On the image of the Saito-Kurokawa lifting over a totally real number field and the Maass relation

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1 Introduction

H. Saito and N. Kurokawa independently conjectured that there exists a lifting from an eigenform \( \varphi \in S_{2k}(SL_2(\mathbb{Z})) \) with \( k \) odd to an eigenform \( \Phi(Z) = \sum_B A(B) \exp(2\pi \sqrt{-1} \text{tr}(BZ)) \in S_{k+1}(Sp_2(\mathbb{Z})) \) such that

\[
L(s, \Phi, sp) = \zeta(s-k)\zeta(s-k+1)L(s, \varphi).
\]

Here \( \zeta(s) \) is the Riemann zeta function, and \( L(s, \Phi, sp) \) is the spin \( L \)-function of \( \Phi \). Moreover they conjectured that the Fourier coefficient \( A(B) \) satisfies the Maass relation

\[
A(B) = \sum_{d|\varepsilon(B)} d^{k}A(B_d)
\]

for any nonzero matrix \( B \in S_2^*(\mathbb{Z}) \), where \( S_2^*(\mathbb{Z}) \) is a set of all half integral symmetric matrix of size \( 2 \times 2 \). Here the summation runs over all positive integer \( d \) which divide \( \varepsilon(B) = \gcd(b_{11}, 2b_{12}, b_{22}) \) for \( B = (b_{ij}) \in S_2^*(\mathbb{Z}) \), and \( B_d \) is defined by \( B_d = \left( \begin{array}{cc} 1 & b_{12}/d \\ b_{12}/d & b_{11}b_{22}/d^2 \end{array} \right) \).

The conjecture was proved by Maass, Andrianov and Zagier (see [2], [17]). Then the lifting is called the Saito-Kurokawa lifting.

Naturally, we can consider the generalization of the Saito-Kurokawa lifting, i.e. we consider the lifting from a Hilbert modular form to a Hilbert-Siegel modular form over a totally real number field. In fact, Piatetski-Shapiro [12] and Schmidt [13] proved the existence of the generalized Saito-Kurokawa lifting using representation theory.

The main purpose of this paper is to give a Fourier coefficient formula of the lifted form and a generalization of the Maass relation.

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The author would like to express his gratitude to Professor Tamotsu Ikeda for his encouragement and valuable advice. He also thanks Professor H. Kojima for his comment about [8].

2 Hilbert-Siegel modular form

In this section, \( K \) is a totally real number field of degree \( d = [K : \mathbb{Q}] \), \( \mathcal{O}_K \) is the ring of integers of \( K \), and \( D_K \) is the different of \( K \) relative to \( \mathbb{Q} \). Let \( \mathfrak{H}_n \) be the Siegel upper half space of degree \( n \), i.e.,

\[
\mathfrak{H}_n = \{ Z \in M_n(\mathbb{C}) \mid \overline{t}Z = Z, \ \text{Im} \ Z > 0 \}.
\]

Let \( \text{GSp}^{+}_n(\mathbb{R}) = \{ g \in GL_{2n}(\mathbb{R}) \mid t g J_n g = \nu(g) J_n, \ \nu(g) > 0 \} \), where \( J_n = \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix} \). For \( Z = (Z_1, \ldots, Z_d) \in \mathfrak{H}_n^d \) and \( M = (M_1, \ldots, M_d) \in \text{GSp}^{+}_n(\mathbb{R})^d \) with \( M_i = \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix} \), put

\[
MZ = (M_1 Z_1, \ldots, M_d Z_d) \in \mathfrak{H}_n^d,
\]

where \( M_i Z_i = (A_i Z_i + B_i)(C_i Z_i + D_i)^{-1} \). Let \( x = (x_1, \ldots, x_d) \in \mathbb{R}^d \) (resp. \( x \in \mathbb{C}^d \)) and \( \kappa = (\kappa_1, \ldots, \kappa_d) \in \mathbb{R}^d \) (resp. \( \kappa \in \mathbb{Z}^d \)). We define a multi-index notation \( x^\kappa \) by

\[
x^\kappa = \prod_{i=1}^{d} x_i^{\kappa_i}.
\]

