ON SPECIAL VALUES OF TENSOR PRODUCT L-FUNCTIONS
OF AN INNER FORM OF GSP(4) AND GL(2)

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ABSTRACT. We consider the Rankin-Selberg integral which represents degree 8
tensor product L-functions for quaternion unitary groups and GL2. Using this
integral representation, we prove the algebraicity of special values.

1. SET UP

Let $F$ be a number field and $E$ a quadratic extension. For each $n \in \mathbb{N}$, we define
the similitude unitary group $G_n = GU(n, n)$:

$$G_n(F) = \{g \in GL(2n, E) | {^t}g^\sigma J_n g = \lambda_n(g) J_n, \lambda_n(g) \in F^{\times}\}$$

where $\sigma$ is non-trivial element in Gal$(E/F)$ and

$$J_n = \begin{pmatrix} 0_n & 1_n \\ -1_n & 0_n \end{pmatrix}.$$  

Let $E \subset D$ be a quaternion algebra over $F$. For $x \in D$, we mean the canonical
involution by $\overline{x}$. For a matrix $A = (a_{ij})$ with entries in $D$, we denote the matrix
$(\overline{a_{ij}})$ by $\overline{A}$.

Let us define the quaternion similitude unitary group $H_D$ by

$$H_D(F) = \left\{g \in GL(2, D) | {^t}\overline{g} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} g = \lambda(g) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \lambda(g) \in F^{\times} \right\}.$$  

When $D \simeq M_2(F)$, we have an isomorphism

$$H_D(F) \simeq GSp(4, F) = G_2(F) \cap GL(4, F).$$

We note that we can take $\varepsilon \in F^{\times}$ such that

$$D \simeq \left\{ \begin{pmatrix} a & \varepsilon b \\ b^\sigma & a^\sigma \end{pmatrix} | a, b \in E \right\}.$$  

Thus we may suppose that $D \subset \text{Mat}_{2 \times 2}(E)$, so that we can consider $H_D$ as a
subgroup of GL$(4, E)$. In fact, $H_D$ can be embedded into $G_2$, and we fix it. Let us define a subgroup $H$ of $G_1 \times G_2$ by

$$H = \{(g_1, h_2) \in G_1 \times H_D | \lambda_1(g_1) = \lambda_2(h_2)\},$$

and we regard $H$ as a subgroup of $G_3$ by the following embedding

$$H \ni \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} a & 0 & b & 0 \\ 0 & A & 0 & B \\ c & 0 & d & 0 \\ 0 & C & 0 & D \end{pmatrix} \in G_3.$$


2. Global integral

Let $P = MN$ denote the Siegel parabolic subgroup of $G_3$ where
\[
M(F) = \left\{ \begin{pmatrix} g & 0 \\ 0 & \lambda \cdot (t^g)^{-1} \end{pmatrix} \mid g \in \text{GL}_3(E), \lambda \in F^\times \right\},
\]
\[
N(F) = \left\{ \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix} \mid tX^\sigma = X \in \text{Mat}_{3 \times 3}(E) \right\}.
\]

Let $\nu$ be a character of $\mathbb{A}_E^\times / E^\times$ and $\tau$ a character of $\mathbb{A}_F^\times / F^\times$. Then we define a character $\nu \otimes \tau$ of $P(\mathbb{A}_F)$ by
\[
(\nu \otimes \tau) \left[ \begin{pmatrix} g & 0 \\ 0 & \lambda \cdot (t^g)^{-1} \end{pmatrix} \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix} \right] = \nu(\det g) \cdot \tau(\lambda).
\]

Let $\delta_P$ denote the modulus character of $P(\mathbb{A}_F)$. Then let $I(s, \nu \otimes \tau)$ denote the normalized degenerate principal series representation $\text{Ind}_{P(\mathbb{A}_F)}^{G(\mathbb{A}_F)}((\nu \otimes \tau) \cdot \delta_P^s)$ of $G(\mathbb{A}_F)$. Here we employ the normalized induction so that $I(s, \nu \otimes \tau)$ is unitarizable when $\text{Re}(s) = 0$. Then for a holomorphic section $f^{(s)}$ of $I(s, \nu \otimes \tau)$ we have the Siegel Eisenstein series defined by
\[
E(g, f^{(s)}) = \sum_{\gamma \in P(F) \backslash G(F)} f^{(s)}(\gamma g).
\]
This series is absolutely convergent in the right half plane $\text{Re}(s) > \frac{1}{2}$ (Langlands [5]).

