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On N-Fractional Calculus of the Function $((z-b)^2-c)^{-4}$

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Abstract

We discuss the N-fractional calculus of $f(z)=((z-b)^2-c)^{-4}$. By applying the fractional calculus, we have three kinds of the representation for γ-th differintegral of the function $((z-b)^2-c)^{-4}$ from the different way. And some identities are reported.

1 Introduction

We adopt the following definition of the fractional calculus.

(I) Definition. (by K. Nishimoto, [1] Vol. 1)

Let $D=\{D_-,D_+\}$, $C=\{C_-,C_+\}$, C_- be a curve along the cut joining two points z and $-\infty+iIm(z)$, C_+ be a curve along the cut joining two points z and $\infty+iIm(z)$, D_- be a domain surrounded by C_- , D_+ be a domain surrounded by C_+ (Here D contains the points over the curve C).

Moreover, let $f=f(z)$ be a regular function in $D(z\in D)$,

$$\begin{aligned} f_\nu &= (f)_\nu = {}_C(f)_\nu \\ &= \frac{\Gamma(\nu+1)}{2\pi i}\int_C \frac{f(\zeta)d\zeta}{(\zeta-z)^{\nu+1}} \quad (\nu\notin Z^-), \end{aligned} \tag{1}$$

$$(f)_{-m}=\lim_{\nu\to-m}(f)_\nu \quad (m\in Z^+), \tag{2}$$

where

$$-\pi\le arg(\zeta-z)\le\pi \ for\ C_-, \quad 0\le arg(\zeta-z)\le 2\pi \ \ for\ C_+,$$

$$\zeta\neq z, \quad z\in C, \quad \nu\in R, \quad \Gamma;\ Gamma\ function,$$

then $(f)_\nu$ is the fractional differintegration of arbitrary order ν (derivatives of order ν for $\nu>0$, and integrals of order $-\nu$ for $\nu<0$), with respect to z , of the function f , if $|(f)_\nu|<\infty$.

(II) On the fractional calculus operator N^ν [3]

Theorem A. Let fractional calculus operator (Nishimoto's Operator) N^ν be

$$N^\nu = \left(\frac{\Gamma(\nu+1)}{2\pi i}\int_C \frac{d\zeta}{(\zeta-z)^{\nu+1}}\right) \quad (\nu \notin Z^-), \quad (Refer\ to[1]) \tag{3}$$

with

$$N^{-m} = \lim_{\nu\to -m} N^\nu \quad (m \in Z^+), \tag{4}$$

and define the binary operation $\circ$ as

$$N^\beta \circ N^\alpha f = N^\beta N^\alpha f = N^\alpha(N^\beta f) \quad (\alpha, \beta \in R), \tag{5}$$

then the set

$$\{N^\nu\} = \{N^\nu | \nu \in R\} \tag{6}$$

is an Abelian product group (having continuous index ν) which has the inverse transform operator $(N^\nu)^{-1} = N^{-\nu}$ to the fractional calculus operator N^ν , for the function f such that $f \in F = \{f; 0 \neq |f_\nu| < \infty, \nu \in R\}$, where $f = f(z)$ and $z \in C$. (vis. $-\infty < \nu < \infty$).

(For our convenience, we call $N^\beta \circ N^\alpha$ as product of N^β and N^α .)

Theorem B. " F.O.G. $\{N^\nu\}$) " is an " Action product group which has continuous index ν " for the set of F . (F.O.G. ; Fractional calculus operator group)

Theorem C. Let

$$S := \{\pm N^\nu\} \cup \{0\} = \{N^\nu\} \cup \{-N^\nu\} \cup \{0\} \quad (\nu \in R). \tag{7}$$

Then the set S is a commutative ring for the function $f \in F$, when the identity

$$N^\alpha + N^\beta = N^\gamma \quad (N^\alpha, N^\beta, N^\gamma \in S) \tag{8}$$

holds. [4]

(III)

In some previous papers, the following result are known as elementary properties.

