A Simplified Characterisation of Provably Computable Functions of the System ID$_1$ of Inductive Definitions (Extended Abstract)

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Abstract

We present a simplified and streamlined characterisation of provably total computable functions of the system ID$_1$ of non-iterated inductive definitions. The idea of the simplification is to employ the method of operator-controlled derivations that was originally introduced by Wilfried Buchholz and afterwards applied by the second author to a streamlined characterisation of provably total computable functions of Peano arithmetic PA.

1 Introduction

As stated by Gödel's first incompleteness theorem, any reasonable consistent formal system has an unprovable $\Pi^0_2$-sentence that is true in the standard model of arithmetic. This means that the total (computable) functions whose totality is provable in a consistent system, which are known as provably (total) computable functions, form a proper subclass of total computable functions. Hence it is natural to ask how we can describe the provably computable functions of a given system. Not surprisingly provably computable functions are closely related to provable well-ordering, i.e., ordinal analysis. Several successful applications of techniques from ordinal analysis to provably computable functions have been provided by B. Blankertz and A. Weiermann

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Modern ordinal analysis is based on the method of *local predicativity*, that was first introduced by W. Pohlers, cf. [10, 11]. Successful applications of local predicativity to provably computable functions contain works by Blankertz and Weiermann [12] and by Weiermann [2]. However, to the authors' knowledge, the most successful way in ordinal analysis is based on the method of *operator-controlled derivations*, an essential simplification of local predicativity, that was introduced by Buchholz [3]. In [13] the second author successfully applied the method of operator-controlled derivations to a streamlined characterisation of provably computable functions of PA. (See also [11, Section 2.1.5].) Technically this work aims to lift up the characterisation obtained in [13] to an impredicative system ID₁ of non-iterated inductive definitions. We introduce an ordinal notation system $\mathcal{O}(\Omega)$ and define a computable function $f^\alpha$ for a starting numerical function $f : \mathbb{N} \to \mathbb{N}$ by transfinite recursion on $\alpha \in \mathcal{O}(\Omega)$. The transfinite definition of $f^\alpha$ stems from [13]. We show that a function is provably computable in ID₁ if and only if it is a Kalmar elementary function in $\{s^\alpha \mid \alpha \in \mathcal{O}(\Omega) \text{ and } \alpha < \Omega\}$, where $s$ denotes the numerical successor function $m \mapsto m + 1$ and $\Omega$ denotes the least non-computable ordinal (Corollary 6.4).

This paper consists of two materials, a technical report [8] by the authors and a draft [14] by the second author. Section 3–6 consist of [8] and Section 7 consists of [14]. We mention in particular that the ordinal notation system $\mathcal{OT}(\mathcal{F})$ stems from [14]. Most of proofs are omitted due to the page limitation. We note however that there is a non-trivial error in the technical report [8, p. 8, Lemma 15.5]. We restate Lemma 4.4.5, provide its proof and discuss in detail about embedding (Section 5) affected by this correction. The full details of missing proofs will appear in [7].

## 2 Preliminaries

In order to make our contribution precise, in this preliminary section we collect the central notions. We write $\mathcal{L}_{PA}$ to denote the standard language of first order theories of arithmetic. In particular we suppose that the constant 0 and the successor function symbol $S$ are included in $\mathcal{L}_{PA}$. For each natural $m$ we use the notation $\underline{m}$ to denote the corresponding numeral built from 0 and $S$. Let a set variable $X$ denote a subset of $\mathbb{N}$. We write $X(t)$ instead of $t \in X$ and $\mathcal{L}_{PA}(X)$ for $\mathcal{L}_{PA} \cup \{X\}$. Let $FV_{1}(A)$ denote the set of free number variables appearing in a formula $A$ and $FV_{2}(A)$ the set of free set variables in $A$. And then let $FV(A) := FV_{1}(A) \cup FV_{2}(A)$. For a fresh set variable $X$ we call an $\mathcal{L}_{PA}(X)$-formula $A(x)$ a positive operator form if $FV_{1}(A(x)) \subseteq \{x\}$, $FV_{2}(A(x)) = \{X\}$, and $X$ occurs only positively in $A$.

