that, for every $\beta : \beta > \alpha$, there exists a constant $c_\beta > 0$, which satisfies

$$|\tau - \frac{q}{p}| \geq \frac{c_\beta}{p^{2+\beta}}$$

for all rational numbers $q/p \in \mathbb{Q}$.

Let $\{n_i/m_i\}$ be a convergents of τ. τ is called an α-order weak Liouville number if there exists an infinite subsequence $\{m_{k_j}\} \subset \{m_k\}$, which satisfies

$$|\tau - \frac{n_{k_j}}{m_{k_j}}| < \frac{c}{m_{k_j}^{2+\alpha}}, \quad \forall j$$

for some constants $c, \alpha > 0$.

We have also shown that the set of α-order Roth numbers is almost equal to the complement set of α-order weak Liouville numbers and so every irrational number has the parameter; say d_0, which specifies the badly or well approximable levels by rational numbers,

$$\begin{aligned} d_0 &:= \sup\{\beta : \tau \text{ is a } \beta\text{-order weak Liouville number}\} \\ &= \inf\{\alpha : \tau \text{ is an } \alpha\text{-order Roth number}\}. \end{aligned}$$

Then we say that τ satisfies a d_0-(D) condition. If an irrational number τ satisfies d_0-(D) condition for $0 \leq d_0 < \infty$, then τ is a Roth number with its order $d_0 + \varepsilon$ for every $\varepsilon > 0$ and also τ is a weak Liouville number with its order $d_0 - \varepsilon$ for every $\varepsilon > 0$. If an irrational number τ does not satisfy the Diophantine condition for a finite value d_0, we say that τ is a Liouville number or $d_0 = \infty$.

Now we consider the rotation on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$

$$f(x) = x + \alpha \pmod 1, \quad x \in \mathbb{T}$$

and we define the discrete quasi-periodic orbit Σ_x, given by the rotation,

$$\Sigma_x = \{f^n(x) : \; n \in \mathbb{N}\}$$

for an irrational frequency α, which satisfies d_0-(D) condition. We estimate the recurrent properties of the q.p. orbits, the gap of recurrent dimensions, by using the Diophantine parameter d_0 of the irrational frequency α.

Definitions of recurrent dimensions:

Define the first ε-recurrent time by

$$M_\varepsilon(x) = \min\{m \in \mathbb{N} : |f^m(x) - x| < \varepsilon\}.$$

The upper recurrent dimension is defined by

$$\overline{D}_x = \limsup_{\varepsilon \to 0} \frac{\log M_\varepsilon(x)}{-\log \varepsilon}$$

and the lower recurrent dimension is defined by

$$\underline{D}_x = \liminf_{\varepsilon \to 0} \frac{\log M_\varepsilon(x)}{-\log \varepsilon}.$$

Then we can define the gaps of recurrent dimensions by $G_x = \overline{D}_x - \underline{D}_x$.

If the gap values G_x take positive values, we cannot exactly determine or predict the ε-recurrent time of the orbits. Thus we propose the value G_x as the parameter, which measures the unpredictability level of the orbit. We obtained the following estimate for G_x in [9], [10].

Theorem 2.1. *Let the irrational frequency α satisfy the d_0-(D) condition for $d_0 \geq 0$.*

Then, for each $x \in \mathbb{T}$, we have

$$\overline{D}_x = 1, \qquad \underline{D}_x = \frac{1}{1 + d_0}.$$

Consequently, we can estimate the gap value by

$$G_x = \frac{d_0}{1 + d_0}.$$

Since almost all irrationals (in the Lebesgue measure sense) satisfy $d_0 = 0$, almost all q.p. orbits given by the rotation are predictable, that is,

$$G_x = 0.$$

In the case where the irrational frequency is a weak Liouville number with its order $d_0 > 0$, its q.p. orbit is unpredictable with its level given by

$$G_x = \frac{d_0}{1 + d_0} > 0.$$

3. Multiple frequencies case

First we prepare some notations to analyze recurrent properties of q.p. orbits with multiple frequencies. For an irrational number τ, let $\{a_j\}$ be the partial quotients of its continued fractions and $\{n_j/m_j\}$ be its convergents. For each positive integer l we can obtain the unique expansion of l by using the denominators $\{m_j\}$, considering the lexicographical order:

$$l = p_k m_k + p_{k-1} m_{k-1} + \cdots + p_u m_u$$

where $p_j \in \mathbb{N}_0 : \ p_j \leq a_{j+1}, \ u \leq j \leq k$ and $p_k, p_u \geq 1$.