Let $\sigma$ be an irreducible cuspidal representation of $\text{GL}_2(\mathbb{A}_F)$ and let $\chi$ be a character of $\mathbb{A}_E^\times / E^\times$ such that
\[
\chi|_{A_F^\times} = \omega_{\sigma}
\]
where $\omega_{\sigma}$ denotes the central character of $\sigma$. Since we have the isomorphism
\[
G_1(F) \simeq (\text{GL}(2, F) \times E^\times) / \{(a, a^{-1}) \mid a \in F^\times \},
\]
we can regard $\sigma \boxtimes \chi$ as the irreducible cuspidal automorphic representation of $G_1(\mathbb{A}_F)$ and we denote it by $\pi$. Let $V_\pi$ be the space of automorphic forms for $\pi$.

Let $(\Pi, V_\Pi)$ be an irreducible cuspidal automorphic representation of $H_D(\mathbb{A}_F)$. Let $\omega_\Pi$ denote the central character of $\Pi$. Then we study a global integral defined by
\[
(2.0.2) \quad Z(f^{(s)}, \psi, \Phi) = \int_{Z(\mathbb{A}_F) H(\mathbb{A}_F)} E(f^{(s)}, h) \psi(g_1) \Phi(h_2) dh
\]
for $f^{(s)} \in I(s, \nu \otimes \tau)$, $\psi \in V_\pi$ and $\Phi \in V_\Pi$, where $Z = Z_G \cap H$, $Z_{G_3}$ denotes the center of $G_3$, and $h = (g_1, h_2) \in H$. Here in order for the integral (2.0.2) to be well-defined, we assume that
\[
\omega_\Pi \cdot \omega_{\sigma} \cdot \tau^2 \cdot (\nu|_{\mathbb{A}_F^\times})^3 = 1.
\]

**Proposition 2.1.** For $\text{Re}(s) > 0$, we have
\[
Z(f^{(s)}, \psi, \Phi) = \int_{S(\mathbb{A}_F) \backslash H(\mathbb{A}_F)} f^{(s)}(\eta h) W_\psi(g_1) B_\Phi(h_2) dh
\]
where $B_\Phi$ is the Bessel model of $\Phi$ with respect to a non-split torus and $W_\Psi$ is the Whittaker model of $\Psi$, and $S$ is defined as follows: Let us define the Bessel subgroups $R$ of $H_D$ by

$$R(F) = \left\{ \begin{pmatrix} a^\sigma & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a^\sigma & 0 \\ 0 & 0 & 0 & a \end{pmatrix} \begin{pmatrix} 1 & 0 & \varepsilon b & c \\ 0 & 1 & c^\sigma & b \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in G_2(F) \mid a \in E^\times, b \in F, c \in E \right\}.$$  

Then a subgroup $S$ of $H$ is defined by

$$S = \{(\varphi(r), r) \mid r \in R\}$$

where we denote

$$\varphi \left[ \begin{pmatrix} a^\sigma & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a^\sigma & 0 \\ 0 & 0 & 0 & a \end{pmatrix} \begin{pmatrix} 1 & 0 & \varepsilon b & c \\ 0 & 1 & c^\sigma & b \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right] = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix}.$$  

Remark. Our integral representation is a generalization to the similitude quaternion unitary case of Saha’s interpretation [11] of Furusawa’s integral [2]. Note that we unfold the Rankin-Selberg integral involving the Siegel Eisenstein series on $G_3$ directly without recourse to the Klingen Eisenstein series on $G_2$. Thus even when $H_D \simeq \text{GSp}(4)$, our local integral is totally different from Saha’s.

In order for our investigation to be non-vacuous, we assume that

$\Pi$ has a Bessel model of non-split type.

We note that by the result of Li [6], any irreducible cuspidal automorphic representation of $H_D(\mathbb{A})$ has a Bessel model of this type if $D$ does not split. Moreover if $D \simeq \text{Mat}_{2 \times 2}(F)$, i.e., $H_D \simeq \text{GSp}(4)$, $\Pi$ has a Whittaker model or a Bessel model of some type. If $\Pi$ is associated to a holomorphic cusp form, it is non-generic, and Pitale-Schmidt [8] shows that it does not have a Bessel model of split type. Thus such automorphic representations satisfy the above assumption.