Lemma. We have [1]

(i)

$$((z-c)^\beta)_\alpha = e^{-i\pi\alpha}\frac{\Gamma(\alpha-\beta)}{\Gamma(-\beta)}(z-c)^{\beta-\alpha} \quad \left(\left|\frac{\Gamma(\alpha-\beta)}{\Gamma(-\beta)}\right| < \infty\right)$$

(ii)

$$(log(z-c))_\alpha = -e^{-i\pi\alpha}\Gamma(\alpha)(z-c)^{-\alpha} \quad (|\Gamma(\alpha)| < \infty)$$

(iii)

$$((z-c)^{-\alpha})_{-\alpha} = -e^{i\pi\alpha}\frac{1}{\Gamma(\alpha)}\log(z-c), \quad (|\Gamma(\alpha)| < \infty)$$

where $z-c \neq 0$ in (i), and $z-c \neq 0, 1$ in (ii) and (iii) ,

(iv)

$$(u \cdot v)_\alpha := \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+1)}{k!\Gamma(\alpha+1-k)} u_{\alpha-k} v_k. \quad (u = u(z), v = v(z))$$

Moreover in the previous works we refer to the next theorem [6].
Theorem D. We have

(i)

$$(((z-b)^\beta - c)^\alpha)_\gamma = e^{-i\pi\gamma}(z-b)^{\alpha\beta-\gamma} \sum_{k=0}^{\infty} \frac{[-\alpha]_k \Gamma(\beta k - \alpha\beta + \gamma)}{k!\Gamma(\beta k - \alpha\beta)} \left(\frac{c}{(z-b)^\beta}\right)^k \tag{9}$$

$$(|\frac{\Gamma(\beta k - \alpha\beta + \gamma)}{\Gamma(\beta k - \alpha\beta)}| < \infty),$$

and

(ii)

$$(((z-b)^\beta - c)^\alpha)_n = (-1)^n (z-b)^{\alpha\beta-n} \sum_{k=0}^{\infty} \frac{[-\alpha]_k [\beta k - \alpha\beta]_n}{k!} \left(\frac{c}{(z-b)^\beta}\right)^k \tag{10}$$

$$(n \in Z_0^+, \quad |\frac{c}{(z-b)^\beta}| < 1),$$

where

$$[\lambda]_k = \lambda(\lambda+1)\cdots(\lambda+k-1) = \Gamma(\lambda+k)/\Gamma(\lambda) \ \ with \ \ [\lambda]_0 = 1,$$

(Pochhammer's Notation).

2 N-Fractional Calculus of the Functions $f(z) = ((z-b)^2-c)^{-4}$

In order to have a representation of N-fractional calculus with γ-order, we directly apply the theorem to the function at the beginning.

Theorem 1. Let

$$f = f(z) = ((z-b)^2-c)^{-4} \quad \left(((z-b)^2-c)^4 \neq 0\right) \tag{1}$$

we have

$$(f)_\gamma = e^{-i\pi\gamma}(z-b)^{-8-\gamma} \sum_{k=0}^{\infty} \frac{[4]_k \Gamma(2k+8+\gamma)}{k!\Gamma(2k+8)} \left(\frac{c}{(z-b)^2}\right)^k \tag{2}$$

Proof. According to Theorem D, we have the equation (2) directly.

Secondly, we consider the function as a product of two functions like as

$$f(z) = ((z-b)^2-c)^{-5} \cdot ((z-b)^2-c),$$

and we have the new representation for $(f)_\gamma$ as follows.

Theorem 2. We set $f = f(z)$, and S, H ,G as follows,

$$S = S(z) = \frac{c}{(z-b)^2}, \quad (|S| < 1) \tag{3}$$

$$H(k,\gamma,m) = \frac{[5]_k \Gamma(2k+10+\gamma-m)}{k!\Gamma(2k+10)} S^k, \tag{4}$$

$$G(\gamma,m) = \sum_{k=0}^{\infty} H(k,\gamma,m). \tag{5}$$

We have

$$\begin{aligned}(f)_\gamma &= e^{-i\pi\gamma}(z-b)^{-8-\gamma}\{(1-S)G(\gamma,0) - 2\gamma G(\gamma,1) \\ &\quad +\gamma(\gamma-1)G(\gamma,2)\}\end{aligned} \tag{6}$$