Let $FV_{1}(A(x)) = \{x\}$. For a formula $F(x)$ such that $x \in FV_{1}(F(x))$ we write $A(F, t)$ to denote the result of replacing in $A(t)$ every subformula $X(s)$ by $F(s)$. The language $\mathcal{L}_{ID_{1}}$, of the system ID₁ of non-iterated inductive definitions is defined by $\mathcal{L}_{ID_{1}} := \mathcal{L}_{PA} \cup \{P_{A} \mid A$ is a positive operator form\} where for each positive operator
form $A$, $P_A$ denotes a new unary predicate symbol. We write $\mathcal{T}(\mathcal{L}_{ID_1}, \mathcal{V})$ to denote the set of $\mathcal{L}_{ID_1}$-terms and $\mathcal{T}(\mathcal{L}_{ID_1})$ to denote the set of closed $\mathcal{L}_{ID_1}$-terms. The axioms of ID$_1$ consist of the axioms of Peano arithmetic PA in the language $\mathcal{L}_{ID_1}$ and the following new axiom schemata (ID$_1$) and (ID$_2$):

(Id1) $\forall x(A(P_A, x) \rightarrow P_A(x))$.

(Id2) (The universal closure of) $\forall x(A(F, x) \rightarrow F(x)) \rightarrow \forall x(P_A(x) \rightarrow F(x))$, where $F$ is an $\mathcal{L}_{ID_1}$-formula.

For each $n \in \mathbb{N}$ we write $\Sigma_n$ to denote the fragment of Peano arithmetic PA with induction restricted to $\Sigma_n$-formulas. Let $k$ be a natural number and $f : \mathbb{N}^k \rightarrow \mathbb{N}$ a numerical function and $T$ be a system of arithmetic containing $\Sigma_1$. Then we say that $f$ is provably total computable in $T$ or provably computable in $T$ for short if there exists a $\Sigma_1$-formula $A_f(x_1, \ldots, x_k, y)$ such that (i) $FV(A_f) = FV_1(A_f) = \{x_1, \ldots, x_k, y\}$, (ii) for all $\vec{m}, n \in \mathbb{N}$, $f(\vec{m}) = n$ holds if and only if $A_f(\vec{m}, n)$ is true in the standard model $\mathbb{N}$ of PA, and (iii) $\forall \vec{x} \exists! y A_f(\vec{x}, y)$ is a theorem in $T$.

3 A non-computable ordinal notation system $\mathcal{OT}(\mathcal{F})$

In this section we introduce a non-computable ordinal notation system $\mathcal{OT}(\mathcal{F}) = \langle \mathcal{OT}(\mathcal{F}), < \rangle$. This new ordinal notation system is employed in the next section. For an element $\alpha \in \mathcal{OT}(\mathcal{F})$ let $\mathcal{OT}(\mathcal{F}) \upharpoonright \alpha$ denote the set \{ $\beta \in \mathcal{OT}(\mathcal{F}) \mid \beta < \alpha$ \}.

Definition 3.1 We define three sets $\mathcal{SC} \subseteq \mathcal{H} \subseteq \mathcal{OT}(\mathcal{F})$ of ordinal terms and a set $\mathcal{F}$ of unary function symbols simultaneously. Let $0, \varphi, \Omega, S, E$ and $+$ be distinct symbols.

1. $0 \in \mathcal{OT}(\mathcal{F})$ and $\Omega \in \mathcal{H}$.

2. $\{S, E\} \subseteq \mathcal{F}$.

3. If $\alpha \in \mathcal{OT}(\mathcal{F}) \upharpoonright \Omega$, then $S(\alpha) \in \mathcal{OT}(\mathcal{F})$ and $E(\alpha) \in \mathcal{H}$.

4. If $\{\alpha_1, \ldots, \alpha_l\} \subseteq \mathcal{H}$ and $\alpha_1 \geq \cdots \geq \alpha_l$, then $\alpha_1 + \cdots + \alpha_l \in \mathcal{OT}(\mathcal{F})$.

