

Isentropic Gas Flow in a Laval Nozzle *–Physical Phenomena of Steady Flow and Time* *Global Existence of Solutions–*

Naoki Tsuge*
Gifu University

Dedicated to Professor Fumioki Asakura on his 60th birthday

Abstract

In this paper, we consider the motion of isentropic gas in the Laval nozzle.

First, we introduce the physical phenomena of steady flow. In particular, we investigate two questions: (1) Why does the rocket engine have convergent-divergent form? (2) Why is the speed of the solar wind large? Considering these questions, we study the properties of the Laval nozzle.

Next, we consider the unsteady flow in the Laval nozzle. Our goal is to prove the time global existence of solutions. Then, the most difficult point is to obtain the bounded estimate of solutions. To solve this, we introduce an invariant region.

1 Introduction

We are concerned with a convergent-divergent nozzle, which is pinched in the middle, making an hourglass-shape (see Figure 1). This is called the *Laval nozzle*. We assume the cross section is a circle and the center lies on a line. Let the cross section at x be $A(x)$. Moreover, we call the left and right of the nozzle the *entry* and *exhaust* section, respectively.

In engineering, it is used for the exhaust port of the engines for rockets and supersonic jet airplanes. On the other hand, the motion of gas in the Laval nozzle is closely relative to that of solar wind, the stream of plasma ejected from the corona of the sun.

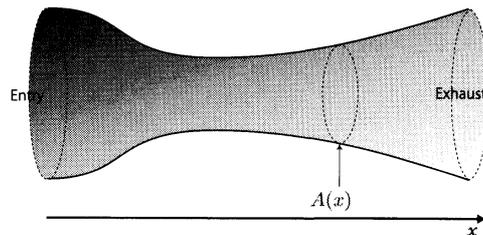


Figure 1: Laval Nozzle

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The Laval nozzle is essential for rockets and supersonic jet airplanes, because the Laval nozzle can accelerate a subsonic gas flow to a supersonic one (see [2]). As a result, the engine equipped with the Laval nozzle has maximal thrust.

2 Steady Solution

The steady flow of the inviscid and compressible gas through a nozzle is governed by the following equation.

$$\begin{cases} m' = -\frac{A'(x)}{A(x)}m, \\ \left(\frac{m^2}{\rho} + p(\rho)\right)' = -\frac{A'(x)}{A(x)}\frac{m^2}{\rho}, \end{cases} \quad x \in \mathbf{R}, \quad ' = \frac{d}{dx}, \quad (2.1)$$

where ρ , m and p are the density, the momentum and the pressure of the gas, respectively. If $\rho > 0$, $v = m/\rho$ represents the velocity. For a barotropic gas, $p(\rho) = \rho^\gamma/\gamma$, where $\gamma \in (1, 5/3]$ is the adiabatic exponent for usual gases.

In this section, we consider the only gas which flows from left to right. Therefore, every velocity, momentum and Mach number are positive in this section.

From (2.1), we obtain the mass flow conservation

$$A(x)m = A(x)\rho v = C_1 \quad (2.2)$$

and the Bernoulli equation

$$\frac{v^2}{2} + \frac{\rho^{\gamma-1}}{\gamma-1} = C_2, \quad (2.3)$$

where C_1 and C_2 are constants.

Next, we introduce some physical terms. We define the sonic speed by $c = \sqrt{p'(\rho)} = \rho^{\frac{\gamma-1}{2}}$. A steady flow is called *subsonic*, *sonic*, or *supersonic* as the speed of gas v is less than, equal to, or greater than the sound speed, i.e., the Mach number

$$M = \frac{v}{c}$$

is less than, equal to, or greater than one. The *thrust* T is defined by

$$T = A(x)\rho v^2. \quad (2.4)$$

The thrust is the force that produced by an engine to push a plane and rocket forward.

2.1 Laval Nozzle

The Laval nozzle is invented by Gustaf de Laval. He was a Swedish engineer. He first researched such a convergent-divergent nozzle. In addition, he made important contribution to dairy machine. In particular, he is famous for inventing milk-cream separator in Europe.

