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SEPARATION THEOREM IN THE CARTESIAN PRODUCT OF A VECTOR SPACE AND A PARTIALLY ORDERED VECTOR SPACE WITH A CHAIN COMPLETENESS
(ベクトル空間と CHAIN 完備な半順序ベクトル空間の直積における分離定理について)

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1. Introduction

A separation theorem for convex sets is one of the most fundamental theorems in the optimization theory and functional analysis theory. Let X be a vector space, $X^{'}$ its algebraic dual space and A a subset of X. We denote ${}^{l}A$ the linear span of A and ${}^{i}A$ denotes the *relatively algebraic interior* of A, where

$$ {}^{i}A=\left\{y\in Y \;\middle|\; \begin{array}{l} \text{any } y^{'}\in{}^{l}A \text{ there exists } \varepsilon>0 \text{ with} \\ y+\lambda(y^{'}-y)\in A \text{ for any } \lambda\in[0,\varepsilon) \end{array}\right\}. $$

If ${}^{l}A=X$, then it is called *algebraic interior*, core (A), of A. Then we can obtain the separation theorem in the vector space as follows, see [1, 12]:

Theorem 1.1. *Let X be a vector space, $X^{'}$ its algebraic dual space, A, B convex subsets of X such that relatively algebraic interior ${}^{i}A$ and ${}^{i}B$ are non-empty. Then there exists $u\in X^{'}$, $u\neq 0$, and $\lambda\in R$ such that $\langle u,x\rangle\leq\lambda\leq\langle u,y\rangle$ for any $x\in A$ and $y\in B$. Moreover, $\langle u,z\rangle\neq\lambda$ for at least one $z\in A\cup B$ if and only if ${}^{i}A\cap{}^{i}B=\emptyset$.*

In [7, 16], this theorem is generalized in the Cartesian product space of a vector space and a Dedekind complete partially ordered vector space. Under certain assumptions, two non-void subsets of a product space can be separated by an affine manifold of that product space. Its proof is due to Hahn Banach's theorem and Dedekind completeness. On the other hand, when we consider the Cartesian product of a vector space and a chain complete partially ordered vector space, two subsets in that product space are not separated by an affine manifold. So we can not rely on the method in [7, 16].

In this paper, we give a separation theorem in the Cartesian product of a vector space and a chain complete partially ordered vector space (Lemma 3.1) using a Gerstewitzs(Tammer) scalarization method [10] for a vector space.

2. Preliminaries

Let R be the set of a real number, N the set of a natural number, I an indexed set. Let X and Y be real vector spaces. We denote by $\mathcal{L}(X,Y)$ a linear mapping from X into Y. In particular, $X^{'}=\mathcal{L}(X,R)$. We give a convex cone K and define its algebraic dual cone $K^{\sharp}$ by

$$K^{\sharp}:=\{x^{'}\in X^{'}\mid\langle x^{'},x\rangle\geq 0 \text{ for all } x\in K\},$$

where $\langle x^{'},x\rangle$ denotes dual pair, and its quasi interior is defined by

$$K^{\sharp 0}:=\{x^{'}\in X^{'}\mid\langle x^{'},x\rangle>0 \text{ for all } x\in K\setminus\{0\}\}.$$

A partial ordering on Y with respect to K is defined by $x \leqq_K y$ if $y - x \in K$ for all x, $y \in Y$. If $y - x \in K \setminus \{0\}$ for all x, $y \in Y$, we denote by $x \leq_K y$. We assume K is a proper convex çone (that is, $K \neq \emptyset$, $K \neq \{\theta\}$, where θ denotes the zero element in Y, $K \neq Y$, $\lambda K \subset K$ for all $\lambda \geq 0$, and $K + K \subset K$). It is well known that $\leqq_K$ is reflexive and transitive. Moreover, $\leqq_K$ has invariable properties to vector space structures as translation and scalar multiplication. In the sequel, we consider $(Y, \leqq_K)$ as a partially ordered vector space, where K is a proper convex cone. In particular, we assume that K is pointed, that is, $K \cap K = \{0\}$, then K is antisymmetric.

Let Z be a subset of Y. The set Z is called a *chain* if any two elements are comparable; that is, $x \leqq_K y$ or $y \leqq_K x$ for any x, $y \in Z$. An element $x \in Y$ is called a *lower bound* (resp., *upper bound*) of Z if $x \leqq_K y$ (resp., $y \leqq_K x$) for any $y \in Z$, *minimum* (resp., *maximum*) of Z if x is a lower bound (resp., *upper bound*) of Z and $x \in Z$. If there exists a lower bound (resp., an upper bound) of Z, then Z is said to be *bounded from below* (resp., *bounded from above*). If the set of all lower bounds of Z has the maximum, then the maximum is called an *infimum* of Z and denoted by $\inf Z$. If the set of all upper bounds of Z has the minimum, then the minimum is called a *supremum* of Z and denoted by $\sup Z$. A partially ordered vector space Y is said to be *chain complete* if every nonempty chain of Y which is bounded from below has an infimum; *Dedekind complete* if every nonempty subset of Y which is bounded from below has an infimum; *Dedekind σ-complete* if every nonempty countable subset of Y which is bounded from below has an infimum. A partially ordered vector space Y is (upward) *directed* if for any x,$y \in Y$ there exists $z \in Y$ such that $x \leqq_K z$ and $y \leqq_K z$. For the further information of a partially ordered vector space and a partially ordered set, see [4, 5, 15, 17, 18].