5. If $\{\alpha, \beta\} \subseteq \mathcal{OT}(\mathcal{F}) \upharpoonright \Omega$, then $\varphi \alpha \beta \in \mathcal{H}$.

6. If $\alpha \in \mathcal{OT}(\mathcal{F})$ and $\xi \in \mathcal{OT}(\mathcal{F}) \upharpoonright \Omega$, then $\Omega^\alpha \cdot \xi \in \mathcal{H}$.

7. If $F \in \mathcal{F}$, $\alpha \in \mathcal{OT}(\mathcal{F})$ and $\xi \in \mathcal{OT}(\mathcal{F}) \upharpoonright \Omega$, then $F^\alpha(\xi) \in \mathcal{SC}$.

8. If $F \in \mathcal{F}$ and $\alpha \in \mathcal{OT}(\mathcal{F})$, then $F^\alpha \in \mathcal{F}$.
We write \( \omega^\alpha \) to denote \( \varphi 0 \alpha \) and \( m \) to denote \( \omega^0 \cdot m = \omega^0 + \cdots + \omega^0 \).

Let \( \text{Ord} \) denote the class of ordinals and \( \text{Lim} \) the class of limit ones. We define a semantic \([\cdot] \) for \( \mathcal{OT}(\mathcal{F}) \), i.e., \([\cdot]: \mathcal{OT}(\mathcal{F}) \rightarrow \text{Ord} \). The well ordering \( < \) on \( \mathcal{OT}(\mathcal{F}) \) is defined by \( \alpha < \beta \Leftrightarrow [\alpha] < [\beta] \). Let \( \Omega_1 \) denote the least non-computable ordinal \( \omega^\text{CK}_1 \). For an ordinal \( \alpha \) we write \( \alpha = \text{NF} \Omega_1^{\alpha_1} \cdot \beta_1 + \cdots + \Omega_1^{\alpha_l} \cdot \beta_l \) if \( \alpha > \alpha_1 > \cdots > \alpha_l, \{\beta_1, \ldots, \beta_l\} \subseteq \Omega_1, \) and \( \alpha = \Omega_1^{\alpha_1} \cdot \beta_1 + \cdots + \Omega_1^{\alpha_l} \cdot \beta_l \). Let \( \varepsilon_\alpha \) denote the \( \alpha \)th epsilon number. One can observe that for each ordinal \( \alpha < \varepsilon_{\Omega_{1}+1} \) there uniquely exists a set \( \{\alpha_1, \ldots, \alpha_l, \beta_1, \ldots, \beta_l\} \) of ordinals such that \( \alpha = \text{NF} \Omega_1^{\alpha_1} \cdot \beta_1 + \cdots + \Omega_1^{\alpha_l} \cdot \beta_l \). For a set \( K \subseteq \text{Ord} \) and for an ordinal \( \alpha \) we will write \( K < \alpha \) to abbreviate \( \forall \xi \in K \xi < \alpha \), and dually \( \alpha \leq K \) to abbreviate \( \exists \xi \in K \alpha \leq \xi \).

**Definition 3.2 (Collapsing operators)** 1. Let \( \alpha \) be an ordinal such that \( \alpha = \text{NF} \Omega_1^{\alpha_1} \cdot \beta_1 + \cdots + \Omega_1^{\alpha_l} \cdot \beta_l < \varepsilon_{\Omega_{1}+1} \). The set \( K_{\Omega} \alpha \) of coefficients of \( \alpha \) is defined by

\[
K_{\Omega} \alpha = \{\beta_1, \ldots, \beta_l\} \cup K_{\Omega} \alpha_1 \cup \cdots \cup K_{\Omega} \alpha_l.
\]