The uniqueness of Bessel model is expected for any irreducible admissible representations of $H_D(F_v)$. However as far as the author knows, there is no reference which proves the uniqueness in general. For example, for unramified representations of $\text{GSp}(4, F_v)$, Sugano [12] proves the uniqueness. Then by the uniqueness of Bessel model and Whittaker model, we obtain

$$Z(s) = \prod_{v \notin S} Z_v(W_{\Psi,v}, B_{\Phi,v}, f_v^{(s)}) \cdot Z_S(W_{\Psi,S}, B_{\Phi,S}, f_S^{(s)}).$$  

Here $S$ is a finite set of places such that any place $v \notin S$ is finite and satisfies

1. $2$ does not divide $v$
2. $E_v/F_v$ is unramified quadratic extension or $E_v \simeq F_v \oplus F_v$
3. $\Pi_v, \pi_v, \nu_v, \tau_v$ are unramified.
4. $D(F_v) \simeq \text{Mat}_{2 \times 2}(F_v)$.

Then Furusawa and Ichino computed unramified local integrals explicitly.
Proposition 2.2 (Furusawa-Ichino, Appendix in [7]). Suppose \( v \not\in S \). For normalized spherical vectors \( W_v, B_v \) and \( f_v^{(s)} \), we have

\[
Z_v(s) = \prod_{i=1}^{3} L \left( 6s + i, \nu|_{F_v^x} \cdot \epsilon_{E_v/F_v}^{i+3} \right)^{-1} \cdot L \left( 3s + \frac{1}{2}, \Pi \times \sigma \times (\nu|_{F_v^x})^2 \times \tau \right)
\]

where we normalize the measure on \( H(F_v) \) suitably, and \( \epsilon_{E_v/F_v} \) is the quadratic character of \( F_v^x \) corresponding to \( E_v \) via local class field theory.

3. MAIN THEOREM

Assume that

\[
H_D(\mathbb{R}) \simeq \text{GSp}(4, \mathbb{R}) \quad \text{and} \quad F = \mathbb{Q}.
\]

We possibly have \( D \simeq \text{Mat}_{2 \times 2}(\mathbb{Q}) \). We suppose that the central characters of \( \Pi \) and \( \pi \) are trivial.

Suppose that the archimedean component \( \Pi_\infty \) of \( \Pi \) is the holomorphic discrete series of \( \text{PGSp}(4, \mathbb{R}) \) with Harish-Chandra parameter \( \ell(e_1 + e_2) \) with even integer \( \ell \) where we define

\[
e_i \left( \begin{array}{cc} t_1 & t_2 \\ t_1^{-1} & t_2^{-1} \end{array} \right) = t_i \quad t_i \in \mathbb{G}_m.
\]

Suppose that \( \sigma \) is a cuspidal automorphic representation associated to a new form of weight \( \ell \). Then we consider an automorphic form \( \Psi \in V_\sigma \) as the automorphic form on \( G_1(\mathbb{A}) \) by extending it trivially, i.e.

\[
\Psi(ag) = \Psi(g)
\]

for \( a \in \mathbb{A}_E^x \) and \( g \in \text{GL}(2, \mathbb{A}_\mathbb{Q}) \).

Theorem 3.1. Suppose that \( \ell > 6 \). Let \( \Phi \in V_\Pi \) and \( \Psi \in V_\sigma \) be arithmetic automorphic forms in the sense of Harris [4]. Then for an integer \( m \) such that \( 2 < m \leq \frac{\ell}{2} - 1 \), we have

\[
\frac{L(m, \Pi \times \sigma)}{\pi^{4m} \langle \Psi \otimes \Phi, \Psi \otimes \Phi \rangle} \in \overline{\mathbb{Q}}
\]

and

\[
\left( \frac{L(m, \Pi \times \sigma)}{\pi^{4m} \langle \Psi \otimes \Phi, \Psi \otimes \Phi \rangle} \right)^\tau = \frac{L(m, \Pi^\tau \times \sigma^\tau)}{\pi^{4m} \langle \Psi^\tau \otimes \Phi^\tau, \Psi^\tau \otimes \Phi^\tau \rangle}
\]

for all \( \tau \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \). Here we define

\[
\langle \Psi \otimes \Phi, \Psi \otimes \Phi \rangle = \int_{Z_H(\mathbb{A}_\mathbb{Q})H(\mathbb{Q}) \backslash H(\mathbb{A}_\mathbb{Q})} |\Psi(g_1)\Phi(h_2)|^2 dh
\]

where we denote \( h = (g_1, h_2) \in H(\mathbb{A}_\mathbb{Q}) \), and \( dh \) is the Tamagawa measure on \( H(\mathbb{A}_\mathbb{Q}) \).