It is clear that if Y is Dedekind complete, then it is chain complete. However, the converse is not true in general. The following example shows this fact.

Example 2.1. The set of all continuous functions on the interval $[0, 1]$, $C([0, 1])$ is chain complete. In fact, when we consider an increasing sequence of continuous functions which is bounded from above, then it is a chain and has a supremum. Since the supremum is also a uniformly convergent limit of a sequence, it is continuous. However, $C([0, 1])$ is not a Dedekind σ-complete space, (see [15, Example 23.3. (ii)]).

The following, we give elementary properties for the vector space and sacalarizing function.

Definition 2.2. A point $x \in X$ is *linearly accessible* from A if there exists $a \in A$ with $a \neq x$ such that $(a, x) \subset A$. We write lina (A) for the set of all such x and put lin $(A) = A \cup$ lina (A). A subset A of X is said to be *algebraically closed* if $A =$ lin (A).

Let $\overline{R} = R \cup \{\infty\}$ and $\varphi : Y \to \overline{R}$. Then we define the domain and epigraph of φ by

$$\text{dom}\ (\varphi) = \{y \in Y \mid \varphi(y) < \infty\},\ \text{epi}\ (\varphi) = \{(y, t) \in Y \times R \mid \varphi(y) \leq t\},$$

respectively. We say that φ is convex if epi (φ) is a convex set; proper if dom $(\varphi) \neq \emptyset$ and $\varphi(y) > -\infty$ for all $y \in Y$. If we take $k^0 \in K$, then we have

$$K + [0, \infty) \cdot k^0 \subset K. \tag{2.1}$$

A function f from X into R is said to be *sublinear* if the following conditions are satisfied.

(S1) For any $x, y \in X$, $f(x + y) \leq f(x) + f(y)$.
(S2) For any $x \in X$ and $\alpha \geq 0$, $f(\alpha x) = \alpha f(x)$.

Gerstewitz (Tammer) [9] considers the sublinear scalarizing function defined by

$$\varphi_{K,k^0}(y) = \inf\{t \in R \mid y \in tk^0 - K\}.$$

For this function, we have the following proposition, see [10, Theorem 2.3.1].

Proposition 2.3. Let K be a closed proper convex cone and $k^0 \in K$. Then we have dom $(\varphi_{K,k^0}) = R \cdot k^0 - K$,

$$\{y \in Y \mid \varphi_{K,k^0}(y) \leq \lambda\} = \lambda k^0 - K \tag{2.2}$$

and

$$\varphi_{K,k^0}(y + \lambda k^0) = \varphi_{K,k^0}(y) + \lambda. \tag{2.3}$$

Moreover, we have the following results:

(i): φ_{K,k^0} is convex.
(ii): K is cone if and only if φ_{K,k^0} satisfies $\varphi_{K,k^0}(\lambda y) = \lambda \varphi_{K,k^0}(y)$ for any $\lambda > 0$.
(iii): φ_{K,k^0} is proper if and only if K does not contain the lines parallel to $k^0 \in Y \setminus \{0\}$, that is,

$$\text{for any } y \in Y \text{ there exists } t \in R \text{ such that } y + tk^0 \notin K. \tag{2.4}$$

(iv): φ_{K,k^0} is finite-valued if and only if K does not contain the lines parallel to $k^0 \in Y \setminus \{0\}$ and

$$R \cdot k^0 - K = Y. \tag{2.5}$$

(v): φ_{K,k^0} is K-monotone, that is, $y_2 - y_1 \in K$ implies $\varphi_{K,k^0}(y_1) \leq \varphi_{K,k^0}(y_2)$.

