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Radii of Starlikeness and Convexity for analytic functions with bounded derivative

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Abstract

We consider two families of analytic functions $|f'(z)-1|<r$ $(|z|<1)$, *and* $|f'(z)-1|<1$ $(|z|<R)$, *then investigate radii of starlikeness and convexity for these families*

1 Introduction

Let $\mathbb{D}_R=\{|z|<R\}$ $(0<R\leq 1)$, and for brevity we write $\mathbb{D}_1=\mathbb{D}$.
Let $\mathcal{A}$ be the class of functions of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n \qquad (1.1)$$

that are analytic in $\mathbb{D}$. And let $\mathcal{S}^*$ and $\mathcal{C}$ denote the subclass of $\mathcal{A}$ consisting of functions which are starlike and convex, respectively. i.e.

$$f(z)\in\mathcal{S}^* \text{ in } \mathbb{D}_{\mathrm{R}}\Leftrightarrow \mathrm{Re}\frac{zf'(z)}{f(z)}>0\ (z\in\mathbb{D}_{\mathrm{R}}), \qquad (1.2)$$

$$f(z)\in\mathcal{C} \text{ in } \mathbb{D}_{\mathrm{R}}\Leftrightarrow \mathrm{Re}\left\{1+\frac{\mathrm{zf''(z)}}{\mathrm{f}'(\mathrm{z})}\right\}>0\ (\mathrm{z}\in\mathbb{D}_{\mathrm{R}}). \qquad (1.3)$$

A function $f(z)\in\mathcal{A}$ is said to be in the class $\mathcal{B}_r$ if it satisfies

$$|f'(z)-1|<r\ (z\in\mathbb{D}). \qquad (1.4)$$

We investigate the following three problems.

Problem 1. Find the maximum value R_1 of R s.t.

$$f\in\mathcal{B}_1\Rightarrow f(z)\in\mathcal{S}^* \text{ in } \mathbb{D}_R$$

Problem 2. Find the maximum value R_2 of r s.t.

$$f\in\mathcal{B}_r\Rightarrow f(z)\in\mathcal{S}^* \text{ in } \mathbb{D}$$

Problem 3. Find the maximum value $\mathrm{R}_3(r)$ of R s.t.

$$f\in\mathcal{B}_r\Rightarrow f(z)\in\mathcal{C} \text{ in } \mathbb{D}_{\mathrm{R}}$$

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2 Known results

For R_1, first T.MacGregor showed in [1] that

$$R_1 \geq \frac{2}{\sqrt{5}} = 0.894\cdots$$

Lator M.Nunokawa showed in [3]

$$R_1 > 0.926\cdots$$

And Nunokawa, Fukui, Owa, Saitoh and Sekine improved in [4]

$$R_1 > 0.933\cdots$$

On the other hand P.Mocanu showed in [2] that

$$R_1 < 1$$

Using the Mocanu's method, the present author[5] have showed that

$$R_1 < 0.9982$$

For R_2, P. Mocanu[2] also proved

$$R_2 \geq 0.894\cdots$$

R.Yamakawa[5] have showed

$$R_2 < 0.9962$$

For R_3, T.MacGregor showed that

$$R_3(1) = \frac{1}{2}$$

Now, for R_1 and R_2, we improve little. And for R_3 we investigate $R_3(r)$.

3 Results

Theorem 1. $R_1 < 0.99815$ *and* $R_2 < 0.9961$. i.e.

(1) *Suppose* $f(z) \in \mathcal{A}$ *satisfies* $|f'(z) - 1| < 1$, *and suppose* $R < 0.99815$ *then*

$$\mathrm{Re}\,\frac{zf'(z)}{f(z)} > 0 \quad (z \in \mathbb{D}_R).$$

(2) *Suppose* $r < 0.9961$, *and suppose* $f(z) \in \mathcal{A}$ *satisfies* $|f'(z) - 1| < r$, *then*

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0 \quad (z \in \mathbb{D}).$$

Proof (1) We only have to show that there exist $f(z) \in \mathcal{B}_1$, and $z_0 \in \{|z| = 0.99815\}$ such that $\mathrm{Re}\dfrac{z_0 f'(z_0)}{f(z_0)} < 0$. Let

$$f'(z) = 1 + z\frac{1+az}{a+z} \quad (a > 1), \tag{3.1}$$

then from $f(0) = 0$, we have

$$f(z) = (2 - a^2)z + \frac{a}{2}z^2 + a(a^2 - 1)\log\left(1 + \frac{z}{a}\right). \tag{3.2}$$