We define the valuation $(l)_\tau$ of a positive integer l by

$$(l)_\tau = \frac{k - u}{k},$$

which shows a relative length of its expansion.

For a pair of irrational numbers τ_1, τ_2, let $\{a_j\}, \{b_k\}$ be the partial quotients of the continued fractions and $\{n_j/m_j\}, \{r_k/l_k\}$ be the convergents, respectively. We use some generalized common multiples of $\{m_j, l_k\}$, called Extended Common Multiples (abr. ECM).

For constants $k, d, s, r \in \mathbb{N} : k \leq d, s \leq r$, we put

$$[M]_d^k := \{m \in \mathbb{N} : m = p_k m_k + \cdots + p_d m_d,\ p_j \in \mathbb{N}_0, p_k \geq 1,\ p_j \leq a_{j+1}\},$$
$$[L]_r^s := \{l \in \mathbb{N} : l = q_s l_s + \cdots + q_r l_r,\ q_j \in \mathbb{N}_0, q_s \geq 1,\ q_j \leq b_{j+1}\}$$

and define

$$[M]_d := \bigcup_{k=d}^{\infty} [M]_d^k, \qquad [L]_r := \bigcup_{s=r}^{\infty} [L]_r^s$$

for $d, r \in \mathbb{N}$.

For positive integers $m \in [M]_d, l \in [L]_r : p_d, q_r \geq 1$, define $\zeta_1, \zeta_2 : \mathbb{N} \to \mathbb{N}$ by

$$\zeta_1(m) = d, \qquad \zeta_2(l) = r.$$

We define the ECM sequence of positive integers $\{T_j\} \subset [M]_1 \cap [L]_1$ as follows. Let

$$T_1 = \min\{m : m \in [M]_1 \cap [L]_1\},\ \ d_1 = \min\{\zeta_1(T_1), \zeta_2(T_1)\},$$
$$T_2 = \min\{m \in [M]_{d_1+1} \cap [L]_{d_1+1}\},\ \ d_2 = \min\{\zeta_1(T_2), \zeta_2(T_2)\}, \cdots,$$

iteratively, let $d_j = \min\{\zeta_1(T_j), \zeta_2(T_j)\}$ and $T_{j+1} = \min\{m \in [M]_{d_j+1} \cap [L]_{d_j+1}\}$.

For the $ECM(\tau_1, \tau_2)$ sequence $\{T_j\}$, we define the following constants

$$\delta_0 = \liminf_{j\to\infty}\ \max\{(T_j)_{\tau_1}, (T_j)_{\tau_2}\},\ \ \delta_1 = \limsup_{j\to\infty}\ \max\{(T_j)_{\tau_1}, (T_j)_{\tau_2}\}.$$

In [4] we obtained the inequality relations between the above constants and the Diophantine constant d_0 for a pair of irrational numbers.

We say that $\{\tau_1, \tau_2\}$ satisfies d_0-(D) condition if there exists a constant $d_0 \geq 2$ such that, for each $d > d_0$, there exists $\gamma_d > 0$, which satisfies

$$|(\tau_1 m_1 + \tau_2 m_2) - n| \geq \frac{\gamma_d}{|m|^d} \tag{3.1}$$

for $\forall m = (m_1, m_2) \in \mathbb{Z}^2$, $\forall n \in \mathbb{Z}$ and furthermore, for each $d : 0 < d < d_0$ and each $\gamma > 0$, there exist integers $m_\gamma = (m_{\gamma,1}, m_{\gamma,2}) \in \mathbb{Z}^2$ and $n_\gamma \in \mathbb{Z}$, which satisfy

$$|(\tau_1 m_{\gamma,1} + \tau_2 m_{\gamma,2}) - n_\gamma| < \frac{\gamma}{|m_\gamma|^d}. \tag{3.2}$$

The constant d_0 specifies the infimum value of d, which satisfies (3.1) and also the supremum value of d, which satisfies (3.2).