Proof. By the definition of φ_{K,k^0}, $\varphi_{K,k^0}(y + \lambda k^0) = \varphi_{K,k^0}(y) + \lambda$ is clear. We only to prove the equation (2.2). The proofs of remainder are similar to that of [10, Theorem 2.3.1]. $\{y \in Y \mid \varphi_{K,k^0}(y) \leq \lambda\} \supset \lambda k^0 - K$ is obvious. For any $y \in \{y \in Y \mid \varphi_{K,k^0}(y) \leq \lambda\}$, we assume that $y \notin \lambda k^0 - K$. Since K is closed, $\lambda k^0 - K$ is also closed, and $y \notin$ lin $(\lambda k^0 - K)$. Then for any $d \in K$, there exists μ with $0 < \mu < 1$ such that $\mu(\lambda k^0 - d) + (1 - \mu)y \notin \lambda k^0 - K$. Take $d = k^0$, we have

$$y \notin \lambda k^0 + \frac{\mu}{1-\mu} k^0 - K.$$

Thus $\varphi_{K,k^0}(y) \geq \lambda + \frac{\mu}{1-\mu} > \lambda$. Contradict the fact $y \in \{y \in Y \mid \varphi_{K,k^0}(y) \leq \lambda\}$. □

Let X be a vector space and $(Y, \leqq_K)$ a partially ordered vector space, where K is a convex cone. A mapping $f : X \to Y$ is called sublinear if for all $x,y \in X$ and all $\lambda \geq 0$, f satisfies (S1) and (S2). We denote $\mathcal{L}(X, Y)$ the real vector space of all linear mapping from X into Y. For a chain complete partially ordered vector space, Fel'dman [8] gives the following theorem.

Theorem 2.4. *Let X be a vector space, $(Y, \leqq_K)$ a chain complete partially ordered vector space, where K is a convex cone. Let f be a sublinear mapping from a X into Y and x_0 a point in X. Then there exists $g \in \mathcal{L}(X, Y)$ such that $g(x) \leqq_K f(x)$ for any $x \in X$ and $g(x_0) = f(x_0)$.*

3. Separation Theorem

Let X be a vector space and $(Y, \leqq_K)$ a chain complete directed partially ordered vector space, where K is a proper closed convex cone. Let $f \in \mathcal{L}(X, Y)$, $g \in \mathcal{L}(Y, Y)$, t_0 a point in R, $k^0 \in$ core (K) and φ_{K,k^0} a scalarizing function from Y into R. Then

$$H = \{(x, y) \in X \times Y \mid \varphi_{K,k^0}(f(x) + g(y)) = t_0\}$$

is a subset in $X \times Y$. Let A, B be nonempty subsets of $X \times Y$. It is said that H separates A and B if

$$H_- = \{(x, y) \in X \times Y \mid \varphi_{K,k^0}(f(x) + g(y)) \leq t_0\} \supset A$$

and

$$H_+ = \{(x, y) \in X \times Y \mid \varphi_{K,k^0}(f(x) + g(y)) \geq t_0\} \supset B$$

hold. The operator P_X defined by $P_X(x, y) = x$ for any $(x, y) \in X \times Y$ is called the projection of $X \times Y$ onto X. Similarly, we define the projection P_Y of $X \times Y$ onto Y by $P_Y(x, y) = y$. Then $P_X \in \mathcal{L}(X \times Y, X)$ and $P_Y \in \mathcal{L}(X \times Y, Y)$. We define

$$P_X(A) = \{x \in X \mid \text{ there exists } y \in Y \text{ such that } (x, y) \in A\}$$

and

$$P_Y(A) = \{y \in Y \mid \text{ there exists } x \in X \text{ such that } (x, y) \in A\}.$$

Then for each $* = X, Y$, we have $P_*(A+B) = P_*(A) + P_*(B)$. We take a chain $C \subset P_Y(A-B)$ and define

$$P_X^C(A-B) = \{x \in X \mid \text{ there exists } y \in C \text{ such that } (x, y) \in A - B\}.$$

The set

$$\text{cone}\,(A) = \{\lambda z \in X \times Y \mid \lambda \geq 0,\ z \in A\}$$

is called a *cone span* of A. If A is convex, then cone (A) is convex. We called a subset Z in X is expansive if for at least one $a \in {}^iZ$ and for each $z \in Z$, it holds that $a + \lambda(z-a) \in {}^iZ$.

We obtain a separation theorem for the Cartesian product of a vector space and a chain complete directed partially ordered vector space, as follows:

Theorem 3.1. *Let X be a vector space, $(Y, \leqq_K)$ a chain complete directed partially ordered vector space, where K is a proper closed convex cone, and $k^0 \in$ core (K). Let A, B be non-empty subsets of $X \times Y$ such that* cone $(A-B)$ *is a convex cone, and C a chain of $P_Y(A-B)$. Assume that the following* (i) *and* (ii) *hold :*
(i) $0 \in {}^iP_X^C(A-B)$ *and* ${}^lP_X^C(A-B) = X$.
(ii) *If $(x, y_1) \in A$ and $(x, y_2) \in B$, then we have $y_2 \leqq_K y_1$.*
Then there exist $f \in \mathcal{L}(X, Y)$ and $t_0 \in R$ such that $H = \{(x, y) \in X \times Y \mid \varphi_{K,k^0}(f(x) - y) = t_0\}$ separates A and B.

