Jacquet-Langlands-Shimizu correspondence for theta lifts to $GSp(2)$ and its inner forms

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with an appendix by Ralf Schmidt

Abstract

As was first pointed out by Ibukiyama [I], the spinor $L$-functions of automorphic forms on the indefinite symplectic group $GSp(1,1)$ or the definite symplectic group $GSp^*(2)$ over $\mathbb{Q}$ right invariant by a (global) maximal compact subgroup are conjectured to be those of paramodular forms of some specified level on the symplectic group $GSp(2)$, which can be viewed as a generalization of the Jacquet-Langlands-Shimizu correspondence to the case of $GSp(2)$ and its inner forms $GSp(1,1)$ and $GSp^*(2)$.

This short note surveys our results presented at the RIMS-conference held during January 16-21 in 2012. They provide evidence of this conjecture by theta lifts from $GL(2) \times B^\times$ to the inner forms and theta lifts from $GL(2) \times GL(2)$ to $GSp(2)$ (considered by [O]), where $B$ denotes a definite quaternion algebra over $\mathbb{Q}$. Our explicit functorial correspondence given by these theta lifts are proved to be compatible with a non-archimedean local Jacquet-Langlands correspondence for $GSp(2)$ (or $GSp(4)$) and its inner forms, which is considered in the appendix by Ralf Schmidt.

1 Basic facts

1.1 Algebraic groups.

Let $B$ be a definite quaternion algebra over $\mathbb{Q}$ with the discriminant $d_B$, and let $B \ni x \mapsto \overline{x} \in B$ be the main involution of $B$. By $n$ and $\text{tr}$ we denote the reduced norm and the reduced trace of $B$ respectively.

Let $G_{nc} = GSp(1,1)$ and $G_{nc}^1 = Sp(1,1)$ be the $\mathbb{Q}$-algebraic groups defined by

$$G_{nc}(\mathbb{Q}) := \{g \in M_2(B) \mid ^t \overline{g} Q_{nc} g = \nu(g) Q_{nc}, \nu(g) \in \mathbb{Q}^\times\}, \quad G_{nc}^1(\mathbb{Q}) := \{g \in G_{nc}(\mathbb{Q}) \mid \nu(g) = 1\},$$

where $Q_{nc} := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Furthermore let $G_c = GSp^*(2)$ and $G_c^1 = Sp^*(2)$ be the $\mathbb{Q}$-algebraic groups defined by

$$G_c(\mathbb{Q}) := \{g \in M_2(B) \mid ^t \overline{g} Q_c g = \mu(g) Q_c, \mu(g) \in \mathbb{Q}^\times\}, \quad G_c^1(\mathbb{Q}) := \{g \in G_c(\mathbb{Q}) \mid \mu(g) = 1\},$$

*Partially supported by Grand-in-Aid for Young Scientists (B) 21740025, JSPS.
where $Q_c := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

On the other hand, let $G' = GSp(2)$ be the $\mathbb{Q}$-algebraic group defined by

$$G'(\mathbb{Q}) := \left\{ g \in GL_4(\mathbb{Q}) \mid t_g \begin{pmatrix} 0_2 & 1_2 \\ -1_2 & 0_2 \end{pmatrix} = \lambda(g) \begin{pmatrix} 0_2 & 1_2 \\ -1_2 & 0_2 \end{pmatrix}, \lambda(g) \in \mathbb{Q}^\times \right\}.$$  

We should note that $G_{nc}$ and $G_c$ are inner $\mathbb{Q}$-forms of $G'$. By $Z_{\mathcal{G}}$ we denote the center of $\mathcal{G} = G_{nc}, G_c$ or $G_s$.

In what follows, we often put $G = G_c$ or $G_{nc}$.

### 1.2 Maximal compact subgroups.

Let $Q = Q_{nc}$ or $Q_c$. We first introduce maximal compact subgroups at the archimedean place. We put $G_{\infty}^1 := \{ g \in M_2(\mathbb{H}) \mid t_{\overline{g}Qg} = Q \}$, where $\mathbb{H} := B \otimes \mathbb{R}$ is the Hamilton quaternion algebra. Then $G_{\infty}^1$ is the maximal compact subgroup itself when $Q = Q_c$, and $K_{\infty}^0 := \{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \in M_2(\mathbb{H}) \mid a \pm b \in \mathbb{H}^1 \}$ forms a maximal compact subgroup of $G_{\infty}^1$ when $Q = Q_{nc}$, where $\mathbb{H}^1 := \{ u \in \mathbb{H} \mid n(u) = 1 \}$. The map $K_{\infty}^0 \ni \begin{pmatrix} a & b \\ b & a \end{pmatrix} \mapsto (a + b, a - b) \in \mathbb{H}^1 \times \mathbb{H}^1$ gives rise to an isomorphism $K_{\infty}^0 \simeq \mathbb{H}^1 \times \mathbb{H}^1$.

We next put $G_{\infty}^{1*} := \left\{ g \in GL_4(\mathbb{R}) \mid t_g \begin{pmatrix} 0_2 & 1_2 \\ -1_2 & 0_2 \end{pmatrix} = \begin{pmatrix} 0_2 & 1_2 \\ -1_2 & 0_2 \end{pmatrix} \right\}$. Then

$$K_{\infty}^{1*} := \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \mid A + \sqrt{-1}B \in U(2) \right\}$$

is a maximal compact subgroup of $G_{\infty}^{1*}$, where $U(2) := \{ X \in M_2(\mathbb{C}) \mid ^t\overline{X}X = 1_2 \}$ denotes the unitary group of degree two. The map $K_{\infty}^{1*} \ni \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \mapsto A + \sqrt{-1}B \in U(2)$ induces an isomorphism $K_{\infty}^{1*} \simeq U(2)$.

Let us introduce maximal compact subgroups at non-archimedean places. We first deal with the case of $G = GSp(1,1)$ or $GSp^*(2)$. We remark that $GSp(1,1)$ and $GSp^*(2)$ are isomorphic to each other over $\mathbb{Q}_p$. We can thus identify $GSp(1,1)(\mathbb{Q}_p)$ with $GSp^*(2)(\mathbb{Q}_p)$.