In this subsection, we investigate why the exhaust port of the rocket engine has the convergent-divergent form.

In the rocket engine, the oxygen (oxidizing agent) and hydrogen (fuel) are mixed and burned in a combustion chamber. The resultant gas flows through the Laval nozzle, which constitutes the exhaust port. We study the motion of gas in this nozzle in particular.

We assume that $A(x)$ satisfies the following.

$$\begin{cases} A'(x) \leq 0, & x < 0, \\ A'(x) \geq 0, & x \geq 0, \\ A'(0) = 0. \end{cases} \quad (2.5)$$

Notice that $A(x)$ is minimum at the origin. We call the origin *throat*. Moreover, let the entry section be x_0 and the Mach number at x_0 be M_0 .

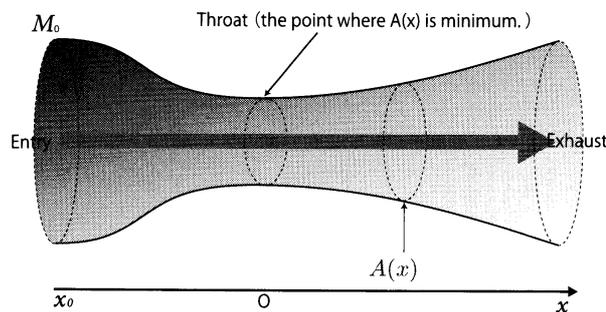


Figure 2: Laval Nozzle

If the Mach number of the exhaust gas at the entry section is lower than one (i.e. subsonic $M_0 < 1$), our goal is to make the thrust (of the exhaust gas at the exhaust section) T maximum.

It follows from (2.2) and (2.3) that

$$T = A(x)\rho v^2 = C_1\sqrt{(\gamma-1)}C_2\frac{1}{\sqrt{\frac{\gamma-1}{2} + \frac{1}{M^2}}}. \quad (2.6)$$

We find that T is the increasing function of M . Therefore, we devote to investigating the behavior of M .

From (2.1), we obtain the following

$$F(M) = \frac{A(x_0)}{A(x)}F(M_0), \quad (2.7)$$

where

$$F(M) = M \left(\frac{\frac{\gamma+1}{2}}{1 + \frac{\gamma-1}{2}M^2} \right)^{\frac{\gamma+1}{2(\gamma-1)}}.$$

Then we find that

$$F(0) = 0, F(1) = 1, \begin{cases} F'(M) \geq 0, & 0 \leq M < 1, \\ F'(M) \leq 0, & M \geq 1, \end{cases} \quad F(M) \rightarrow 0 \quad (M \rightarrow \infty).$$

In the view of (2.7), there only exists a solution, if the following holds

$$\frac{A(x_0)}{A(0)} F(M_0) \leq 1.$$

Varying M_0 , we consider the case where the equal sign is valid.

$$\frac{A(x_0)}{A(0)} F(M_0) = 1.$$

We denote M_0 in this case by \hat{M}_0 . We call \hat{M}_0 the *critical value* of the Mach number.

We investigate the increase and decrease of the Mach number M .

(I) Noncritical Case

We first consider the case where

$$A(x_0)F(M_0)/A(0) < 1.$$

(I) (i) $x_0 \leq x \leq 0$

In this case, since $A(x)$ is decreasing, the right hand side of (2.7) is increasing. Therefore, we find that M is increasing (see Figure 3).

(I) (ii) $x > 0$

In this case, since $A(x)$ is increasing in the region, the right hand side of (2.7) is decreasing. Therefore, we find M is decreasing (see Figure 4).

As a consequence, the graph of M is as follows.