Put

\[
j(M, Z)^\kappa = \det(M)^{-\kappa/2} \det(CZ + D)^\kappa,
\]

using the multi-index notation. Here we use following abbreviations:

\[
\det(M) = (\det M_1, \ldots, \det M_d),
\]

\[
CZ + D = (C_1 Z_1 + D_1, \ldots, C_d Z_d + D_d),
\]

for \( Z = (Z_1, \ldots, Z_d) \in \mathfrak{H}_n^d \) and \( M = (M_1, \ldots, M_d) \in \text{GSp}^{+}_n(\mathbb{R})^d \) with \( M_i = \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix} \). Fix the mutually different real embeddings \( K \ni x \mapsto x^{(i)} \in \mathbb{R} \) (resp. \( x \in \mathbb{C} \)) for all \( v \mid \infty \). Let \( K_{\infty} \) and \( \text{GSp}^{+}_n(K_{\infty}) \) be the archimedian parts of \( \mathbb{A}_K \) and \( \text{GSp}^{+}_n(\mathbb{A}_K) \). Then we identify \( \mathbb{R}^d \) (resp. \( \text{GSp}^{+}_n(\mathbb{R})^d \)) with \( K_{\infty} \) (resp. \( \text{GSp}^{+}_n(K_{\infty}) \)) by \( K \ni x \mapsto (x^{(1)}, \ldots, x^{(d)}) \in \mathbb{R}^d \).

Let \( \Phi \) be a function on \( \mathfrak{H}_n^d \), and \( M \in \text{GSp}^{+}_n(K) \subset \text{GSp}^{+}_n(\mathbb{R})^d \). Define a function \( \Phi_{|\kappa} M \) by

\[
(\Phi_{|\kappa} M)(Z) = j(M, Z)^{-\kappa} \Phi(MZ).
\]

Put \( W = \text{GSp}^{+}_n(K_{\infty}) \times \prod_{v<\infty} W_v \), where \( W_v = \text{GSp}^{+}_n(\mathcal{O}_{K_v}) \). Take \( h \) elements \( t_1, \ldots, t_h \) of \( \mathcal{O}_K^\times \) so that \( t_{i,v} = 1 \) for all \( v \mid \infty \) and \( t_1 \mathcal{O}_K, \ldots, t_h \mathcal{O}_K \) form a complete set of representatives for narrow ideal classes of \( K \). Here we denote by \( y \mathcal{O}_K \) the fractional ideal
of $K$ associated to $y \in \mathbb{A}_K^\times$. Let $x_\lambda = \text{diag}(t_\lambda^{-1}I_n,I_n)$. Then we have $\text{GSp}_n(\mathbb{A}_K) = \bigsqcup_{\lambda=1}^{\h} \text{GSp}_n(K)x_\lambda W$, where $\bigsqcup$ is the disjoint union. Put $I_\lambda = x_\lambda Wx_\lambda^{-1} \cap \text{GSp}_n(K)$.

Now we assume $K \neq \mathbb{Q}$ or $n \neq 1$. Then a Hilbert-Siegel modular form of weight $\kappa \in \mathbb{Z}^d$ with respect to $I_\lambda$ is a holomorphic function $\Phi : \mathfrak{H}_n^d \to \mathbb{C}$ such that $\Phi|_\kappa M = \Phi$ for any $M \in I_\lambda$. We denote the space of Hilbert-Siegel modular forms of weight $\kappa$ and degree $n$ with respect to $I_\lambda$ by $M_\kappa^{(n)}(I_\lambda)$. Let $\Phi \in M_\kappa^{(n)}(I_\lambda)$ and $M \in \text{GSp}_n^+(K)$. It is known that $\Phi|_\kappa M$ has a Fourier expansion

$$(\Phi|_\kappa M)(Z) = \sum_B A_M(B) \exp(2\pi i \text{Tr}(BZ)).$$

Here $\text{Tr}(BZ) = \text{tr}(\sum_{i=1}^{d} B^{(i)} Z_i)$, and the summation runs over $B \in S_n^*(t_{\lambda}\mathcal{D}_K^{-1})$ such that $B^{(i)}$ $(i = 1, \ldots, d)$ are all positive semi-definite. We define a space of cusp forms $S_\kappa^{(n)}(I_\lambda)$ by

$$S_\kappa^{(n)}(I_\lambda) = \left\{ \Phi \in M_\kappa^{(n)}(I_\lambda) \mid A_M(B) = 0 \text{ unless } M \gg 0 \text{ for any } M \in \text{GSp}_n^+(K) \right\}.$$  