We let $D$ be a divisor of $d_B$ and fix a maximal order $\mathfrak{O}$ of $B$. For $p|d_B$ let $\mathfrak{P}_p$ be the maximal ideal of the $p$-adic completion $\mathfrak{O}_p$ of $\mathfrak{O}$ and let

$$L_p := \begin{cases} t(\mathfrak{O}_p \oplus \mathfrak{O}_p) & (p \nmid d_B \text{ or } p|D), \\ t(\mathfrak{O}_p \oplus \mathfrak{P}_p^{-1}) & (p|d_B). \end{cases}$$

Then $K_p := \{ k \in G_p \mid kL_p = L_p \}$ is a maximal compact subgroup of $G_p$ for each finite prime $p$ when $G = GSp(1,1)$ or $GSp^*(2)$. Every maximal compact subgroup of $G_p$ is conjugate to some $K_p$ by $G_p$. 


Let us next deal with the case of $GSp(2)$. When $p$ does not divide $d_B$, we put $K'_p := GSp(2)(\mathbb{Z}_p)$. When $p | d_B$ we put 

\[ K'_p := \left\{ \begin{array}{c c c c} Z_p & Z_p & p^{-1}Z_p & Z_p \\
pZ_p & Z_p & Z_p & Z_p \\
pZ_p & pZ_p & Z_p & pZ_p \\
pZ_p & Z_p & Z_p & pZ_p \\
p^2Z_p & Z_p & Z_p & Z_p \\
p^2Z_p & p^2Z_p & Z_p & p^2Z_p \\
p^2Z_p & p^2Z_p & Z_p & p^2Z_p \end{array} \right\} \cap GSp(2)(\mathbb{Q}_p) (p | D). \]

We call this open compact subgroup of $GSp(2)(\mathbb{Q}_p)$ a paramodular subgroup of $GSp(2)(\mathbb{Q}_p)$ of level $p$ or $p^2$, which is maximal when the level is $p$. We remark that $K_p \cong K'_p$ for $p \nmid d_B$.

We note that we can identify $G_{nc}(\mathbb{A}_f)$ with $G_c(\mathbb{A}_f)$ since $G_{nc}$ is isomorphic to $G_c$ over $\mathbb{Q}_p$.

Every maximal compact subgroup of $G(\mathbb{A}_f) = G_{nc}(\mathbb{A}_f) = G_c(\mathbb{A}_f)$ is $G(A_f)$-conjugate to $K_f(D) := \prod_{p<\infty} K_p$ with $D | d_B$. In addition, we put $K'_f(D) := \prod_{p<\infty} K'_p$, which is an open compact subgroup of $G'(A_f)$.

2 Theta lifts to $GSp(1, 1)$, $GSp^*(2)$ and $GSp(2)$.

Let $H$ and $H'$ be $\mathbb{Q}$-algebraic groups defined by

\[ H(\mathbb{Q}) = GL_2(\mathbb{Q}), \quad H'(\mathbb{Q}) := B^\times \]

respectively. For a positive integer $\kappa$ we let $S_\kappa(D)$ be the space of elliptic cusp forms of weight $\kappa$ with level $D$ (cf. [M-N-2, Section 3.1]). For a non-negative integer $\kappa'$ we let $A_{\kappa'}$ be the space of automorphic forms of weight $\sigma_{\kappa'}$ with respect to $\prod_{p<\infty} \Omega_p^\times$ (cf. [M-N-2, Section 3.2]), where $\Omega_p^\times$ denotes the unit group of $\Omega_p$.

For Hecke eigenforms $(f, f') \in S_\kappa(D) \times A_{\kappa_2}$ let $\pi(f)$ be the automorphic representation of $GL_2(\mathbb{A})$ generated by $f$ and $JL(\pi(f'))$ be the Jacquet-Langlands lift of the automorphic representation $\pi(f')$ generated by $f'$. The Hecke equivariant isomorphism between $A_{\kappa_2}$ and the space of new forms in $S_{\kappa_2+2}(d_B)$ (Eichler [E-1], [E-2], Shimizu [Sh]) sends a Hecke eigenform $f'$ to a primitive form $JL(f')$. The automorphic representation $JL(\pi(f'))$ is nothing but that generated by $JL(f')$.

2.1 Theta lift to $G$

For every finite prime $p < \infty$ let $V_p$ be the space of locally constant compactly supported functions on $B_p^2 \times Q_p^2$. Let $S(\mathbb{H}^2)$ stand for the space of Schwartz functions on $\mathbb{H}^2$. When $G = G_{nc}$ (respectively $G = G_c$) we then introduce the space $V_\infty$ of smooth
functions \( \varphi \) on \( \mathbb{H}^2 \times \mathbb{R}^+ \) such that, for each fixed \( t \in \mathbb{R}^+ \), \( \mathbb{H}^2 \ni X \mapsto \varphi(X, t) \) belongs to \( S(\mathbb{H}^2) \otimes \text{End}(V_{\kappa_1 + \kappa_2} \otimes V_{\kappa_1 - \kappa_2}) \) for \( (\kappa_1, \kappa_2) \in (2\mathbb{Z}_{\geq 0})^2 \) with \( \kappa_1 \leq \kappa_2 \) (respectively \( S(\mathbb{H}^2) \otimes \text{End}(H_{\kappa_1 - 4}) \) for \( \kappa_1 \in 2\mathbb{Z}_{\geq 0} \) with \( \kappa_1 \geq 4 \)), where \( H_{\kappa_1 - 4} \) denotes the space of homogeneous harmonic polynomials of degree \( \kappa_1 - 4 \) on \( \mathbb{H}^2 \). We let \( \varphi_{0,p} \in \mathbb{V}_p \) be the characteristic function of \( L_p \times \mathbb{Z}_p^+ \).

Let \( G = G_{nc} \). For \( (\kappa_1, \kappa_2) \in (2\mathbb{Z}_{\geq 0})^2 \) with \( \kappa_1 \leq \kappa_2 \) we define \( \varphi_{0,\infty}^{nc} = \varphi_{0,\infty}^{nc, (\kappa, \kappa)} \in \mathbb{V}_\infty \) by

\[
\varphi_{0,\infty}^{nc}(X, t) := \begin{cases} 
\frac{t^{\frac{\kappa_2 + 3}{2}}}{\sigma_{\frac{\kappa_2 + 3}{2}}(X_1 + X_2)} \otimes \frac{\sigma_{\frac{\kappa_2 - 1}{2}}(X_1 - X_2)}{\varphi(X, t)} \exp(-2\pi t^t X^{-} X) & (t > 0), \\
0 & (t < 0).
\end{cases}
\]