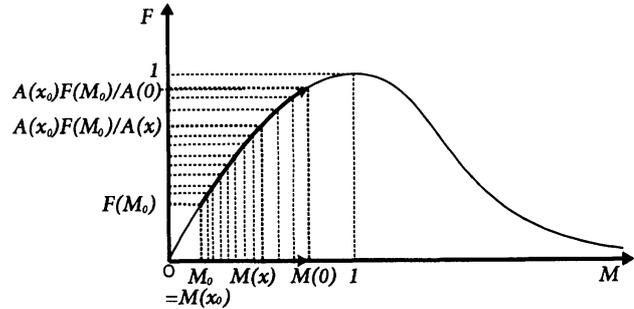


Figure 3: Case (I) (i)

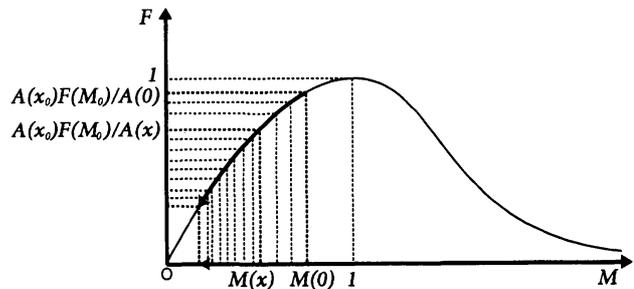


Figure 4: Case (I) (ii)

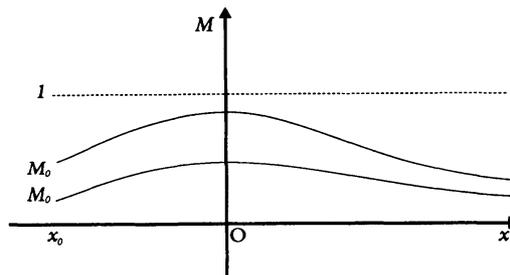


Figure 5: Graph of the Mach Number in Case (I)

(II) Critical Case

We next consider the case where

$$\frac{A(x_0)}{A(0)} F(\hat{M}_0) = 1.$$

(II) (i) $x_0 \leq x \leq 0$

In this case, the situation is similar to Case (I) (i). Notice that the right hand side at the origin is 1, the top of the graph (see Figure 6). This fact yields a bifurcation on the region where $x > 0$.

(II) (ii) $x > 0$

The first solution is the almost same as the previous case (see Figure 7).

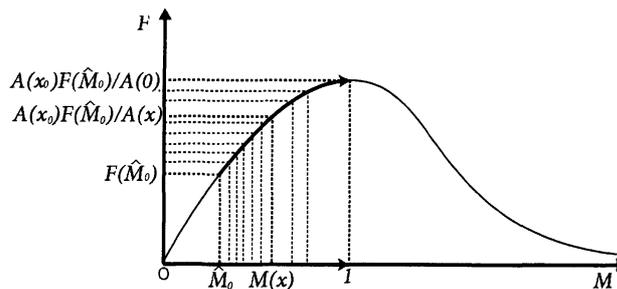


Figure 6: (II) (i)

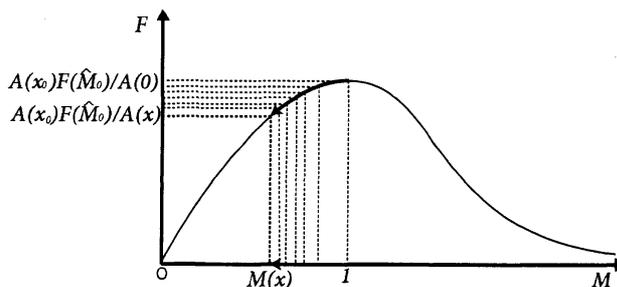


Figure 7: (II) (ii) Solution I

However, another solution arises in the critical case. Choosing intersections on the opposite side, we can find another solution. In this case, M is increasing (see Figure 8).

The graph of the above solution is as follows. Passing through the throat, this solution becomes supersonic. This is an important solution, *subsonic-supersonic flow*.