2. Let \( F : \text{Ord} \rightarrow \text{Ord} \) be an ordinal function. Then a function \( F^\alpha : \text{Ord} \rightarrow \text{Ord} \) is defined by transfinite recursion on \( \alpha \in \text{Ord} \) by

\[
\begin{align*}
F^0(\xi) &= F(\xi), \\
F^\alpha(\xi) &= \min\{\gamma \in \text{Ord} \mid \omega^\gamma = \gamma, \ K_{\Omega} \alpha \cup \{\xi\} < \gamma \text{ and } (\forall \eta < \gamma)(\forall \beta < \alpha)(K_{\Omega} \beta < \gamma \Rightarrow F^\beta(\eta) < \gamma)\}.
\end{align*}
\]

**Corollary 3.3** Let \( F : \text{Ord} \rightarrow \text{Ord} \) be an ordinal function. Then \( F^\beta(\eta) < F^\alpha(\xi) \) holds if \( (\beta < \alpha \land K_{\Omega} \beta \cup \{\eta\} < F^\alpha(\xi)) \) or \( (\alpha < \beta \land F^\beta(\eta) \leq K_{\Omega} \alpha) \).

**Proposition 3.4** Suppose that \( \alpha < \varepsilon_{\Omega_{1}+1} \), a function \( F : \text{Ord} \rightarrow \text{Ord} \) has a \( \Sigma_1 \)-definition in the \( \Omega_1 \)th stage \( L_{\Omega_1} \) of the constructible hierarchy \( (L_\alpha)_{\alpha \in \text{Ord}} \) and that \( F(\xi) < \Omega_1 \) for all \( \xi < \Omega_1 \). Then \( F^\alpha \) also has a \( \Sigma_1 \)-definition in \( L_{\Omega_1} \) and \( F^\alpha(\xi) < \Omega_1 \) holds for all \( \xi < \Omega_1 \).

**Proposition 3.5** For any \( \alpha \in \text{Ord} \), for any \( \eta, \xi < \Omega_1 \) and for any ordinal function \( F : \Omega_1 \rightarrow \Omega_1 \), if \( \eta < F^\alpha(\xi) \), then \( F^\alpha(\eta) \leq F^\alpha(\xi) \).

**Definition 3.6** We define the value \([\alpha] \in \text{Ord} \) of an ordinal term \( \alpha \in \mathcal{OT}(\mathcal{F}) \) by recursion on the length of \( \alpha \).

1. \([0] = 0 \) and \([\Omega] = \Omega_1 \).
2. \([\alpha + \beta] = [\alpha] + [\beta] \).
3. \([\varphi \alpha \beta] = [\varphi] [\alpha] [\beta], \) where \([\varphi] \) is the standard Veblen function, i.e.,

\[
\begin{array}{ll}
[\varphi] 0 \beta &= \omega^\beta, \\
[\varphi] (\alpha + 1) 0 &= \sup\{( [\varphi] \alpha)^n 0 \mid n \in \omega \}, \\
[\varphi] 0 \gamma &= \sup\{ [\varphi] \alpha 0 \mid [\varphi] \alpha \gamma \}\text{ if } \gamma \in \text{Lim}, \\
[\varphi] (\alpha + 1) (\beta + 1) &= \sup\{( [\varphi] \alpha)^n ([\varphi] (\alpha + 1) \beta + 1) \mid n \in \omega \}, \\
[\varphi] \gamma (\beta + 1) &= \sup\{ [\varphi] \alpha ([\varphi] \gamma \beta + 1) \mid [\varphi] \alpha \gamma \}\text{ if } \gamma \in \text{Lim}, \\
[\varphi] \alpha \gamma &= \sup\{ [\varphi] \alpha \beta \mid [\varphi] \alpha \gamma \}\text{ if } \gamma \in \text{Lim}.
\end{array}
\]
4. \[ \Omega^\alpha \cdot \xi = \Omega_1^\alpha \cdot \xi. \]

5. \[ [S(\alpha)] = [S](\alpha), \]

where \( S \) denotes the ordinal successor \( \alpha \mapsto \alpha + 1 \). Clearly \( \{ [S](\xi) \mid \xi < \Omega_1 \} \subseteq \Omega_1 \).