Proof. According to Lemma (iv), we have

$$(f)_\gamma = \left(((z-b)^2-c)^{-5} \cdot ((z-b)^2-c)\right)_\gamma \tag{7}$$

$$= \sum_{k=0}^{\infty} \frac{\Gamma(\gamma+1)}{k!\Gamma(\gamma+1-k)} (((z-b)^2-c)^{-5})_{\gamma-k} \cdot ((z-b)^2-c)_k \tag{8}$$

and applying Theorem D.(i) to

$$(((z-b)^2-c)^{-5})_{\gamma-k}, \tag{9}$$

we obtain

$$\begin{aligned}
(f)_\gamma &= \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1)}\left(((z-b)^2-c)^{-5}\right)_\gamma((z-b)^2-c)_0 \\
&\quad+\frac{\Gamma(\gamma+1)}{\Gamma(\gamma)}\left(((z-b)^2-c)^{-5}\right)_{\gamma-1}(2(z-b)) \\
&\quad+\frac{\Gamma(\gamma+1)}{2!\Gamma(\gamma-1)}\left(((z-b)^2-c)^{-5}\right)_{\gamma-2}\cdot 2 \\
&= \left(((z-b)^2-c)^{-5}\right)_\gamma((z-b)^2-c)+2\gamma\left(((z-b)^2-c)^{-5}\right)_{\gamma-1}\cdot(z-b) \\
&\quad+\gamma(\gamma-1)\left(((z-b)^2-c)^{-5}\right)_{\gamma-2} \\
&= e^{-i\pi\gamma}(z-b)^{-10-\gamma}((z-b)^2-c)\sum_{k=0}^{\infty}\frac{[5]_k\Gamma(2k+10+\gamma)}{k!\Gamma(2k+10)}\left(\frac{c}{(z-b)^2}\right)^k \\
&\quad+2\gamma(z-b)e^{-i\pi(\gamma-1)}(z-b)^{-10-\gamma+1}\sum_{k=0}^{\infty}\frac{[5]_k\Gamma(2k+10+\gamma-1)}{k!\Gamma(2k+10)}\left(\frac{c}{(z-b)^2}\right)^k \\
&\quad+\gamma(\gamma-1)e^{-i\pi(\gamma-2)}(z-b)^{-10-\gamma+2}\sum_{k=0}^{\infty}\frac{[5]_k\Gamma(2k+10+\gamma-2)}{k!\Gamma(2k+10)}\left(\frac{c}{(z-b)^2}\right)^k
\end{aligned} \tag{10}$$

Then we have the representation

$$(f(z))_\gamma = e^{-i\pi\gamma}(z-b)^{-8-\gamma}\{(1-S)G(\gamma,0)-2\gamma G(\gamma,1)+\gamma(\gamma-1)G(\gamma,2)\}. \tag{11}$$

This is the same one as the equation (6).

Next, we choose another process of the fractional calculus which is devided into two stages as like as

$$(f(z))_\gamma = ((f(z))_1)_{\gamma-1}\,. \tag{12}$$

We have an another result.

Theorem 3. We set $f=f(z)$, and S, R ,W as follows,

$$S = S(z) = \frac{c}{(z-b)^2}, \quad (|S|<1) \tag{13}$$

$$R(k,\gamma,m)=\frac{[5]_k\Gamma(2k+10+\gamma-m)}{k!\Gamma(2k+10)}S^k, \tag{14}$$

$$W(\gamma,m)=\sum_{k=0}^{\infty}R(k,\gamma,m). \tag{15}$$

Then we have

$$(f)_\gamma=8e^{-i\pi\gamma}(z-b)^{-8-\gamma}\{W(\gamma,1)-(\gamma-1)W(\gamma,2)\}. \tag{16}$$

Proof. We have

$$\begin{aligned}\left(((z-b)^2-c)^{-4}\right)_1 &= -4((z-b)^2-c)^{-5}\cdot 2(z-b)\\ &= -8((z-b)^2-c)^{-5}\cdot(z-b)\end{aligned} \tag{17}$$