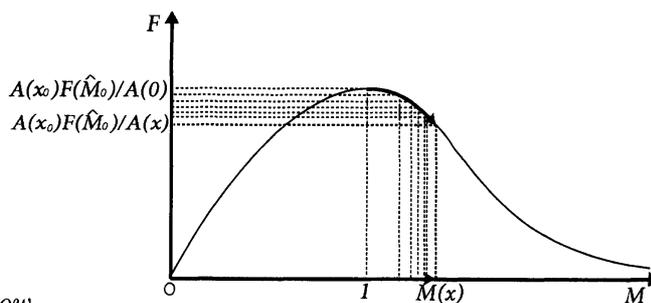


Figure 8: (II) (ii) Solution II

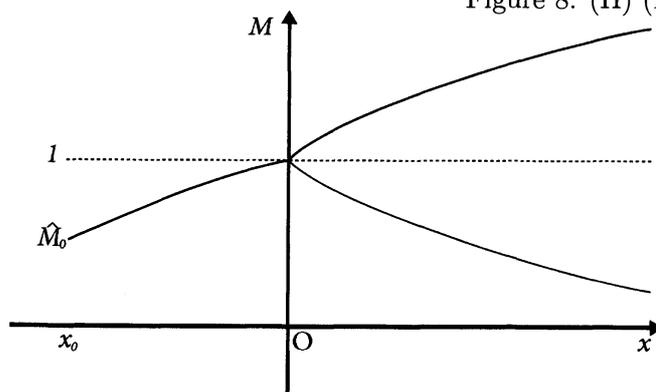


Figure 9: Graph of the Mach Number in Case (II)

By using the Laval nozzle, we can obtain the supersonic gas ($M > 1$) from the subsonic gas ($M < 1$). As a result, the gas has the maximal thrust at the exhaust section. Therefore, the Laval nozzle is essential for the rocket and the supersonic jet plane.

2.2 Solar Wind

The solar wind is the stream of the plasma ejected from the corona of the sun. It consists of electrons and positive ions. The solar wind collides with the magnetosphere of the earth. As a result, influencing its magnetic field, the solar wind causes the outbreak of auroras and the electromagnetic interference. Although these phenomena are well-known, we pay attention to another one. That is, the speed of the solar wind near the orbit of the earth is $300 \sim 700$ km/s. In this subsection, we investigate why the speed of the solar wind is so large.

We consider the spherically symmetric flow around the sun. We denote the radius of the sun by r_0 . Let the distance from the center of the sun be r .

For simplicity, we consider the isothermal case ($\gamma = 1$). The motion of the solar wind is governed by the following equation.

$$\begin{cases} m' = -\frac{2}{r}m, \\ \left(\frac{m^2}{\rho} + \rho\right)' = -\frac{2m^2}{r\rho} - \frac{GM_s}{r^2}\rho, \quad r \geq r_0, \quad ' = \frac{d}{dr}, \end{cases} \quad (2.8)$$

where G is the gravitational constant and M_s is the mass of the sun. Since $c = 1$ in this case, we find $v = M$.

From (2.8), we have the following.

$$(F_s(M)A_s(r))' = 0, \quad (2.9)$$

where

$$F_s(M) = Me^{-\frac{1}{2}M^2}, \quad A_s(r) = r^2 e^{\frac{GM_s}{r}}.$$

$F_s(M)$ and $A_s(r)$ satisfy the following.

$$\begin{cases} F_s'(M) \geq 0, & 0 \leq M \leq 1, \\ F_s'(M) \leq 0, & 1 < M, \\ F_s(0) = 0, \\ F_s(M) \rightarrow 0 & (M \rightarrow \infty), \end{cases} \quad \begin{cases} A_s'(r) \leq 0, & r_0 \leq r < GM_s/2, \\ A_s'(r) \geq 0, & r \geq GM_s/2, \\ A_s'(GM_s/2) = 0. \end{cases} \quad (2.10)$$

Their decrease and increase is the same as $F(M)$ and $A(x)$ in (2.7).

Therefore, (2.8) has the same structure as the Laval nozzle. This implies that the motion of solar wind is the same as that of the gas in (2.1). As a result, the solar wind can be supersonic.