Here $B \gg 0$ if $B^{(i)} > 0$ for $i = 1, \ldots, d$. Put $M_\kappa^{(n)} = \prod_{\lambda=1}^{\h} M_\kappa^{(n)}(I_\lambda)$ and $S_\kappa^{(n)} = \prod_{\lambda=1}^{\h} S_\kappa^{(n)}(I_\lambda)$. Let $(\Phi_1, \ldots, \Phi_h) \in M_\kappa^{(n)}$. We define a function $\tilde{\Phi}$ on $\text{GSp}_n(\mathbb{A})$ by $\tilde{\Phi}(\alpha x_\lambda w) = \Phi|_\kappa w_{\infty}(i)$, where $\alpha \in \text{GSp}_n(K)$, $w \in W$, and $i = (\sqrt{-1}I_n, \ldots, \sqrt{-1}I_n) \in \mathfrak{H}_n^d$. We identify $\tilde{\Phi}$ with $(\Phi_1, \ldots, \Phi_h) \in M_\kappa^{(n)}$.

### 3 Siegel series

In this section $K$, $\mathcal{O}$, $\mathfrak{p}$ and $\varpi$ denote a finite algebraic extension of $\mathbb{Q}_p$, the integral closure of $\mathbb{Z}_p$ in $K$, maximal ideal of $\mathcal{O}$, and a prime element of $K$, respectively. Let $q = [\mathcal{O} : \mathfrak{p}]$, and $| \cdot |$ be the normalized absolute value on $K$, i.e. $|\varpi| = q^{-1}$. Let $R$ be a fractional ideal of $K$. Let $S_n(R)$ denote the set of symmetric matrices of size $n$ with entries in $R$, and put $S_n^*(R) = \{ (b_{ij}) \in S_n(K) \mid b_{ii}, 2b_{ij} \in R, (i,j = 1, \ldots, n) \}$. Let $\chi(x) = \exp(-2\pi \sqrt{-1}y)$ be a character of $K$ with $y \in \mathbb{Z}[1/p]$ such that $\text{tr} \left( K_{\mathbb{Q}_p}(x) - y \right) \in \mathbb{Z}_p$. Then we have $\chi(x) = 1 \in S_n^*(\mathcal{D}_K^{-1})$. Here $\mathcal{D}_K$ is the different of $K$ relative to $\mathbb{Q}_p$. For $S \in S_n(K)$, we put $\nu(S) = [S\mathcal{O}^n + \mathcal{O}^n : \mathcal{O}^n]$.

Given $B \in S_n^*(\mathcal{D}_K^{-1})$, we define a formal Dirichlet series $b(B,s)$ by

$$b(B,s) = \sum_{R \in S_n(K)/S_n(\mathcal{O})} \chi(\text{tr}(BR))\nu(R)^{-s}.$$  

We call the series $b(B,s)$ a Siegel series of degree $n$ over $K$. Here $\nu(R)$ and $\chi(\text{tr}(BR))$ depend only on the class $R$ and then the sum is formaly well-defined. It is known that $b(B,s)$ is convergent if $\text{Re}(s)$ sufficiently large for any given $B \in S_n^*(\mathcal{D}_K^{-1})$. 


Now we consider the Siegel series in the case $n = 2$. We fix $F = \text{diag}(\varpi^{l_1}, \varpi^{l_1+l_2}) \in \mathcal{F}$ and nonzero matrix $B \in S_2(D_K^{-1})$. Let $\varepsilon(B)$ be the minimal integral ideal $\mathfrak{a}$ satisfying $B \in S_2(D_K^{-1}\mathfrak{a})$, and $\alpha_1 = \text{ord}_p \varepsilon(B)$. We put $\delta \in K$ so that $\delta O = D_K$. For $N \in K^\times$, we define $\alpha(N)$ and $\xi_N$ by $\alpha(N) = \frac{1}{2}(\text{ord}_p N - \text{ord}_\mathfrak{P} D_{K(\sqrt{N})/K}) + \text{ord}_p \delta$ and

$$\xi_N = \begin{cases} 1 & \text{if } N \in K^{\times 2}, \\ -1 & \text{if } K(\sqrt{N})/K \text{ is unramified extension}, \\ 0 & \text{if } K(\sqrt{N})/K \text{ is ramified extension}, \end{cases}$$

respectively. Here $D_{K(\sqrt{N})/K}$ is different of $K(\sqrt{N})$ relative to $K$, and $\mathfrak{P}$ is maximal ideal of $K(\sqrt{N})$. Put $t = -\det(2\delta B)$, $\alpha = \alpha(t) - \text{ord}_p \delta$, and $\xi_B = \xi_t$.