Then

$$\begin{aligned}&\left((((z-b)^2-c)^{-4}))_1\right)_{\gamma-1}=-8\left(((z-b)^2-c)^{-5}(z-b)\right)_{\gamma-1}\\ &= -8\sum_{k=0}^{\infty}\frac{\Gamma(\gamma)}{k!\Gamma(\gamma-k)}\left(((z-b)^2-c)^{-5}\right)_{\gamma-1-k}(z-b)_k\\ &= -8\{\frac{\Gamma(\gamma)}{\Gamma(\gamma)}(((z-b)^2-c)^{-5})_{\gamma-1}(z-b)+\frac{\Gamma(\gamma)}{\Gamma(\gamma-1)}(((z-b)^2-c)^{-5})_{\gamma-1-1}\}\\ &= -8\{e^{-i\pi(\gamma-1)}(z-b)^{-10-\gamma+2}\sum_{k=0}^{\infty}\frac{[5]_k\Gamma(2k+10+\gamma-1)}{k!\Gamma(2k+10)}\left(\frac{c}{(z-b)^2}\right)^k\\ &\quad +(\gamma-1)e^{-i\pi(\gamma-2)}(z-b)^{-10-\gamma+2}\sum_{k=0}^{\infty}\frac{[5]_k\Gamma(2k+10+\gamma-2)}{k!\Gamma(2k+10)}\left(\frac{c}{(z-b)^2}\right)^k\end{aligned} \tag{18}$$

And we put

$$R(k,\gamma,m)=\frac{[5]_k\Gamma(2k+10+\gamma-m)}{k!\Gamma(2k+10)}\left(\frac{c}{(z-b)^2}\right)^k,$$

$$W(\gamma,m)=\sum_{k=0}^{\infty}R(k,\gamma,m).$$

So we have

$$(f(z))_\gamma=8e^{-i\pi\gamma}(z-b)^{-8-\gamma}\{W(\gamma,1)-(\gamma-1)W(\gamma,2)\},\quad(\gamma\notin Z^-). \tag{19}$$

We have the equation (16) from above equation directly.

3 Identities

We have three kinds of representation on N-fractional calculus of the function $f(z) = ((z-b)^2 - c)^{-4}$ like as Theorem 1, 2 and 3. Accordingly we have the following identities with using S and G and W given in §3.

Theorem 4. We have

(i)

$$\sum_{k=0}^{\infty} \frac{[4]_k \Gamma(2k+8+\gamma)}{k!\Gamma(2k+8)} S^k = (1-S)G(\gamma,0) - 2\gamma G(\gamma,1) + \gamma(\gamma-1)G(\gamma,2), \quad (\gamma \notin Z^-) \tag{1}$$

and

(ii)

$$\sum_{k=0}^{\infty} \frac{[4]_k \Gamma(2k+8+\gamma)}{k!\Gamma(2k+8)} S^k = 8\{W(\gamma,1) - (\gamma-1)W(\gamma,2)\}. \quad (\gamma \notin Z^-) \tag{2}$$

Proof. From Theorems 2 and 3 we can obtain above equations directly.

4 A Special Case

In order to make sure of the formulations of Theorem 2 and 3, we consider the case of the integer $\gamma = 1$. When $\gamma = 1$, from Theorem 2 , we have

$$\begin{aligned}\left(((z-b)^2 - c)^{-4}\right)_1 &= -(z-b)^{-9}\{(1-S)G(1,0) - 2G(1,1)\} \\ &= -(z-b)^{-9}\{(1-S)\sum_{k=0}^{\infty} \frac{[5]_k \Gamma(2k+10+1)}{k!\Gamma(2k+10)} S^k \\ &\quad +2\sum_{k=0}^{\infty} \frac{[5]_k \Gamma(2k+10)}{k!\Gamma(2k+10)} S^k\}.\end{aligned} \tag{1}$$