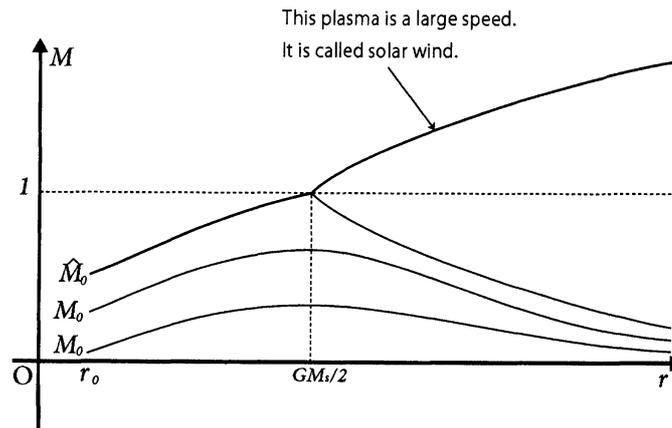


Figure 10: Graph of Solar Wind

3 Unsteady Flow

3.1 Relative Results and the Main Theorem

The motion of isentropic gas flow through a nozzle is governed by the following compressible Euler equations.

$$\begin{cases} \rho_t + m_x = a(x)m, \\ m_t + \left(\frac{m^2}{\rho} + p(\rho) \right)_x = a(x) \frac{m^2}{\rho}, \end{cases} \quad x \in \mathbf{R}, \quad (3.1)$$

where the given function $a(x)$ is represented by

$$a(x) = -A'(x)/A(x), \quad A(x) = e^{-\int^x a(y)dy},$$

where $A \in C^2(\mathbf{R})$ is a slowly varying cross section at x in the nozzle. Since we are devoted to the Laval nozzle, we assume that $A(x)$ satisfies (2.5).

We then consider the Cauchy problem (3.1) with initial data

$$(\rho, m)|_{t=0} = (\rho_0(x), m_0(x)) = (\rho_0(x), \rho_0(x)v_0(x)). \quad (3.2)$$

We survey the relative results for the Cauchy problem (3.1)–(3.2).

LIU [3] is a classical result celebrated for inhomogeneous conservation laws including (3.1). In this result, the global existence of solutions is proved by the Glimm scheme, provided that

- (C1) the total variation of initial data is small enough,
- (C2) initial data are away from sonic states,
- (C3) inhomogeneous terms are small enough in L^1 and L^∞ .

However, as mentioned above, the important subsonic-supersonic flow attains the sonic state at the throat. In addition, since gas turns from subsonic to supersonic, the value changes a great deal. Therefore, the condition (C1) and (C2) are restrictive for physics and engineering.

To state our main theorem, we define the Riemann invariants.

$$z = \frac{m}{\rho} - \frac{\rho^\theta}{\theta} = v - \frac{\rho^\theta}{\theta}, \quad w = \frac{m}{\rho} + \frac{\rho^\theta}{\theta} = v + \frac{\rho^\theta}{\theta} \quad \left(\theta = \frac{\gamma-1}{2} \right). \quad (3.3)$$

These Riemann invariants satisfy the following.

Remark 1.

$$|w| \geq |z|, \quad w \geq 0, \quad \text{when } v \geq 0. \quad |w| \leq |z|, \quad z \leq 0, \quad \text{when } v \leq 0. \quad (3.4)$$

$$v = \frac{w+z}{2}, \quad \rho = \left(\frac{\theta(w-z)}{2} \right)^{1/\theta}, \quad m = \rho v. \quad (3.5)$$

From the above, the lower bound of z and the upper bound of w yield the bound of ρ and $|v|$.

Then, the main theorem is as follows.

Theorem 1 ([6]). *We assume that, for any fixed positive constant M , initial density and velocity data $(\rho_0, v_0) \in L^\infty(\mathbf{R})$ satisfy*

$$\begin{aligned} \rho_0(x) \geq 0, \quad -M\{A_-(x)\}^{-\frac{\gamma-1}{\gamma+1}} \leq z_0(x) := v_0(x) - \frac{\{\rho_0(x)\}^\theta}{\theta}, \\ w_0(x) := v_0(x) + \frac{\{\rho_0(x)\}^\theta}{\theta} \leq M\{A_+(x)\}^{-\frac{\gamma-1}{\gamma+1}}. \end{aligned} \quad (3.6)$$

Then the Cauchy problem (3.1)–(3.2) has a global entropy weak solution, where

$$A_-(x) = \begin{cases} A(x), & x \geq 0, \\ A(0), & x < 0, \end{cases} \quad A_+(x) = \begin{cases} A(0), & x \geq 0, \\ A(x), & x < 0. \end{cases}$$

Remark 2. We make two remarks for the condition (3.6).