**Theorem 3.1** Let $B \in S_2(D_K^{-1})$, and $\det B \neq 0$, then

$$b(B, s) = \frac{(1-q^{-s})(1-q^{2-2s})}{1-\xi_B q^{1-s}} F(B, q^{-s})$$

where $F(B, X)$ is a polynomial of $X$ with integral coefficients:

$$F(B, X) = \sum_{l=0}^{\alpha_1} (q^{2}X)^l \left\{ \sum_{m=0}^{\alpha-l} (q^{3}X^{2})^{m} - \xi_B qX \sum_{m=0}^{\alpha-l-1} (q^{3}X^{2})^{m} \right\}.$$

**Corollary 3.2** Let $\tilde{F}(B, X) = X^{-\alpha} F(B, q^{-3/2}X)$, then

$$\tilde{F}(B, X) = \sum_{l=0}^{\alpha_1} q^{l/2} \left( \frac{X^{\alpha-l+1} - X^{-\alpha+l-1}}{X - X^{-1}} - \xi_B q^{-1/2} \frac{X^{\alpha-l} - X^{-\alpha+l}}{X - X^{-1}} \right).$$

### 4 Main results

Assume $(\phi_\lambda) \in S_{2\kappa}^{(1)}$ is a Hecke eigenform with the Satake parameter $\{\alpha_v, \alpha_v^{-1}\}$ for $v < \infty$. Let $(\Phi_\lambda) \in S_{\kappa+1}^{(2)}$ be the image of the Saito-Kurokawa lifting of $(\phi_\lambda)$, and $A_\lambda(B)$ the $B$th Fourier coefficient of

$$\Phi_\lambda = \sum_B A_\lambda(B) \exp(2\pi\sqrt{-1} \text{Tr}(BZ)) \in S_{\kappa+1}^{(2)}(\Gamma_\lambda).$$

Here $\text{Tr}(BZ) = \text{tr}(\sum_{i=1}^{d} B^{(i)}Z_i)$, and the summation runs over $B \in S_2(t_\lambda D_K^{-1})$ such that $B^{(i)} \ (i = 1, \ldots, d)$ are all positive semi-definite. Then the first main result is as follows:

**Theorem 4.1 (Fourier coefficient formula)** The following assertion holds:

$$A_\lambda(B) = C_B N(t_\lambda O_K)^{3/2} \det B^{\kappa/2-1/4} \prod_{v < \infty} \tilde{F}_v(t_\lambda^{-1} B, \alpha_v)$$
using the multi-index notation. Here $\tilde{F}_v(B, X)$ is the Laurent polynomial in Corollary 3.2, and $t_\lambda = (t_{\lambda,v}) \in A_K^\times$ are as in §2.

Moreover the constant $C_B$ satisfies $C_{(A_B | A)} = \text{sgn}(\det A)^{\kappa+1} C_B$ for any $A \in \text{GL}_2(K)$, $r \in K^\times$. Thus the constant $C_B$ depends only on det $B \mod (K^\times)^2$.

For given $B \in S^2_2(t_{\lambda} D_K^{-1})$ and $\delta_v \in K_v$ such that $\delta_v \mathcal{O}_{K_v} = D_{K_v}$, let $t_B = -\det(2B)$, and $\alpha_{\lambda,v}(B) = \frac{1}{2}(\text{ord } t_B - \text{ord } D_{K_v(\sqrt{t_B})/K_v}) + \text{ord}_v \delta_v - \text{ord}_v t_{\lambda,v}$. Let $\varepsilon_\lambda(B)$ be the minimal integral ideal $\mathfrak{a}$ satisfying $B \in S^2_2(t_{\lambda} D_K^{-1} \mathfrak{a})$. Put $f_{\lambda,B} = \prod_{\mathfrak{p} \in \mathfrak{p}_v} \mathfrak{p}_v^{\alpha_{\lambda,v}(B)}$, where $\mathfrak{p}_v$ is the maximal ideal of $K_v$. For an integral ideal $\mathfrak{a} | \varepsilon(B)$, we take a fractional ideal $t_\mu \mathcal{O}_{K}$ and $\eta \in K^\times = \{ x \in K^\times | x \gg 0 \}$ so that $t_{\lambda} \mathfrak{a} = t_\mu(\eta)$. We put $A_0^\lambda(B/\mathfrak{a}) = A_0^\mu(\eta^{-1} B)$. Then $A_0^\lambda(B/\mathfrak{a})$ is independent of the choice of $\eta$. Let $\mu(\mathfrak{a})$ be the Möbius function, and $S^{(2),SK}_{\kappa+1}$ the subspace of $S^{(2)}_{\kappa+1}$ spanned by the Saito-Kurokawa lifting of all Hecke eigenforms in $S^{(2)}_{2\kappa+1}$. Then the second main result is as follows:

**Theorem 4.2 (Maass relation)** Let $\Phi = (\Phi_\lambda) \in S^{(2),SK}_{\kappa+1}$, and $A_\lambda(B)$ the $B^{th}$ Fourier coefficient of $\Phi_\lambda$. Put $A_0^\lambda(B) = N(t_{\lambda} \mathcal{O}_{K})^{-3/2} \det B^{-\kappa/2+1/4} A_\lambda(B)$. Then for $N \in K^\times/K^{\times 2}$ and an integral ideal $\mathfrak{a}$, there exists a $\mathbb{C}$-valued function $T_\Phi(N, \mathfrak{a})$ such that

$$A_0^\lambda(B) = \sum_{\mathfrak{a} | \varepsilon_\lambda(B)} N(\mathfrak{a})^{1/2} T_\Phi(t_B, f_{\lambda,B} \mathfrak{a}^{-1}),$$

for any $B \in S^2_2(t_{\lambda} D_K^{-1})$ and $\lambda$. Here $\mathfrak{a}$ runs over all integral ideals dividing $\varepsilon_\lambda(B)$.

Next we assume that the narrow class number of $K$ is one. Take $0 \ll \delta \in \mathcal{O}_K$ so that $D_K = \delta \mathcal{O}_K$. For $0 \ll B = \delta^{-1}(b_{ij}) \in S^2_2(D_K^{-1})$ and $0 \ll d \in \mathcal{O}_K$, put $B_d = \delta^{-1} \left( \begin{array}{cc} 1 & b_{12}/d \\ b_{12}/d & b_{11}b_{22}/d^2 \end{array} \right)$. Let $\Phi = \Phi_1 \in S^{(2),SK}_{\kappa+1}$, $A(B) = A_1(B), A^0(B) = A_1^0(B)$ and $t_1 \mathcal{O}_K = \mathcal{O}_K$. Then we can take $T_\Phi(t_B, f_{1,B}(d)^{-1}) = A^0(B_d)$. Thus we have the same formulation as the classical Maass relation.

**Corollary 4.3** Assume above setting. Then the Fourier coefficients satisfy a linear relation

$$A(B) = \sum_{d \mathcal{O}_K | \varepsilon(B)} d^\kappa A(B_d)$$

for any $B \in S^2_2(D_K^{-1})$ using the multi-index notation. Here $0 \ll d \in \mathcal{O}_K$, and for given $B$, $d^\kappa A(B_d)$ depends only on the ideal $d \mathcal{O}_K$. 

5  An example

In this section, we give an example of the Saito-Kurokawa lifting, and we see the lifted form satisfies the Maass relation. Let $K = \mathbb{Q}(\sqrt{5})$, $Z = (Z_1, Z_2) \in \mathfrak{H}_2^2$, $\varepsilon = \frac{1+\sqrt{5}}{2}$ and $Q \in M_8(\mathcal{O}_K)$ which is positive definite and even-integral. Then we define a theta function $\Theta_Q(Z)$ by

$$\Theta_Q(Z) = \sum_{X \in M_{8,2}(\mathcal{O}_K)} \exp(\pi \sqrt{-1} \sigma \left( \frac{Q[X]Z}{\varepsilon\sqrt{5}} \right)).$$

Here $\sigma \left( \frac{Q[X]Z}{\varepsilon\sqrt{5}} \right) = \text{tr}(\frac{Q[X]}{\varepsilon\sqrt{5}})^{(1)} Z_1 + \text{tr}(\frac{Q[X]}{\varepsilon\sqrt{5}})^{(2)} Z_2$ , and $x^{(i)}$ is a real embeddings $K \ni x \to x^{(i)} \in \mathbb{R}$ ($i = 1, 2$). Note that the narrow class number of $K = \mathbb{Q}(\sqrt{5})$ is one, and the different $\mathcal{D}_K$ is generated by a totally positive element $\varepsilon\sqrt{5}$ of $K$. Then $\Theta_Q(Z)$ is a Hilbert-Siegel modular form of weight $\kappa = (4,4)$ for $\text{Sp}_2(\mathcal{O}_K)$, and the Fourier expansion is

$$\Theta_Q(Z) = \sum_{X \in M_{8,2}(\mathcal{O}_K)} \exp(\pi \sqrt{-1} \sigma \left( \frac{Q[X]Z}{\varepsilon\sqrt{5}} \right)),
= \sum_{B \in S_2^*(\mathcal{D}^{-1}_K)} A(Q, B) \exp(2\pi \sqrt{-1} \sigma(BZ)).$$