Let \( G = G_c \). For \( \kappa_1 \in 2\mathbb{Z}_{\geq 0} \) with \( \kappa_1 \geq 4 \), following [Lo, Definition 6.1], we define \( \varphi_{0,\infty}^{c} = \varphi_{0,\infty}^{c, (\kappa, \kappa)} \in \mathbb{V}_\infty \) by

\[
\varphi_{0,\infty}^{c}(X, t) := \begin{cases} 
\frac{t^{\frac{\kappa_1 - 1}{2}}}{\exp(-2\pi t^t X^{-} X)} C(X) & (t > 0), \\
0 & (t < 0),
\end{cases}
\]

where \( C \) is the \( \text{Hom}(H_{\kappa_1 - 4}, H^*_{\kappa_1 - 4}) \simeq \text{End}(H_{\kappa_1 - 4}) \)-valued function on \( \mathbb{H}^2 \) defined by

\[
C(X)(h) := h(X) \quad (h \in \mathcal{H}_{\kappa_1 - 4}),
\]

where \( H^*_{\kappa_1 - 4} \) denotes the dual space of \( H_{\kappa_1 - 4} \).

Following [M-N-1, Section 3] we introduce a metaplectic representation \( r = \otimes_{v \leq \infty} r_v \) of \( G(\mathbb{A}) \times H(\mathbb{A}) \times H'(\mathbb{A}) \) on the restricted tensor product \( \mathbb{V} = \otimes_{v < \infty} \mathbb{V}_v \) with respect to \( \{ \varphi_{0,p} \}_{p \leq \infty} \). It is associated with the standard additive character \( \psi \) of \( \mathbb{A} \). For \( G = G_{nc} \) (respectively \( G = G_c \)) we define the \( \text{End}(V_{\kappa_1 + \kappa_2} \otimes V_{\kappa_1 - \kappa_2}) \)-valued theta function (respectively \( \text{End}(H_{\kappa_1 - 4}) \)-valued theta function) \( \theta_{\kappa_1, \kappa_2}(g, h, h') \) by

\[
\sum_{(X, t) \in \mathbb{B}^2 \times \mathbb{Q}^+} r(g, h, h') \varphi_{0}(X, t),
\]

where \( \varphi_0 := \prod_{v \leq \infty} \varphi_{0,v} \) with

\[
\varphi_{0,\infty} := \begin{cases} 
\varphi_{0,\infty}^{nc} & (G = G_{nc}), \\
\varphi_{0,\infty}^{c} & (G = G_c).
\end{cases}
\]

When \( G = G_{nc} \) (respectively \( G = G_c \)), for \( (\kappa_1, \kappa_2) \in (2\mathbb{Z}_{\geq 0})^2 \) with \( \kappa_1 \leq \kappa_2 \) (respectively with \( \kappa_1 \geq \kappa_2 \) and \( \kappa_1 \geq 4 \)), we consider the theta lift

\[
S_{\kappa_1}(D) \times A_{\kappa_2} \ni (f, f') \mapsto \mathcal{L}(f, f')(g)
\]

with

\[
\mathcal{L}(f, f')(g) := \int_{\mathbb{R}_+^2 (H \times H')(\mathbb{Q}) \backslash (H \times H')(\mathbb{A})} \overline{f(h)} \theta_{\kappa_1, \kappa_2}(g, h, h') f'(h') dh dh'.
\]
By an argument similar to the proof of [M-N-1, Theorem 4.1], we verify that this is convergent on any compact subset of \( G(\mathbb{A}) \) when \( G = G_{nc} \). On the other hand, when \( G = G_{c} \), the theta function \( \theta_{\kappa_{1}, \kappa_{2}}(g, h, h')f'(h') \) with a fixed \( (g, h') \) can be viewed as an elliptic modular form of weight \( \kappa_{1} \) and level \( D \) (cf. [He, Section 6]). The convergence of the integral is thus reduced to that of the Petersson inner product of an elliptic modular form and an elliptic cusp form.

**Theorem 2.1.** Let \( (\kappa_{1}, \kappa_{2}) \in (2\mathbb{Z}_{\geq 0})^{\mathbb{Z}_{2}} \).

1. The theta lift \( \mathcal{L}(f, f') \) defines an automorphic forms, more precisely, it is left-\( G(\mathbb{Q}) \)-invariant, right \( K_{f}(D) \)-invariant and right \( K_{\infty}^{0} \)-equivariant (respectively \( G_{\infty}^{1} \)-equivariant) with respect to the irreducible representation with highest weight \((\frac{\kappa_{2} - \kappa_{1}}{2}, \frac{\kappa_{1} + \kappa_{2}}{2})\) (respectively \((\frac{3\kappa_{1} + \kappa_{2}}{2} - 1, \frac{\kappa_{2} - \kappa_{1}}{2} - 1)\)) when \( G = G_{nc} \) (respectively \( G = G_{c} \)). Furthermore \( \mathcal{L}(f, f') \) has the trivial central character.

2. Suppose that \( (f, f') \) are Hecke eigenforms. Then \( \mathcal{L}(f, f') \) is also a Hecke eigenform. Furthermore, for each \( p | D \), let \( \epsilon_{p} \) (respectively \( \epsilon_{p}' \)) be the eigenvalue for the involutive action of \( \begin{pmatrix} 0 & 1 \\ -p & 0 \end{pmatrix} \) (resp. a prime element \( \varpi_{B, p} \in B_{p} \)) on \( f \) (resp. \( f' \)). Then \( \mathcal{L}(f, f') \equiv 0 \) unless \( \epsilon_{p} = \epsilon_{p}' \).

3. Assume furthermore that \( 1 < \kappa_{1} < \kappa_{2} + 2 \) when \( G = G_{nc} \) (respectively \( 1 < \kappa_{2} + 2 < \kappa_{1} \) when \( G = G_{c} \)). Then \( \mathcal{L}(f, f') \) is a cusp form on \( G_{nc}(\mathbb{A}) \) generating, at the archimedean place, the discrete series representation with Harish-Chandra parameter \( \lambda = (\frac{\kappa_{2} - \kappa_{1}}{2} + 1, \frac{\kappa_{1} + \kappa_{2}}{2}) \) (respectively automorphic forms on \( G_{c}(\mathbb{A}) \) generating, at the archimedean place, the discrete series representation with Harish-Chandra parameter \( \lambda = (\frac{3\kappa_{1} + \kappa_{2}}{2}, \frac{\kappa_{2} - \kappa_{1}}{2} - 1) \)) as a \((\mathfrak{g}, K_{\infty}^{0})\)-module, where \( \mathfrak{g} \) denotes the Lie algebra of \( G_{\infty}^{1} \).