- If $A(x)$ is uniformly bounded, (3.6) implies that we can give arbitrary L^∞ data.
- Solutions of Theorem 1 satisfy

$$\begin{aligned} 0 \leq \rho(x, t), \quad -M\{A_-(x)\}^{-\frac{\gamma-1}{\gamma+1}} \leq z(x, t) := v(x, t) - \frac{\{\rho(x, t)\}^\theta}{\theta}, \\ w(x, t) := v(x, t) + \frac{\{\rho(x, t)\}^\theta}{\theta} \leq M\{A_+(x)\}^{-\frac{\gamma-1}{\gamma+1}}. \end{aligned}$$

That is, the region

$$\Delta_x = \left\{ (\rho, v); \rho \geq 0, -M\{A_-(x)\}^{-\frac{\gamma-1}{\gamma+1}} \leq z, w \leq M\{A_+(x)\}^{-\frac{\gamma-1}{\gamma+1}} \right\} \quad (3.7)$$

is the invariant region for solutions of Theorem 1. The region depends on the space variable x and is closely related to the geometric structure of the source term $A(x)$. These points are quite different from those of Chueh, Conley and Smoller's invariant region theory (see [1]).

The most difficult point in the proof is to obtain the bounded estimate of solutions. For the one-dimensional case (i.e. $A'(x) = 0$), the invariant regions [1] are standard methods to obtain the estimate. However, we cannot apply their method to the nozzle flow (3.1). In TSUGE [5], the global existence of solutions is proved for the case $A(x) = x^2$ (spherical symmetry). If $A(x)$ is monotone decreasing or increasing, we can apply the method [5] to the present problem. In fact, LU [4] applies this method to (3.1) with a monotone $A(x)$. In this paper, we improve the method and apply it to the Laval nozzle. The main point is to prove that (3.7) is an invariant region for the Cauchy problem (3.1)–(3.2). This yields the desired bounded estimate of solutions.

3.2 Outline of the Proof (Formal Argument)

In this subsection, by formal argument, we deduce the bounded estimate of solutions.

If (3.1) has a smooth solution, we can diagonalize (3.1) into

$$\begin{aligned} z_t + \lambda_1 z_x &= -a(x)\rho^\theta v, \\ w_t + \lambda_2 w_x &= a(x)\rho^\theta v. \end{aligned} \quad (3.8)$$

Remark 3. (3.1) has discontinuous solutions. Therefore, (3.8) does not hold in general.

We prove that Δ_x is an invariant region, that is, for any fixed $M > 0$, if initial data satisfy

$$-M\{A_-(x)\}^{-\frac{\gamma-1}{\gamma+1}} \leq z_0(x), \quad w_0(x) \leq M\{A_+(x)\}^{-\frac{\gamma-1}{\gamma+1}},$$

solutions to the Cauchy problem (3.1)–(3.2) satisfy

$$-M\{A_-(x)\}^{-\frac{\gamma-1}{\gamma+1}} \leq z(x, t), \quad w(x, t) \leq M\{A_+(x)\}^{-\frac{\gamma-1}{\gamma+1}}. \quad (3.9)$$

We set $\tilde{z} = \{A_-(x)\}^{\frac{\gamma-1}{\gamma+1}} z$, $\tilde{w} = \{A_+(x)\}^{\frac{\gamma-1}{\gamma+1}} w$. We shall prove the following holds

$$-M \leq \tilde{z}(x, t), \quad \tilde{w}(x, t) \leq M,$$

provided that $-M \leq \tilde{z}_0(x)$, $\tilde{w}_0(x) \leq M$.