Proof. By assumption (i) and the definition of ${}^iP_X^C(A-B)$, for any $x \in X$, there exists $\varepsilon > 0$ and $y \in C$ such that $(\lambda x, y) \in A - B$ for any $\lambda \in [0, \varepsilon)$. For any $x \in X$, we define $C_x = \{y \in C \mid (x, y) \in \text{cone}\,(A-B)\}$, where C is a chain in Y. Since $\lambda^{-1}y \in C_x$ for any $\lambda \in (0, \varepsilon)$, we have $C_x \neq \emptyset$ for all $x \in X$. Moreover, for any $y \in C_0$ with $y \neq 0$, there exist $\lambda > 0$, $(x_1, y_1) \in A$ and $(x_2, y_2) \in B$ such that $(0, y) = \lambda\{(x_1, y_1) - (x_2, y_2)\}$. Then we have $x_1 = x_2$ and $y = \lambda(y_1 - y_2)$. By assumption (ii), we obtain $0 \leqq_K \lambda(y_1 - y_2) = y$. Thus $y \in Y_+ = \{y \in Y \mid 0 \leqq_K y\}$. Since cone $(A-B)$ is a convex cone, we have $C_x + C_{x'} \subset C_{x+x'}$ for any $x, x' \in X$. For any $x \in X$, there exists $y' \in Y$ with $-y' \in C_{-x}$ by the definition. Then we have $y - y' \in C_x + C_{-x} \subset C_0 \subset Y_+$ for any $y \in C_x$. Thus we have $y' \leqq_K y$ for any $y \in C_x$. Put $p(x) = \inf\{y \mid y \in C_x\}$, then p is sublinear. Since Y is chain complete, by Theorem 2.4, there exists $f \in \mathcal{L}(X, Y)$ such that $f(x) \leqq_K p(x)$ for all $x \in X$. For any $(x_1, y_1) \in A$ and $(x_2, y_2) \in B$, if we take $x = x_1 - x_2$, then we have

$$f(x_1 - x_2) \leqq_K p(x_1 - x_2) \leqq_K y_1 - y_2.$$

Therefore we have

$$f(x_1) - y_1 \leqq_K f(x_2) - y_2.$$

Since

$$(f(x_2) - y_2) - (f(x_1) - y_1) \in K,$$

there exists $t_0 \in R$ such that

$$\varphi_{K,k^0}(f(x_1) - y_1) \leq t_0 \leq \varphi_{K,k^0}(f(x_2) - y_2)$$

for any $(x_1, y_1) \in A$ and $(x_2, y_2) \in B$. $\square$

Let X be a vector space and two linear subspaces A and B of X are called *algebraically complementary* to each other if each $x \in X$ can be represented in one and only one way as a sum $x = y + z$ with $y \in A$ and $z \in B$. Then by Theorem 3.1, we obtain the following theorem.

Corollary 3.2. *Let X be a vector space, $(Y, \leqq_K)$ a chain complete directed partially ordered vector space, where K is a proper closed convex cone, and $k^0 \in \mathrm{core}\,(K)$. Let A, B be subsets of $X \times Y$ such that* cone $(A - B)$ *is a convex cone and C a chain of $P_Y(A - B)$. We assume that $P_X^C(A - B)$ is expansive. We also assume that the following* (i) *and* (ii) *hold :*
(i) $0 \in {}^i P_X^C(A - B)$.
(ii) *If $(x, y_1) \in A$ and $(x, y_2) \in B$, then we have $y_2 \leqq_K y_1$.*
Then there exist $f \in \mathcal{L}(X, Y)$ and $t_0 \in R$ such that $H = \{(x, y) \in X \times Y \mid \varphi_{K,k^0}(f(x) - y) = t_0\}$ separates A and B.

Proof. Since $P_X^C(A - B)$ is expansive, ${}^l P_X^C(A - B) = {}^{li} P_X^C(A - B)$ hold, see [2]. We put $X_1 = {}^l P_X^C(A - B) = {}^{li} P_X^C(A - B)$. Then X_1 is a subspace of X. The sets A, B, $A - B$ and cone $(A - B)$ are subsets of X_1. By Theorem 3.1, there exists $f_1 \in \mathcal{L}(X_1, Y)$ such that

$$f_1(x_1 - x_2) \leqq_K y_1 - y_2$$

for any $(x_1, y_1) \in A$ and $(x_2, y_2) \in B$. Let X_2 be an algebraical complementary space of X_1. Then an arbitrary $z \in X$ has a unique representation $z = x + y$ with $x \in X_1$ and $y \in X_2$, see [13, page 51 and 54]. We define $f \in \mathcal{L}(X, Y)$ by $f(z) = f_1(x)$ for all $z \in X$. Then f satisfies the assertion of Corollary. □

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