For $\{n_j/m_j\}$ and $\{r_j/l_j\}$, the convergents of τ_1, τ_2, respectively, we consider the case where the sequences $\{(m_j)^{\frac{1}{j}}\}, \{(l_j)^{\frac{1}{j}}\}$ are bounded. We denote the upper and the lower Lévy constants of τ_1, τ_2 by $\lambda^*(\tau_1), \lambda^*(\tau_2)$ and by $\lambda_*(\tau_1), \lambda_*(\tau_2)$, respectively, as follows.

$$\limsup_{j\to\infty}(m_j)^{\frac{1}{j}} = \lambda^*(\tau_1),\ \ \liminf_{j\to\infty}(m_j)^{\frac{1}{j}} = \lambda_*(\tau_1)$$

and similarly we define $\lambda^*(\tau_2), \lambda_*(\tau_2)$ by using the sequence $\{(l_j)^{\frac{1}{j}}\}$.

We also say that an irrational number τ has a Lévy constant if $\lambda^*(\tau) = \lambda_*(\tau)$. In 1935 Khinchin proved that almost all irrational numbers have the same Lévy constant value and in 1936 Lévy found the explicit expression for this constant; $e^{\frac{\pi^2}{12\log 2}} \sim 3.27582..$ (see [1]). Hereafter we use the following notations.

$$E_1 = \min\{\lambda_*(\tau_1), \lambda_*(\tau_2)\}, \quad E_2 = \max\{\lambda^*(\tau_1), \lambda^*(\tau_2)\}.$$

Theorem 3.1. *Let τ_1, τ_2 have the upper and lower Lévy constants and belong to a d_0-(D) class. Then for the constants d_0, δ_0 we have*

$$1 - \frac{d_0 - 1}{2} \cdot \frac{\log E_2}{\log E_1} \le \delta_0 \le 1 - \frac{d_0}{d_0 + 2} \cdot \frac{\log E_1}{\log E_2}$$

Remark 3.2. It is known that almost all pairs satisfy

$$d_0 = 2, \ E_1 = E_2 = e^{\frac{\pi^2}{12\log 2}},$$

which yield $\frac{1}{2} = \delta_0 \le \delta_1$.

4. Reccurent properties of q.p. orbits

For an irrational pair $\{\tau_1, \tau_2\}$ as frequency we consider the following discrete quasi-periodic orbit in $\mathbb{T} = \mathbb{R}/\mathbb{Z}$:

$$\Sigma = \{\varphi(n) : \ n = 0, 1, 2, ...\}, \quad \varphi(n) = \max\{\{n\tau_1\}, \{n\tau_2\}\}$$

where $\{a\}$ denotes the fractional part of a. The first ε-recurrent time M_ε to 0 is defined by

$$M_\varepsilon = \min\{n \in \mathbb{N} : \varphi(n) < \varepsilon\}$$

and the upper and the lower recurrent dimensions are defined by

$$\overline{D}(\Sigma) = \limsup_{\varepsilon \to 0} \frac{\log M_\varepsilon}{-\log \varepsilon}, \qquad \underline{D}(\Sigma) = \liminf_{\varepsilon \to 0} \frac{\log M_\varepsilon}{-\log \varepsilon}.$$

The gap of the recurrent dimensions, which gives the unpredictability level of orbits, is defined by

$$G(\Sigma) = \overline{D}(\Sigma) - \underline{D}(\Sigma).$$

Since we can show the following estimates by applying the argument in [4]

$$\underline{D}(\Sigma) \le \frac{\log E_2}{(1 - \delta_0)\log E_1}, \quad \overline{D}(\Sigma) \ge \frac{\log E_1}{(1 - \delta_1)\log E_2},$$

we have

$$G(\Sigma) \ge \frac{\log E_1}{(1 - \delta_1)\log E_2} - \frac{\log E_2}{(1 - \delta_0)\log E_1}.$$

In [8] we investigated an example of the irrational frequencies given by extreme numbers (see [11]) and show the positivity of the gap values. Here we consider the case where the irrational frequencies are weak Liouville numbers. We construct a pair of irrationals $\{\tau_1, \tau_2\}$ by defining the partial quotients of continued fractions $\{a_j\}, \{b_k\}$, iteratively, as follows.