For any positive constants ε and T , we set

$$\check{z}(x, t) = \tilde{z}(x, t) + \frac{\varepsilon}{T-t}, \quad \check{w}(x, t) = \tilde{w}(x, t) - \frac{\varepsilon}{T-t}.$$

Then we find

$$-M < \check{z}_0(x), \quad \check{w}_0(x) < M.$$

For simplicity, we assume $\check{z}(x, t)$ and $\check{w}(x, t)$ have compact supports on $0 \leq t < T$. Then, on $0 \leq t < T$, our goal is to prove

$$-M < \check{z}(x, t), \quad \check{w}(x, t) < M.$$

We assume that there exists a point $x = x_*, t = t_*$ ($t_* < T$) such that

$$-M = \check{z}(x_*, t_*), \quad -M < \check{z}(x, t), \quad \check{w}(x, t) < M, \quad 0 \leq t < t_* \quad (3.10)$$

or

$$\check{w}(x_*, t_*) = M, \quad -M < \check{z}(x, t), \quad \check{w}(x, t) < M, \quad 0 \leq t < t_*. \quad (3.11)$$

Then we shall deduce a contradiction. For symmetry, we hereafter consider the only case where $x_* \geq 0$. Then, we notice that $a(x_*) \leq 0$. We can similarly treat with the case where $x_* < 0$.

First, we consider the case (3.10). Then, since \check{z} attains the minimum at $x = x_*, t = t_*$, we find $\check{z}_t \leq 0, \check{z}_x = 0$. On the other hand, from

$$\check{z}_t + \lambda_1(u)\check{z}_x = -\frac{\gamma-1}{\gamma+1}a(x) \{A(x)\}^{\frac{\gamma-1}{\gamma+1}} \left((v)^2 + \frac{(\rho)^{2\theta}}{\theta} \right)$$

and $\check{z}(x, t) = \tilde{z}(x, t) + \varepsilon/(T-t)$, at $x = x_*, t = t_*$ we have

$$\check{z}_t + \lambda_1(u)\check{z}_x = -\frac{\gamma-1}{\gamma+1}a(x) \{A(x)\}^{\frac{\gamma-1}{\gamma+1}} \left((v)^2 + \frac{(\rho)^{2\theta}}{\theta} \right) + \frac{\varepsilon}{(T-t)^2} > 0.$$

This is a contradiction.

Next, we consider the case (3.11). Since \check{w} attains the maximum at $x = x_*, t = t_*$, we find $\check{w}_t \geq 0, \check{w}_x = 0$. On the other hand, at $x = x_*, t = t_*$, we notice that

$$\begin{aligned} w + z &= \{A(0)\}^{-\frac{\gamma-1}{\gamma+1}} \tilde{w} + \{A(x_*)\}^{-\frac{\gamma-1}{\gamma+1}} \check{z} \\ &= \{A(0)\}^{-\frac{\gamma-1}{\gamma+1}} \left(\tilde{w} + \frac{\varepsilon}{T-t} \right) + \{A(x_*)\}^{-\frac{\gamma-1}{\gamma+1}} \left(\check{z} - \frac{\varepsilon}{T-t} \right) \\ &\geq \left(M + \frac{\varepsilon}{T-t} \right) \left[\{A(0)\}^{-\frac{\gamma-1}{\gamma+1}} - \{A(x_*)\}^{-\frac{\gamma-1}{\gamma+1}} \right] \geq 0. \end{aligned}$$

Then, from

$$\tilde{w}_t + \lambda_2 \tilde{w}_x = a(x) \{A(0)\}^{\frac{\gamma-1}{\gamma+1}} \rho^\theta \frac{w+z}{2}$$

and $\check{w}(x, t) = \tilde{w}(x, t) - \varepsilon/(T-t)$, we have

$$\check{w}_t + \lambda_2 \check{w}_x = a(x) \{A(0)\}^{\frac{\gamma-1}{\gamma+1}} \rho^\theta \frac{w+z}{2} - \frac{\varepsilon}{(T-t)^2} < 0.$$

This is a contradiction.

Since ε and T are arbitrary, we can prove (3.9).

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