Outline of proof:

1. The assertion is essentially due to [M-N-1, Section 4].
2. This follows from [M-N-1, Theorem 5.1] and [M-N-1, Remark 5.2 (ii)].
3. The fact that \( \mathcal{L}(f, f') \) is cuspidal when \( G = G_{nc} \) is shown in a manner similar to [M-N-2, Section 13.4]. To determine the representation type of \( \mathcal{L}(f, f') \) at the Archimedean place, we use the result by Li-Paul-Tan-Zhu [L-P-T-Z, Theorem 5.1] on the archimedean theta correspondence, in which \( \mathcal{L}(f, f') \) is involoved. When \( G = G_{c} \) this assertion then follows immediately. In view of the archimedean theta correspondence and the discrete decomposability of the cuspidal spectrum (cf. [G-G-P]), we thus see that, when \( G = G_{nc} \), the archimedean component of the \( G_{\infty}^{1}(\mathbb{A}) \)-module generated by \( \mathcal{L}(f, f')(gf^{*}) \) with any fixed \( gf \in G(A_{f}) \) is isomorphic to the discrete series representation in the statement as a \((\mathfrak{g}, K_{\infty}^{0})\)-module.

### 2.2 Theta lift to \( G' \)

We next consider the theta lift from \( S_{\kappa_{1}}(D_{1}) \times S_{\kappa_{2}}(D_{2}) \) to automorphic forms on \( G'(\mathbb{A}) \). As in [O], we formulate the lift using the metaplectic representation \( r' \) of \( G' \times H^{1} \) considered by Harris-Kudla [Ha-K] and Roberts [R], where \( H^{1} \) denotes the \( \mathbb{Q} \)-algebraic group defined by

\[ \{(h_{1}, h_{2}) \in GL_{2} \times GL_{2} \mid \det(h_{1}) = \det(h_{2})\}. \]
Now let us introduce a quadratic space \((M_2(\mathbb{Q}), \det)\) and note that the action of \(H^1(\mathbb{Q})\) on \(M_2(\mathbb{Q})\) defined by
\[
h \cdot X = h_1^{-1}Xh_2 \quad (X \in M_2(\mathbb{Q}), \ h = (h_1, h_2) \in H^1(\mathbb{Q}))
\]
induces a well-known isomorphism
\[
H_1(\mathbb{Q})/\{(z, z) \mid z \in \mathbb{Q}^\times\} \simeq GSO(2, 2)(\mathbb{Q}).
\]
We assume that \(r'\) is associated with the additive character \(\psi(\frac{1}{2}*)\) on \(\mathbb{A}\).

To construct the theta lift we now recall the choice of the Schwartz function on \(M_2(\mathbb{A})^{\oplus 2}\) in [O]. At a finite place \(v = p < \infty\), we let \(\varphi_{p}'\) be the Schwartz function on \(M_2(\mathbb{Q}_p)^{\oplus 2}\) given by the characteristic function of
\[
\left\{ \left( \begin{array}{cc} a_{x_1} & b_{x_1} \\ c_{x_1} & d_{x_1} \end{array} \right), \left( \begin{array}{cc} a_{x_2} & b_{x_2} \\ c_{x_2} & d_{x_2} \end{array} \right) \right\} \mid \begin{array}{c}
a_{x_1} \in D_2\mathbb{Z}_p, \\
b_{x_1} \in \mathbb{Z}_p, \\
c_{x_1} \in D_1D_2\mathbb{Z}_p, \\
d_{x_1} \in D_1\mathbb{Z}_p,
\end{array} \\
a_{x_2}, b_{x_2}, c_{x_2}, d_{x_2} \in \mathbb{Z}_p,
\]
For the choice of the Schwartz function at the archimedean place we need two functions \(P_1\) and \(P_2\) on \(M_2(\mathbb{R})\) defined as follows:
\[
P_1(X) := \text{tr}(X \begin{pmatrix} -\sqrt{-1} & -1 \\ -1 & \sqrt{-1} \end{pmatrix}), \quad P_2(X) := \text{tr}(X \begin{pmatrix} -\sqrt{-1} & 1 \\ -1 & -\sqrt{-1} \end{pmatrix}) \quad (X \in M_2(\mathbb{R}))
\]
Let \(\mathbb{C}[s_1, s_2]\) denote the polynomial ring of two variables \(s_1\) and \(s_2\) over \(\mathbb{C}\). As our choice of the test function at \(v = \infty\) we take the \(\mathbb{C}[s_1, s_2]\)-valued Schwartz function \(\varphi_{\infty, 0}\) on \(M_2(\mathbb{R})^{\oplus 2}\) as follows:
\[
\varphi_{\infty, 0}'(X_1, X_2) := \exp(-\pi \text{tr}(tX_1X_1 + tX_2X_2))P_1(s_1X_1 + s_2X_2)\Rightarrow_{\kappa + \kappa \ldots (\kappa_1 \leq \kappa_2)}^{\Lambda'}
\]
Put \(\varphi' := \otimes_{v \leq \infty} \varphi'_{v, 0}\) and define the theta series \(\theta'_{\kappa_1, \kappa_2}(g, h)\) on \(G'(\mathbb{A}) \times H^1(\mathbb{A})\) as
\[
\sum_{(X_1, X_2) \in M_2(\mathbb{Q})^{\oplus 2}} r'(g, h) \varphi'_{0}(X_1, X_2).
\]
We view \(f_1 \otimes f_2 := f_1f_2\) as an automorphic form on \(H^1(\mathbb{A})\) or \((H \times H)(\mathbb{A})\) for \((f_1, f_2) \in S_{\kappa_1}(D_1) \times S_{\kappa_2}(D_2)\). We embed \(\mathbb{A}^\times\) into \(H^1(\mathbb{A})\) by
\[
\mathbb{A}^\times \ni a \mapsto (a \cdot 1_2, a \cdot 1_2) \in H^1(\mathbb{A}).
\]
For \((\kappa_1, \kappa_2) \in (2\mathbb{Z}_{\geq 0})^{\oplus 2}\) we then define the theta lifting from \(S_{\kappa_1}(D_1) \times S_{\kappa_2}(D_2)\) to \(G'(\mathbb{A})\) by
\[
S_{\kappa_1}(D_1) \times S_{\kappa_2}(D_2) \ni (f_1, f_2) \mapsto \mathcal{L}'(f_1, f_2)(g),
\]
where \(\Lambda' = \frac{\kappa_1 + \kappa_2}{2}, -\frac{\kappa_1 - \kappa_2}{2}\) and
\[
\mathcal{L}'(f_1, f_2)(g) := \int_{\mathbb{A}^\times H^1(\mathbb{Q}) \setminus H^1(\mathbb{A})} \theta'_{\kappa_1, \kappa_2}(g, hh')f_1 \otimes f_2(hh')dh.
\]
with an invariant measure $dh$ on $A^\times H^1(Q) \backslash H^1(A)$. Here for each $g \in G'(A)$, we take $h' = (h'_1, h'_2) \in (H \times H)(A)$ so that $\nu'(g) = \det(h'_1) \det(h'_2)^{-1}$. We note that this theta lift does not depend on the choice of $h'$.