Putting

$$a = 1.06559, \quad z_0 = re^{i\theta}$$

where

$$r = 0.99815, \quad \theta = \pi + \cos^{-1} 0.9479$$

we obtain

$$\mathrm{Re}\frac{z_0 f'(z_0)}{f(z_0)} = -1.77497 \times 10^{-6} < 0$$

■

(2) Similarly we set

$$f(z) = (1 + (1 - a^2)\mathrm{R})\mathrm{z} + \frac{\mathrm{aR}}{2}\mathrm{z}^2 + \mathrm{a}(\mathrm{a}^2 - 1)\mathrm{R}\log\left(1 + \frac{\mathrm{z}}{\mathrm{a}}\right). \tag{3.3}$$

Then

$$|f'(z) - 1| = \mathrm{R}|\mathrm{z}|\left|\frac{1 + \mathrm{az}}{\mathrm{a} + \mathrm{z}}\right| < \mathrm{R} \quad (\mathrm{z} \in \mathbb{D}).$$

Putting

$$a = 1.065, \quad \mathrm{R} = 0.9961, \quad \mathrm{z}_0 = 0.949\,\mathrm{e}^{\mathrm{i}\theta}$$

where

$$\theta = \pi + \cos^{-1} 0.949.$$

We have

$$\mathrm{Re}\frac{\mathrm{z}_0 \mathrm{f}'(\mathrm{z}_0)}{\mathrm{f}(\mathrm{z}_0)} = -0.000159\cdots < 0$$

So

$$\mathrm{R}_2 < 0.9961$$

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Theorem 2.

$$R_3(r)\begin{cases} = \alpha(r) & \left(0 < r \le \dfrac{1+\sqrt{5}}{4}\right) \\ = \dfrac{1}{2r} & \left(\dfrac{1+\sqrt{5}}{4} < r \le 1\right), \end{cases} \quad \text{where} \quad \alpha(r) = \left\{\frac{\sqrt{(1-r)(1+3r)} - (1-r)}{2r}\right\}^{\frac{1}{2}}.$$

Proof From

$$|f'(z) - 1| < r \quad (0 < r \le 1) \quad (z \in \mathbb{D}),$$

we can write

$$f'(z) = 1 + rzg(z). \tag{3.4}$$

Where $g(z)$ is analytic and

$$|g(z)| \le 1, \tag{3.5}$$

$$|g'(z)| \le \frac{1 - |g(z)|^2}{1 - |z|^2} \tag{3.6}$$

in $\mathbb{D}$. And evidently

$$\left|\frac{zf''(z)}{f'(z)}\right| = \frac{|rz\{g(z)+zg'(z)\}|}{|1+rzg(z)|}.$$

So,from (3.6), setting

$$|z| = s, \quad |g(z)| = t \ (0 \leq s,\ t \leq 1)$$

we have

$$\left|\frac{zf''(z)}{f'(z)}\right| \leq rs\frac{(1-s^2)t+s(1-t^2)}{(1-rst)(1-s^2)}.$$

Let

$$\phi(t) = rs^2t^2 - 2rs(1-s^2)t + 1 - s^2 - rs^2,$$

then

$$\phi(t) > 0 \Rightarrow \left|\frac{zf''(z)}{f'(z)}\right| < 1.$$

Since

$$\left|\frac{zf''(z)}{f'(z)}\right| < 1 \Rightarrow \mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} > 0,$$

$$\phi(t) > 0 \Longrightarrow f(z) \in C. \tag{3.7}$$

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References

[1] T.MacGregor, *A class o univalent functions,* Proc. Amer.Math. Soc., **15** (1964), 311 - 317.

[2] P.T.Mocanu, *Some starlikeness conditions for analytic functions,* Rev. Roumaine Math. Pures. Appl.,**33**(1988), 117 - 1264.

[3] M.Nunokawa, *On the starlike boundary of univalent functions,* Suugaku, **31**(1979), 255 - 256 (Japanese).

[4] M.Nunokawa, S.Fukui, S.Owa, S.Satoh, T.Sekine, *On the starlikeness bound of univalent functions,* Math. Japonica, **33**, No.5(1988), 763 - 767.

[5] R.Yamakawa, *Notes for starlikeness conditions of analytic functions,* Koukyuroku of RIMS, **821**(1993), 112 - 116.

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