We can prove this by a similar way with Garrett-Harris [3]. For a detail of the proof, we refer to [7].
3.1. **Period Relation.** Let $(\Pi, V_\Pi)$ be an irreducible cuspidal automorphic representation of $\mathrm{GSp}(4, \mathbb{A}_\mathbb{Q})$ as in Theorem 3.1. Further we assume that $\Pi$ is tempered and non-endoscopic. We suppose that there exists an irreducible cuspidal automorphic representation $(\Pi_D, V_{\Pi_D})$ of $H_D(\mathbb{A}_\mathbb{Q})$ such that for every place $v$ such that $H_D(\mathbb{Q}_v) \simeq \mathrm{GSp}(4, \mathbb{Q}_v)$,

$$\Pi_v \simeq \Pi_{D,v}.$$  

Then $\Pi_D$ satisfies the condition in Theorem 3.1. Comparing the equations in Theorem 3.1 for $\Pi$ and $\Pi_D$, we obtain the following relation.

**Corollary 3.1.** For any arithmetic forms $\Phi \in V_\Pi$ and $\Phi_D \in V_{\Pi_D}$, we have

$$\langle \Phi, \Phi \rangle / \langle \Phi_D, \Phi_D \rangle \in \overline{\mathbb{Q}}$$

and

$$\langle \langle \Phi, \Phi \rangle / \langle \Phi_D, \Phi_D \rangle \rangle^\tau = \langle \Phi^\tau, \Phi^\tau \rangle / \langle \Phi_D^\tau, \Phi_D^\tau \rangle$$

for any $\tau \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

Here we define the pairing $\langle \Phi_D, \Phi_D \rangle$ by

$$\langle \Phi, \Phi \rangle = \int_{Z_{H_D}(\mathbb{A}_\mathbb{Q})\backslash H_D(\mathbb{Q})H_D(\mathbb{A}_\mathbb{Q})} |\Phi_D(h)|^2 \, dh$$

where $dh$ is the Tamagawa measure on $H_D(\mathbb{A}_\mathbb{Q})$, and we define $\langle \Phi, \Phi \rangle$ similarly.

3.2. **Remarks on Theorem 3.1.**

3.2.1. **critical point.** The critical points in Theorem 3.1 does not cover all critical points on the right half plane $\text{Re}(s) > 0$. Indeed the critical points for $s = \frac{1}{2}$ and $\frac{1}{6}$ are not included due to the analytic property of Eisenstein series.

3.2.2. **Split case.** When $H_D \simeq \mathrm{GSp}(4)$, similar results are proved by many people. Furusawa [2] discovered an integral representation of this $L$-function and he proved the algebraicity at the rightmost critical point for Siegel cusp forms and elliptic cusp form of full level. Pitale-Schmidt [9] extended his result with respect to the level of elliptic cusp forms, and Saha [10] extended with respect to both of levels of Siegel cusp forms and elliptic cusp form. Saha [11] also proved the algebraicity for other critical points combining the pull-back formula and differential operators. On the other hand, Böcherer-Heim [1] showed the algebraicity at all critical points in the full modular balanced mixed weight case using Heim’s integral representation.

3.2.3. **Yoshida’s Conjecture.** When the irreducible cuspidal automorphic representation of $\mathrm{GSp}(4, \mathbb{A}_\mathbb{Q})$ is associated to a Siegel cusp form, our result is compatible with Yoshida’s calculation [13] on Deligne period.

**Acknowledgements.** This article is based on a talk delivered by the author at the 2012 RIMS conference *Automorphic forms and automorphic L-functions*. The author thanks the organizer of the conference, Tomonori Moriyama and Atsushi Ichino, for the opportunity to speak.

The research of the author was supported in part by Grant-in-Aid for JSPS Fellows (23-6883) and JSPS Institutional Program for Young Researcher Overseas Visits project: Promoting international young researchers in mathematics and mathematical sciences led by OCAMI.
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