We now quote the following theorem (cf. [O]):

**Theorem 2.2.** For two non-zero primitive cusp forms $(f_1, f_2) \in S_{\kappa_1}(D_1) \times S_{\kappa_2}(D_2)$, $\mathcal{L}'(f_1, f_2)$ is a non-zero generic cusp form on $G'(A) = GSp(2)(A)$ with the trivial central character satisfying the following properties:

1. $\mathcal{L}'(f_1, f_2)$ is a paramodular form of level $D_1D_2$, namely, at a prime $p|N := D_1D_2$,
   
   $\nu'(g) = \det(h'_1) \det(h'_2)^{-1}$.

2. We note that this theta lift does not depend on the choice of $h'$.

We now quote the following theorem (cf. [O]):

$\mathcal{L}'(f_1, f_2)$ is a paramodular form of level $D_1D_2$, namely, at a prime $p|N := D_1D_2$,

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$\mathcal{L}'(f_1, f_2)$ is a paramodular form of level $D_1D_2$, namely, at a prime $p|N := D_1D_2$,

This theta lift does not depend on the choice of $h'$.
• $F$ is right $K_f(D)$-invariant and right $K^0_{\infty}$-equivariant with respect to the irreducible representation of highest weight $(\frac{\kappa_2 - \kappa_1}{2}, \frac{\kappa_1 + \kappa_2}{2})$,

• $F$ generates, as a $(\mathfrak{g}, K_{\infty}^0)$-module, the discrete series representation with Harish Chandra parameter $(\frac{\kappa_2 - \kappa_1 + 1}{2}, \frac{\kappa_1 + \kappa_2}{2})$ with $(\kappa_1, \kappa_2) \in 2\mathbb{Z}^\oplus 2$ such that $1 < \kappa_1 < \kappa_2 + 2$, where recall that $\mathfrak{g}$ denotes the Lie algebra of $G^1_{\infty}$ (cf. Theorem 2.1 (3)).

The definition is as follows:

$$L(F, \text{spin}, s) := \prod_{v \leq \infty} L_v(F, \text{spin}, s),$$

where

$$L_v(F, \text{spin}, s) := \begin{cases} Q_{F,p}(p^{-s})^{-1} & (v = p < \infty), \\ \Gamma_C(s + \frac{\kappa_1 - 1}{2})\Gamma_C(s + \frac{\kappa_2 + 1}{2}) & (v = \infty). \end{cases}$$

By virtue of Theorem 2.1 (3) we can use this definition for $F = \mathcal{L}(f, f')$ when $(f, f')$ are Hecke eigenforms.

We generalize [M-N-3, Proposition 2.9] to have the following:

**Proposition 3.1.** The spinor $L$-function for $\mathcal{L}(f, f')$ decomposes into

$$L(\mathcal{L}(f, f'), \text{spin}, s) = L(\pi(f), s)L(\pi(f'), s),$$

where $L(\pi(f), s)$ (resp. $L(\pi(f'), s)$) denotes the standard $L$-function of $\pi(f)$ (resp. $\pi(f')$).

Of course, we thus see that $L(\mathcal{L}(f, f'), \text{spin}, s)$ has the meromorphic continuation and satisfies the functional equation between $s$ and $1 - s$ since so do $L(\pi(f), s)$ and $L(\pi(f'), s)$.

We now recall that there is Novodvorsky's zeta integral of the spinor $L$-function for a generic cusp form on $G'(\mathbb{A})$ (cf. [No]). By means of the zeta integral, the theorem as follows (cf. [O]) describes the spinor $L$-function for a generic form $\mathcal{L}'(f_1, f_2)$.

**Theorem 3.2.** Let the notations be as in Theorem 2.2. Then the global spinor $L$-function of $\mathcal{L}'(f_1, f_2)$ decomposes into

$$L(\pi(f_1), s)L(\pi(f_2), s).$$

As an immediate consequence of Proposition 3.1 and this theorem we obtain the following:

**Corollary 3.3.** Let $f \in S_{\kappa_1}(D)$ be a primitive form and $f' \in A_{\kappa_2}$ be a Hecke eigenform. Then we have

$$L(\mathcal{L}(f, f'), \text{spin}, s) = L(\mathcal{L}'(f, f'), \text{spin}, s).$$
3.2 Automorphic representations generated by the theta lifts

We study locally and globally the representation \( \pi(\mathcal{L}(f, f')) \) of \( G(\mathbb{A}) = GSp(1,1)(\mathbb{A}) \) or \( GSp^*(2)(\mathbb{A}) \) generated by \( \mathcal{L}(f, f') \) (respectively the representation \( \pi(\mathcal{L}(f, JL(f'))) \) of \( G'(\mathbb{A}) = GSp(2)(\mathbb{A}) \) generated by \( L'(f, JL(f')) \)).