Here

$$A(Q, B) = \# \{X \in M_{8,2}(\mathcal{O}_K) \mid Q[X] = 2\varepsilon\sqrt{5}B \}.$$

Thus we can compute the $B^{th}$ Fourier coefficient of $\Theta_Q$ counting the solutions of $Q[X] = 2\varepsilon\sqrt{5}B$.

By Maass [10], there exist exactly two inequivalent classes of even quadratic form with determinant one and eight variables over $\mathbb{Q}(\sqrt{5})$. These two classes are represented by following matrices: $F_4^2 = F_4 \oplus F_4$ and $E_8$.

Here $F_4 = \begin{pmatrix} 2 & -1 & 0 & 1-\varepsilon \\ -1 & 2 & -1 & \varepsilon - 1 \\ 0 & -1 & 2 & \varepsilon \\ 1-\varepsilon & \varepsilon - 1 & \varepsilon & 2 \end{pmatrix}$, and $E_8 \in \text{GL}_8(\mathbb{Z})$ is a positive definite even unimodular matrix over $\mathbb{Z}$.

Put

$$s_4(Z) = \frac{1}{2880} (\Theta_{F_4^2}(Z) - \Theta_{E_8}(Z)).$$
Proposition 5.1 $s_4(Z) \in S_{(4,4)}(\text{Sp}_2(\mathcal{O}_K))$.

Let $S_{\kappa+1}^{SK}(\text{Sp}_2(\mathcal{O}_K))$ be the subspace of $S_{\kappa+1}(\text{Sp}_2(\mathcal{O}_K))$ spanned by the Saito-Kurokawa lifting of all Hecke eigenforms in $S_{2\kappa}(\text{SL}_2(\mathcal{O}_K))$.

Proposition 5.2 The following assertion holds:

1. $\dim S_{(4,4)}^{SK}(\text{Sp}_2(\mathcal{O}_K)) = \dim S_{(4,4)}(\text{Sp}_2(\mathcal{O}_K)) = 1$.
2. $s_4(Z) \in S_{(4,4)}^{SK}(\text{Sp}_2(\mathcal{O}_K))$.

We show some Fourier coefficients $A(F_4^2, B)$, $A(E_8, B)$ and $A(B)$ of $\Theta_{F_4^2}$, $\Theta_{E_8}$ and $s_4$, respectively:

<table>
<thead>
<tr>
<th>$N$</th>
<th>$(a, b, c)$</th>
<th>$A(F_4^2, B)$</th>
<th>$A(E_8, B)$</th>
<th>$A(B)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>$(2, \varepsilon, 2)$</td>
<td>2880</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>80</td>
<td>$(2, 2\varepsilon, 8)$</td>
<td>1918080</td>
<td>1814400</td>
<td>36</td>
</tr>
<tr>
<td>80</td>
<td>$(4, 2\varepsilon, 4)$</td>
<td>2102400</td>
<td>1814400</td>
<td>100</td>
</tr>
<tr>
<td>9</td>
<td>$(2, 1, 2)$</td>
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<td>13440</td>
<td>−3</td>
</tr>
<tr>
<td>144</td>
<td>$(2, 2, 8)$</td>
<td>8083200</td>
<td>8117760</td>
<td>−12</td>
</tr>
<tr>
<td>144</td>
<td>$(4, 2, 4)$</td>
<td>8390400</td>
<td>8977920</td>
<td>−204</td>
</tr>
</tbody>
</table>

Here $N = N_{K/\mathbb{Q}}(\det(2\varepsilon \sqrt{5}B))$, and $(a, b, c)$ is an abbreviation for $B$ so that $2\varepsilon \sqrt{5}B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Put $(a, b, c) = B$, $A(a, b, c) = A(B)$ using above abbreviation. Then we see the Maass relation (Corollary 4.3) holds:

\[ A(4, 2\varepsilon, 4) = 100 = A(2, 2\varepsilon, 8) + 64 A(2, \varepsilon, 2), \]
\[ A(4, 2, 4) = -204 = A(2, 2, 8) + 64 A(2, 1, 2). \]

参考文献