First, for a constant $K > 0$, choose a suitable pair of irrationals $\{\tau_1^{(1)}, \tau_2^{(1)}\}$ satisfying $a_j, b_k \le K, \ \forall j, k$ and find $t_1 = \mathrm{lcm}(m_{j_1}, l_{k_1})$, which satisfies

$$t_1 = m' m_{j_1} \sim m_{j_1}^{1+\alpha_1}, \quad t_1 = l' l_{k_1} \sim l_{k_1}^{1+\beta_1}$$

for sufficiently small $\alpha_1, \beta_1 : 0 < \alpha_1, \beta_1 < \sqrt{2} - 1 - \varepsilon$.

Next, put (substitute)

$$a_{j_1+1} = \lceil m_{j_1}^{\alpha_1} \rceil, \quad b_{k_1+1} = \lceil l_{k_1}^{\beta_1} \rceil$$

where $\lceil r \rceil$ is the smallest integer not less than r and define a suitable pair of irrationals $\{\tau_1^{(2)}, \tau_2^{(2)}\}$ satisfying $a_j, b_k \leq K, \ \forall j, k : j > j_1 + 1, k > k_1 + 1$ and find $t_2 = \mathrm{lcm}(m_{j_2}, l_{k_2})$, which satisfies

$$t_2 = m' m_{j_2} \sim m_{j_2}^{1+\alpha_2}, \quad t_1 = l' l_{k_2} \sim l_{k_2}^{1+\beta_2}$$

for sufficiently small $\alpha_2, \beta_2 < \sqrt{2} - 1 - \varepsilon$.

Iteratively, for a pair of irrationals $\{\tau_1^{(i)}, \tau_2^{(i)}\}$: $\ a_j, b_k \leq K, \ \forall j, k :$ $j > j_s + 1, k > k_s + 1, s = 1, \ldots, i-1$, find $t_i = \mathrm{lcm}(m_{j_i}, l_{k_i})$, which satisfies

$$t_i = m' m_{j_i} \sim m_{j_i}^{1+\alpha_i}, \quad t_i = l' l_{k_i} \sim l_{k_i}^{1+\beta_i}$$

and put $a_{j_i+1} = \lceil m_{j_i}^{\alpha_i} \rceil, \ \ b_{k_i+1} = \lceil l_{k_i}^{\beta_i} \rceil$ for sufficiently small $\alpha_i, \beta_i < \sqrt{2} - 1 - \varepsilon$.

Thus we can obtain the pair of irrational $\{\tau_1, \tau_2\}$, by defining the partial quotients of continued fractions $\{a_j\}, \{b_k\}$. Then $\{\tau_1, \tau_2\}$ are weak Liouville numbers with its order $\alpha, \beta \leq \sqrt{2} - 1 - \varepsilon =: c_0$ and we can obtain an ECM subsequence $\{t_i\} \subset ECM(\tau_1, \tau_2)$ satisfying $\max\{(t_i)_{\tau_1}, (t_i)_{\tau_2}\} = 0, \ \forall i$, which yields that $\delta_0 = 0$. It follows from Theorem 3.1 that $\delta_1 \geq \frac{1}{2}$ for almost all pairs of irrationals, but, since I have not yet shown this estimate in this case, we assume that $\delta_1 \geq \frac{1}{2}$.

For constants E_1, E_2, we can estimate

$$E_1 \sim \min\{(m_j)^{\frac{1}{j}}, (l_k)^{\frac{1}{k}}\}, \quad E_2 \sim \max\{(m_{j_i})^{\frac{1+\alpha}{j_i+1}}, (l_{k_i})^{\frac{1+\beta}{k_i+1}}\}$$

and then we have

$$\frac{\log E_1}{\log E_2} \geq \frac{1}{1+c_0}, \quad \frac{\log E_2}{\log E_1} \leq 1 + c_0.$$

Considering the above all constants, we can estimate the positive gap values:

$$G(\Sigma) \geq \frac{2}{1+c_0} - (1+c_0) \geq \frac{2 - (\sqrt{2} - \varepsilon)^2}{1+c_0} > 0.$$

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