(1) The case of \( G \):

We first discuss the case of \( G \). We note that the Lie algebra of the group \( G_\infty/ZG_\infty \) is isomorphic to the Lie algebra \( g \) of \( G^1_\infty \). The group \( G_\infty/ZG_\infty \) is isomorphic to \( G^1_\infty \) when \( G = G_c \), but it is neither connected or isomorphic to \( G^1_\infty \) when \( G = G_{nc} \). For \( G = G_{nc} \) let \( K_{\infty} \) be a maximal compact subgroup of \( G_\infty/ZG_\infty \). We can regard \( K_{\infty}^0 \) as the connected component of the identity for \( K_{\infty} \). Take \( \sigma \) to be the matrix \( \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) \) \( \in G(\mathbb{R}) \). We can then identify \( K_{\infty} \) with \( K_{\infty}^0 \cup K_{\infty}^0 \sigma \). For \( (\kappa_1, \kappa_2) \in (2\mathbb{Z}_{\geq 0})^{\oplus 2} \) with \( 1 < \kappa_1 + 2 < \kappa_2 \) let \( \pi^{(\kappa_1, \kappa_2)}_{\infty} \) be the discrete series representation of \( G^1_\infty \) with Harish Chandra parameter \( \left( \frac{\kappa_2 - \kappa_1}{2} + 1, \frac{\kappa_1 + \kappa_2}{2} \right) \). Then we introduce another representation \( \pi^{(\kappa_1, \kappa_2)}_{\infty, \sigma} \) of \( G^1_\infty \) defined by

\[
\pi^{(\kappa_1, \kappa_2)}_{\infty, \sigma}(g) = \pi^{(\kappa_1, \kappa_2)}_{\infty}(sg\sigma^{-1}) \quad \forall g \in G^1_\infty.
\]

This is equivalent to the discrete series representation with Harish Chandra parameter \( \left( \frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_2 - \kappa_1}{2} + 1 \right) \), which is not isomorphic to \( \pi^{(\kappa_1, \kappa_2)}_{\infty} \). There is an irreducible \((g, K_\infty)\)-module \( V^{(\kappa_1, \kappa_2)}_{\infty} \) which is equivalent to \( \pi^{(\kappa_1, \kappa_2)}_{\infty} \oplus \pi^{(\kappa_1, \kappa_2)}_{\infty, \sigma} \) as \((g, K_{\infty}^0)\)-modules.

**Proposition 3.4.** Suppose that \( f \) and \( f' \) are Hecke eigenforms and that \( 1 < \kappa_1 + 2 < \kappa_2 \) for \( G = G_{nc} \) (respectively \( 1 < \kappa_2 + 2 < \kappa_1 \) for \( G = G_c \)). Then the representation \( \pi(\mathcal{L}(f, f')) \) of \( G(\mathbb{A}_\mathbb{Q}) \) is irreducible.

The point of proof is to use [N-P-S, Theorem 3.1]. We then reduce the global irreducibility to the local irreducibility at the archimedean place. When \( G = G_{nc} \), we can verify that the archimedean component of \( \pi(L(f, f')) \) is isomorphic to \( V^{(\kappa_1, \kappa_2)}_{\infty} \).

We can therefore decompose \( \pi(\mathcal{L}(f, f')) \) into the restricted tensor product \( \prod'_{\nu \leq \infty} \pi_{\nu} \) and are able to determine each local component \( \pi_{\nu} \). To state our result on it we need several notation.

For a primitive cusp form \( f \in S_{\kappa_1}(D) \) let \( \pi(f) \) be an irreducible cuspidal representation of \( GL_2(\mathbb{A}) \), which admits a decomposition into the restricted tensor product \( \pi(f) = \prod_{\nu \leq \infty} \pi(f)_{\nu} \). Then, for \( \nu = p \nmid D \), \( \pi(f)_{\nu} \) is an unramified principal series representation of \( GL_2(\mathbb{Q}_p) \). Let \( \chi_{f, p} \) denote the unramified character of \( \mathbb{Q}_p^\times \) which induces \( \pi(f)_{\nu} \).

For a Hecke eigenform \( f' \in A_{\kappa_2} \) let \( \pi(f') \) be the irreducible automorphic representation of \( H'(\mathbb{A}) \) generated by \( f' \), and let \( \pi(f') = \prod'_{\nu \leq \infty} \pi(f')_{\nu} \) be the decomposition into the restricted tensor product of local representations. When \( p \nmid d_B, \pi(f')_p \) is an unramified principal series representation of \( B^\times_p = GL_2(\mathbb{Q}_p) \). We let \( \chi_{f', p} \) be the unramified character of \( \mathbb{Q}_p^\times \) inducing \( \pi(f')_p \). When \( p|d_B, \pi(f')_p \) is a character of \( B^\times_p \) of order at most two. Thus we have

\[
\pi(f')_p = \delta_p \circ n
\]
with a character $\delta_p$ of $\mathbb{Q}_p^\times$ of order at most two, where recall that the notation $n$ stands for the reduced norm of $B$ (cf. Section 1.1). In view of Theorem 2.1 (2), $\delta_p(p) = \epsilon'_p = \epsilon_p$ is necessary for $p|D$ in order that $\mathcal{L}(f, f') \neq 0$.

Following the notation of the appendix, let $\nu$ be the $p$-adic absolute value of $\mathbb{Q}_p$ and let $\xi$ be the non-trivial unramified character of $\mathbb{Q}_p^\times$ of order two for $p|d_B$. We further note that, in the appendix, the notation $\chi_1B \times \sigma$ is used for the induced representation of $GSp(1,1)(\mathbb{Q}_p)$ defined by two quasi-character $\chi$ and $\sigma$ of $\mathbb{Q}_p^\times$ when $p|d_B$. On the other hand, with three unramified quasi-characters $\chi_1, \chi_2$ and $\sigma$ of $\mathbb{Q}_p^\times$, $\chi_1 \times \chi_2 \times \sigma$ denotes the unramified principal series representation of $GSp(2)(\mathbb{Q}_p)$, which is referred to as "type I" on the table of the appendix.

**Proposition 3.5.** Let the notation be as above.

(1) Let $v = p \mid d_B$. Then $\pi_p$ is an unramified principal series representation of $\mathcal{L}(f, JL(f')) \simeq \sigma$, where $\sigma = \chi_{f, p}$ and $\chi = \chi_{f, p}^{-1} \times \xi$. For each finite place $v = p$, $v \mid d_B$, $\pi_p$ is isomorphic to the irreducible representation of $GSp(1,1)(\mathbb{Q}_p)$ given by $(\chi^{-1}_{f, p} \times \chi^{-1}_{f, p})$.

(2) Let $v = p \not\mid d_B$. Then $\pi_p$ is isomorphic to the irreducible representation of $GSp(1,1)(\mathbb{Q}_p)$ given by $(\chi^{-1}_{f, p} \times \chi^{-1}_{f, p})$.

(3) When $v = \infty$ and $G = G_{nc}$, $\pi_{\infty}$ is isomorphic to $\mathbb{Q}_p^\times$. When $v = \infty$ and $G = G_{nc}$, $\pi_{\infty}$ is isomorphic to $\mathbb{Q}_p^\times$. When $v = \infty$ and $G = G_{nc}$, $\pi_{\infty}$ is isomorphic to $\mathbb{Q}_p^\times$.