6. \[ [E(\alpha)] = [E](\alpha), \]

where the function \( E : \text{Ord} \to \text{Ord} \) is defined by \( [E](\alpha) = \min\{ \xi \in \text{Ord} \mid \omega^\xi = \xi \text{ and } \alpha < \xi \} \). It is also clear that \( \{ [E](\xi) \mid \xi < \Omega_1 \} \subseteq \Omega_1 \).

7. \[ [F^\alpha(\xi)] = [F]^\alpha([\xi]). \]

**Definition 3.7** For all \( \alpha, \beta \in \mathcal{O}(\mathcal{F}), \) \( \alpha < \beta \) if \( [\alpha] < [\beta] \), and \( \alpha = \beta \) if \( [\alpha] = [\beta] \).

We will identify each element \( \alpha \in \mathcal{O}(\mathcal{F}) \) with its value \( [\alpha] \in \text{Ord} \). Accordingly we will write \( K_\Omega \alpha \) instead of \( K_\Omega[\alpha] \) for \( \alpha \in \mathcal{O}(\mathcal{F}) \). Further for a finite set \( K \subseteq \text{Ord} \) we write \( K_\Omega K \) to denote the finite set \( \bigcup_{\xi \in K} K_\Omega \xi \). By this identification, \( \mathbb{H} \) is the set of additively indecomposable ordinals and \( \text{SC} \) is the set of strongly critical ordinals, i.e, \( \text{SC} \subseteq \mathbb{H} \subseteq \text{Lim} \cup \{1\} \subseteq \text{Ord} \).

**Corollary 3.8** \( F^\alpha(\xi) < \Omega \) for any \( F \in \mathcal{F} \) and \( \xi < \Omega \).

**Proof.** Proof by induction over the build-up of \( F \in \mathcal{F} \). \( \square \)

**Corollary 3.9**

1. \( K_\Omega 0 = K_\Omega \Omega = \emptyset. \)

2. If \( K_\Omega \alpha < \xi \) and \( \xi \in \text{SC}, \) then \( K_\Omega S(\alpha) < \xi. \)

3. \( K_\Omega E(\alpha) = \{ E(\alpha) \} \) (since \( \alpha < \Omega \)).

4. If \( K_\Omega \alpha \cup K_\Omega \beta < \xi \) and \( \xi \in \text{SC}, \) then \( K_\Omega (\alpha + \beta) < \xi. \)

5. \( K_\Omega \varphi \alpha \beta = \{ \varphi \alpha \beta \} \) (since \( \alpha, \beta < \Omega \)). Further, if \( \alpha, \beta < \xi \) and \( \xi \in \text{SC}, \) then \( \varphi \alpha \beta < \xi. \)

6. \( K_\Omega F^\alpha(\xi) = \{ F^\alpha(\xi) \} \) (since \( \xi < \Omega \)).

By Corollary 3.8 each function symbol in \( \mathcal{F} \) defines a weakly increasing function \( F : \Omega \to \Omega \) such that \( \xi < F(\xi) \) holds for all \( \xi \in \Omega \). In the rest of this section let \( F \) denote such a function. For a finite set \( K \subseteq \text{Ord} \) we will use the notation \( F[K](\xi) \) to abbreviate \( F(\max(K \cup \{\xi\})) \).

**Lemma 3.10** Let \( K \subseteq \text{Ord} \) be a finite set such that \( K < \Omega \). Then \( (F[K])^\alpha(\xi) \leq F^\alpha[K](\xi) \) for all \( \xi < \Omega \).