And we noteice following relations,

$$\sum_{k=0}^{\infty} \frac{[\lambda]_k}{k!} z^k = (1-z)^{-\lambda} \tag{2}$$

$$\sum_{k=0}^{\infty} \frac{[\lambda]_k k}{k!} T^k = \sum_{k=0}^{\infty} \frac{[\lambda]_k}{(k-1)!} T^k$$
$$= \sum_{k=0}^{\infty} \frac{[\lambda]_{k+1}}{k!} T^{k+1}$$
$$= \lambda T \sum_{k=0}^{\infty} \frac{[\lambda+1]_k}{k!} T^k = \lambda T (1-T)^{-1-\lambda} \tag{3}$$

$$[\lambda]_{k+1} = \frac{\Gamma(\lambda+1+k)}{\Gamma(\lambda)} = \lambda[\lambda+1]_k \tag{4}$$

Then, we have the following relations with applying to the above euations.

$$(f)_1 = -(z-b)^{-9}\{(1-S)\sum_{k=0}^{\infty} \frac{[5]_k(2k+10)}{k!} S^k$$
$$-2\sum_{k=0}^{\infty} \frac{[5]_k \Gamma(2k+10)}{k!\Gamma(2k+10)} S^k\}$$
$$= -(z-b)^{-9}\{2(1-S)\sum_{k=0}^{\infty} \frac{[5]_k k}{k!} S^k + 10(1-S)\sum_{k=0}^{\infty} \frac{[5]_k}{k!} S^k$$
$$-2\sum_{k=0}^{\infty} \frac{[5]_k}{k!} S^m\}$$
$$= -(-b)^{-9}\{10S(1-S)^{-5} + 10(1-S)^{-4} - 2(1-S)^{-5}\}$$
$$= -8(z-b)^{-9}(1-S)^{-5}$$
$$= \frac{-8(z-b)}{((z-b)^2-c)^5} \tag{5}$$

And from Theorem 3, we have

$$(f)_1 = 8e^{-i\pi}(z-b)^{-9}\{W(1,1)\}$$
$$= 8e^{-i\pi}(z-b)^{-9}\sum_{k=0}^{\infty} \frac{[5]_k \Gamma(2k+10)}{k!\Gamma(2k+10)} S^k$$
$$= 8e^{-i\pi}(z-b)^{-9}\sum_{k=0}^{\infty} \frac{[5]_k}{k!} S^k$$
$$= 8e^{-i\pi}(z-b)^{-9}(1-S)^{-5}$$
$$= -8\frac{(z-b)}{((z-b)^2-c)^5}. \tag{6}$$

Therefore we have the same results from two different forms of N-fractional calculus for the function $((z-b)^2-c)^{-4}$.

Now these results are consistent with the one of the classical calculus of

$$\frac{d}{dz}((z-b)^2-c)^{-4}. \tag{7}$$

Here we confirm again the result for Theorem 1.

When $\gamma=1$, from Theorem 1.(2), we have

$$
\begin{aligned}
\left(((z-b)^2-c)^{-4}\right)_1 &= -(z-b)^{-9}\sum_{k=0}^{\infty}\frac{[4]_k[2k+8]_1}{k!}S^k \\
&= -(z-b)^{-9}\sum_{k=0}^{\infty}\frac{[4]_k(2k+8)}{k!}S^k \\
&= -(z-b)^{-9}\{2\sum_{k=0}^{\infty}\frac{[4]_k k}{k!}S^k+8\sum_{k=0}^{\infty}\frac{[4]_k}{k!}S^k\} \\
&= -(z-b)^{-9}\{8S\sum_{k=0}^{\infty}\frac{[5]_k}{k!}S^k+8\sum_{k=0}^{\infty}\frac{[4]_k}{k!}S^k\} \\
&= -(z-b)^{-5}\{8S(1-S)^{-5}+8(1-S)^{-4}\} \\
&= -(z-b)^{-9}\frac{8}{(1-S)^5} \\
&= -8(z-b)((z-b)^2-c)^{-5}
\end{aligned} \tag{8}
$$

This result also coincides with the one cbtained by the classical calculus.

So we conclude that according to the definition of fractional differintegration , we have three forms for γ-th differintegrate of the function $\frac{1}{((z-b)^2-c)^4}$ by Theorem 1 . 2 and 3.

We made sure that they have the same results as the classical result when the differential order is in the case of $\gamma=1$.

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