The archimedean component of $\pi(\mathcal{L}(f, f'))$ is already determined in the proof of Proposition 3.4. It thus suffices to consider the non-archimedean components. For every finite prime $p$, $\pi_p$ is a spherical representation of $G_p = GSp(1,1)(\mathbb{Q}_p)$ or $GSp(2)(\mathbb{Q}_p)$ (cf. [C]). As we see in [C], $\pi_p$ is uniquely determined by the Hecke eigenvalues. We calculate Hecke eigenvalues of $\mathcal{L}(f, f')$ explicitly in terms of eigenvalues for $(f, f')$ to obtain the assertion.

**The case of $G'$:**

We next deal with the automorphic representation $\pi(\mathcal{L}(f, JL(f'))) \simeq GSp(2)(\mathbb{A})$ generated by $\mathcal{L}(f, JL(f'))$. According to [R, Theorem 8.3], $\pi'(f, JL(f'))$ is an irreducible cuspidal representation. It thereby admits a decomposition into the restricted tensor product $\pi(\mathcal{L}(f, JL(f'))) = \prod_{\nu \in \infty} \pi'_\nu$. For each finite prime $v = p$, $\pi_p$ is involved in the local theta correspondence for $GSO(2,2)(\mathbb{Q}_p)$ and $GSp(2)(\mathbb{Q}_p)$, which is explicitly described in Gant-Takeda [G-T-2]. To describe each $\pi_p$ we use the notation of the appendix. To describe the archimedean component $\pi'_\infty$, we need to introduce, for two even integers $(\kappa_1, \kappa_2)$ with $1 < \kappa_1 + 2 < \kappa_2$, the irreducible admissible representation $V'_{\kappa_1\kappa_2}$ of $GSp(2)(\mathbb{R})$ whose restriction to $Sp(2)(\mathbb{R})$ is the direct sum of the two large discrete series representation of $Sp(2)(\mathbb{R})$ with Harish Chandra parameters $(\kappa_1 + \kappa_2, -\kappa_2 - 1)$ and $(\kappa_2 - 1, -\kappa_2 - 1)$.

**Proposition 3.6.** Let the notation be as above.

(1) Let $v = p \not\mid d_B$. Then $\pi'_p$ is isomorphic to $\pi_p$, namely an unramified principal series representation of $GSp(1,1)(\mathbb{Q}_p) \simeq GSp^{*}(2)(\mathbb{Q}_p) \simeq GSp(2)(\mathbb{Q}_p)$ given by $(\chi^{-1}_{f, p} \times \chi^{-1}_{f, p})$.
(2) Let $v = p |d_B|$. When $v = p |d_B|$, $\pi'_p$ is isomorphic to the irreducible representation of $GSp(2)(\mathbb{Q}_p)$ of type $I_{II}$ with $\sigma = \chi_{f,p}$ and $\chi = \chi_{f,p}^{-1} \cdot \delta_p$. When $v = p |D|$ and $\delta_p$ is trivial (respectively non-trivial), $\pi'_p$ is isomorphic to the irreducible representation of $GSp(2)(\mathbb{Q}_p)$ of type $V_0$ with $\sigma = \xi$ (respectively $\sigma = 1$), where, for the representations of $GSp(2)(\mathbb{Q}_p)$ of type $I_{II}$ and $V_0$, see the appendix.

(3) When $v = \infty$, $\pi'_\infty$ is isomorphic to $V^{(\kappa_1, \kappa_2)}$.

Using Przebinda [Prz], the representation $\pi'_\infty$ at the infinite prime $v = \infty$ is determined by the same reasoning as in the case of $GSp(1,1)(\mathbb{R})$. The representation $\pi_p$ is included in the table 2 of Section 14 or Theorem 8.2 (iv), (v), (vi) of Gan-Takeda [G-T]. Then, looking also at the table of the appendix, we have the assertion on $\pi_p$.

### 3.3 Conjecture and conclusion

Let $\mathcal{A}_G$ and $\mathcal{A}_{G'}$ denote the equivalence classes of irreducible automorphic representations of $G(\mathbb{A})$ and $G'(\mathbb{A})$ respectively. We note that the $L$-group $L^G$ of $G$ is the same as the $L$-group $L^{G'}$ of $G'$, where $L^G = L^{G'}$ is the direct product of $GSp(2)(\mathbb{C})$ and the Weil group of $\mathbb{Q}$ (for the notion of $L$-group see [La] and [B] et al). As the choice of the $L$-morphism between $L^G$ and $L^{G'}$ we can take the identity map. The Langlands principle of functoriality predicts the following:

**Conjecture 3.7 (Langlands).** The $L$-morphism induced by the identity map would give rise to a natural transfer from $\mathcal{A}_G$ to $\mathcal{A}_{G'}$ which preserves $L$-functions, namely an $L$-function of an irreducible automorphic representation of $G(\mathbb{A})$ is one of some irreducible automorphic representation of $G'(\mathbb{A})$.

Let us now introduce

$$\mathcal{A}_G(K_f(D)) := \{ \pi = \prod_{v \leq \infty} \pi_v \in \mathcal{A}_G \mid \pi_p \text{ has a } K_p\text{-fixed vector for } v = p < \infty \},$$

$$\mathcal{A}_{G'}(K'_f(D)) := \{ \pi' = \prod_{v \leq \infty} \pi'_v \in \mathcal{A}_{G'} \mid \pi'_p \text{ has a } K'_p\text{-fixed vector for } v = p < \infty \},$$

where see Section 1.2 for $K_f(D)$ and $K'_f(D)$.