**Lemma 3.11** \( (F^\alpha)^\beta(\xi) \leq F^{\alpha+\beta}(\xi) \) for all \( \xi < \Omega \).
4 An infinitary proof system $\text{ID}_1^{\infty}$

In this section we introduce the main definition of this paper, a new infinitary proof system $\text{ID}_1^{\infty}$, to which the new ordinal notation system $\mathcal{OT}(\mathcal{F})$ is connected, and into which every (finite) proof in $\text{ID}_1$ can be embedded in good order. For each positive operator form $A$ and for each ordinal term $\alpha \in (\mathcal{OT}(\mathcal{F}) \upharpoonright \Omega) \cup \{\Omega\}$ let $P_A^{<\alpha}$ be a new unary predicate symbol. Let us define an infinitary language $\mathcal{L}^*$ of $\text{ID}_1^{\infty}$ by $\mathcal{L}^* = \mathcal{L}_{PA} \cup \{\neq, \notin\} \cup \{P_A^{<\alpha}, \neg P_A^{<\alpha} \mid \alpha \in (\mathcal{OT}(\mathcal{F}) \upharpoonright \Omega) \cup \{\Omega\}\}$ and $A$ is a positive operator form.

Let us write $P_A^{<\Omega}$ to denote $P_A$ to have the inclusion $\mathcal{L}_{ID_1} \subseteq \mathcal{L}^*$. We write $\mathcal{T}(\mathcal{L}^*)$ to denote the set of closed $\mathcal{L}^*$-terms. Specifically, the language $\mathcal{L}^*$ contains complementary predicate symbol $\neg P$ for each predicate symbol $P \in \mathcal{L}^*$. We note that the negation $\neg$ nor the implication $\to$ is not included as a logical symbol. The negation $\neg A$ is defined via de Morgan’s law by $\neg(\neg P(t)) := P(t)$ for an atomic formula $P(t)$, $\neg (A \land B) := \neg A \lor \neg B$, $\neg (A \lor B) := \neg A \land \neg B$, $\forall x A := \exists x \neg A$ and $\exists x A := \forall x \neg A$. The implication $A \to B$ is defined by $\neg A \lor B$. We start with technical definitions.

**Definition 4.1 (Complexity measures $lh$, $rk$, $k^\Pi$, $k$ of $\mathcal{L}^*$-formulas)**

1. The length $lh(A)$ of an $\mathcal{L}^*$-formula $A$ is the number of the symbols $P_A^{<\alpha}$, $\neg P_A^{<\alpha}$, $\lor$, $\land$, $\exists$ and $\forall$ occurring in $A$.

2. The rank $rk(A)$ of an $\mathcal{L}^*$-formula $A$.

   (a) $rk(P_A^{<\alpha}(t)) := rk(\neg P_A^{<\alpha}(t)) := \omega \cdot \alpha$.

   (b) $rk(A) := 0$ if $A$ is an $\mathcal{L}_{ID_1}$-literal.

   (c) $rk(A \land B) := rk(A \lor B) := \max\{rk(A), rk(B)\} + 1$.

   (d) $rk(\forall x A) := rk(\exists x A) := rk(A) + 1$.

3. The set $k^\Pi(A)$ of $\Pi$-coefficients of an $\mathcal{L}^*$-formula $A$.

   (a) $k^\Pi(P_A^{<\alpha}(t)) := \{0\}$, $k^\Pi(\neg P_A^{<\alpha}(t)) := \{0, \alpha\}$.

   (b) $k^\Pi(A) := \{0\}$ if $A$ is an $\mathcal{L}_{ID_1}$-literal.

   (c) $k^\Pi(A \land B) := k^\Pi(A \lor B) := k^\Pi(A) \cup k^\Pi(B)$.

   (d) $k^\Pi(\forall x A) := k^\Pi(\exists x A) := k^\Pi(A)$.

4. The set $k^\Sigma(A)$ of $\Sigma$-coefficients of an $\mathcal{L}^*$-formula $A$.

   $k^\Sigma(A) := k^\Pi(\neg A)$.

5. The set $k(A)$ of all the coefficients of an $\mathcal{L}^*$-formula $A$.

   $k(A) := k^\Pi(A) \cup k^\Sigma(A)$.

6. The set $k^\Pi_0(A)$ of $\Pi$-coefficients of an $\mathcal{L}^*$-formula $A$ less than $\Omega$.

   $k^\Pi_0(A) := k^\Pi(A) \upharpoonright \Omega$.