Based on the observation by R. Schmidt including the table of irreducible admissible representations of $G(\mathbb{Q}_p) = G_{nc}(\mathbb{Q}_p) = G_{c}(\mathbb{Q}_p)$ and $G'(\mathbb{Q}_p) = G_{s}(\mathbb{Q}_p)$ in the appendix (see also [RS, Section A.8]), we can formulate the conjecture as follows:

**Conjecture 3.8.** The above transfer would map $\mathcal{A}_G(K_f(D))$ into $\mathcal{A}_{G'}(K'_f(D))$ and an $L$-function of $\pi \in \mathcal{A}_G(K_f(D))$ is one of some $\pi' \in \mathcal{A}_{G'}(K'_f(D)).$

We remark that this was first pointed out by Ibukiyama [I] for the case of $G = G_c$ and $D = 1$. As a consequence of Corollary 3.3, Propositions 3.5 and 3.6 we have known that our theta lifts $\mathcal{L}(f, f')$ and $\mathcal{L}'(f, JL(f'))$ provide evidence of Conjecture 3.8. We state it as follows:
Theorem 3.9. Suppose that two even integers \((\kappa_1, \kappa_2)\) satisfy \(1 < \kappa_1 + 2 < \kappa_2\) when \(G = G_{nc}\) (respectively \(1 < \kappa_2 + 2 < \kappa_1\) when \(G = G_c\)). For any given primitive form \(f \in S_{\kappa_1}(D)\) and Hecke eigenform \(f' \in A_{\kappa_2}\), the map

\[ \mathcal{A}_G(K_f(D)) \ni \pi(L(f, f')) \mapsto \pi(L'(f, JL(f'))) \in \mathcal{A}_{G'}(K_f'(D)) \]

preserves the coincidence of the global spinor \(L\)-functions and is compatible with the non-archimedean local Jacquet-Langlands correspondence for \(G\) and \(G' = GSp(2)\) (cf. Appendix). Namely, this map satisfies the expected properties of the transfer in the conjecture.

A Appendix: The spherical representations of \(GSp(1,1)\) and local Langlands parameters for \(GSp(4)\) (by Ralf Schmidt)

Let \(F\) be a non-archimedean local field of characteristic zero. Let \(B\) be the non-split quaternion algebra over \(F\), and let \(x \mapsto \bar{x}\) be its standard involution. We consider \(GSp(1,1)\) and \(GSp(4)\) (or \(GSp(2)\)) over \(F\). Let \(o_B\) be a maximal order in \(B(F)\), and let \(p_B\) be the unique maximal ideal of \(o_B\). Let

\[
K_1 = \{ g \in GSp(1,1)(F) \cap \left[ \begin{array}{cc} o_B & o_B \\ o_B & o_B \end{array} \right] : \nu(g) \in o^\times \},
\]

\[
K_2 = \{ g \in GSp(1,1)(F) \cap \left[ \begin{array}{cc} o_B & p_B \\ p_B^{-1} & o_B \end{array} \right] : \nu(g) \in o^\times \}.
\]

We remark that these groups \(K_1\) and \(K_2\) are maximal compact subgroups of \(GSp(1,1)(F)\), and every maximal compact subgroup is conjugate to either \(K_1\) or \(K_2\).

The following table lists all irreducible, admissible representations of \(GSp(1,1)(F)\) which are constituents of representations of the form \(\chi 1_{B^\times} \rtimes \sigma\), where \(\chi\) and \(\sigma\) are characters of \(F^\times\). The table also lists all the irreducible, admissible representations of \(GSp(4,F)\) supported in the Borel subgroup, using the notations and classification scheme of [R-S]. Representations with the same \(L\)-parameter \(W_F \rightarrow GSp(4,C)\) appear in the same row; this is nothing but the Langlands functorial transfer from \(GSp(1,1)\) to \(GSp(4)\) coming from the natural inclusion of dual groups. The actual \(L\)-parameters can be found in Table A.7 of [R-S].

The columns labeled \(K_1\) and \(K_2\) indicate, in the case when the inducing characters are unramified, the dimension of the space of \(K_1\) resp. \(K_2\) invariant vectors in a representation of \(G(F)\).
The notation $\nu$ stands for the valuation of $F$. For the IIa type representation, $\chi$ is such that $\chi^2 \neq \nu^{\pm 1}$ and $\chi \neq \nu^{\pm 3/2}$. For the representations in group V, the character $\xi$ is assumed to be non-trivial and quadratic.

<table>
<thead>
<tr>
<th></th>
<th>GSp(1,1)</th>
<th>GSp(4)</th>
<th>$K_1$</th>
<th>$K_2$</th>
</tr>
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<tbody>
<tr>
<td>I</td>
<td>—</td>
<td>$\chi_1 \times \chi_2 \times \sigma$ (irreducible)</td>
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<td></td>
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<tr>
<td>II</td>
<td>$a\chi_1^{B_x} \times \sigma$</td>
<td>$\chi \text{St}_{\text{GL}(2)} \times \sigma$</td>
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<td>1</td>
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<tr>
<td></td>
<td>$b$</td>
<td>$\chi_1 \text{St}_{\text{GL}(2)} \times \sigma$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>III</td>
<td>$a$</td>
<td>$\chi \times \sigma \text{St}_{\text{GSp}(2)}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$b$</td>
<td>$\chi \times \sigma_1 \text{GSp}(2)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>IV</td>
<td>$a\sigma \text{St}_{\text{GSp}(1,1)}$</td>
<td>$\sigma \text{St}_{\text{GSp}(4)}$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$b$</td>
<td>$L(\nu^2, \nu^{-1} \sigma \text{St}_{\text{GSp}(2)})$</td>
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<td></td>
</tr>
<tr>
<td></td>
<td>$c\sigma_1 \text{GSp}(1,1)$</td>
<td>$L(\nu^{3/2} \text{St}_{\text{GL}(2)}, \nu^{-3/2} \sigma)$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>V</td>
<td>$a\delta(\nu^{1/2} \xi_1 B_x, \nu^{-1/2} \sigma)$</td>
<td>$\delta([\xi, \nu \xi], \nu^{-1/2} \sigma)$</td>
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<td>0</td>
</tr>
<tr>
<td></td>
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<td>$L(\nu^{1/2} \xi_1 \text{St}_{\text{GL}(2)}, \nu^{-1/2} \sigma)$</td>
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<td>1</td>
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<tr>
<td></td>
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<td>$L(\nu^{1/2} \xi_1 \text{St}_{\text{GL}(2)}, \xi \nu^{-1/2} \sigma)$</td>
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<td>1</td>
</tr>
<tr>
<td></td>
<td>$d$</td>
<td>$L(\nu, \nu \xi, \xi \times \nu^{-1/2} \sigma)$</td>
<td></td>
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</tr>
<tr>
<td>VI</td>
<td>$a$</td>
<td>$\tau(S, \nu^{-1/2} \sigma)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$b$</td>
<td>$\tau(T, \nu^{-1/2} \sigma)$</td>
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<tr>
<td></td>
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<td>$L(\nu^{1/2} \text{St}_{\text{GL}(2)}, \nu^{-1/2} \sigma)$</td>
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<td>$d$</td>
<td>$L(\nu, 1 F_x \times \nu^{-1/2} \sigma)$</td>
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