   The set $k^\Pi_0(A)$ and $k_0(A)$ are defined accordingly.
By definition $\text{rk}(A) = \text{rk}(\neg A)$, $k(A) = k(\neg A)$ and $k_{\Omega}(A) = k_{\Omega}(\neg A)$.

**Definition 4.2 (Complexity measures val, ord, N of $\mathcal{L}^*$-terms)**

1. The value $\text{val}(t)$ of a term $t \in \mathcal{T}(\mathcal{L}_{ID_1}) = \mathcal{T}(\mathcal{L}_{PA})$ is the value of the closed term $t$ in the standard model $\mathbb{N}$ of the Peano arithmetic $PA$.

2. A complexity measure $\text{ord} : \mathcal{T}(\mathcal{L}^*) \to (\mathcal{OT}(\mathcal{F}) \uparrow \Omega) \cup \{\Omega\}$ is defined by
   
   $$
   \begin{cases}
   \text{ord}(t) := 0 & \text{if } t \in \mathcal{T}(\mathcal{L}_{ID_1}), \\
   \text{ord}(\alpha) := \alpha & \text{if } \alpha \in \mathcal{OT}(\mathcal{F}).
   \end{cases}
   $$

3. The norm $N(\alpha)$ of $\alpha \in \mathcal{OT}(\mathcal{F})$.
   
   $$(a) N(0) = 0 \text{ and } N(\Omega) = 1.
   $$
   $$(b) N(S(\alpha)) = N(\alpha) + 1.
   $$
   $$(c) N(E(\alpha)) = N(\alpha) + 1.
   $$
   $$(d) N(\alpha + \beta) = N(\alpha) + N(\beta).
   $$
   $$(e) N(\varphi \alpha \beta) = N(\alpha) + N(\beta) + 1,
   $$
   $$(f) N(\Omega^\alpha \xi) = N(\alpha) + N(\xi) + 1.
   $$
   $$(g) N(F^\alpha(\xi)) = N(F(\xi)) + N(\alpha). \quad \text{(Note that } F(\xi) \in \mathcal{OT}(\mathcal{F}) \text{ if } F^\alpha(\xi) \in \mathcal{OT}(\mathcal{F}).)$$

The norm is extended to a complexity measure $N : \mathcal{T}(\mathcal{L}^*) \to \mathbb{N}$ by

$$
\begin{cases}
N(t) := \text{val}(t) & \text{if } t \in \mathcal{T}(\mathcal{L}_{ID_1}), \\
N(\alpha) := N(\alpha) & \text{if } \alpha \in \mathcal{OT}(\mathcal{F}).
\end{cases}
$$

By definition $N(\omega^\alpha) = N(\varphi 0 \alpha) = N(\alpha) + 1$ and $N(\omega) = N(0 \cdot m) = m$ for any $m < \omega$. This seems to be a good point to explain why we contain the constant $\Omega$ in $\mathcal{OT}(\mathcal{F})$. Having that $N(\Omega) = 1$ removes some technicalities.

**Definition 4.3** We define a relation $\simeq$ between $\mathcal{L}^*$-sentences and (infinitary) propositional $\mathcal{L}^*$-sentences.

1. $\neg P_{\mathcal{A}}^{<\alpha}(t) : \simeq \bigwedge_{\xi \in \mathcal{OT}(\mathcal{F}) \uparrow \alpha} \neg A(P_{\mathcal{A}}^{<\xi},t)$ and $P_{\mathcal{A}}^{<\alpha}(t) : \simeq \bigvee_{\xi \in \mathcal{OT}(\mathcal{F}) \uparrow \alpha} \neg A(P_{\mathcal{A}}^{<\xi},t)$.

2. $A \land B : \simeq \bigwedge_{\iota \in \{0,1\}} A_{\iota}$ and $A \lor B : \simeq \bigvee_{\iota \in \{0,1\}} A_{\iota}$ where $A_{0} \equiv A$ and $A_{1} \equiv B$.

3. $\forall x A(x) : \simeq \bigwedge_{t \in \mathcal{T}(\mathcal{L}_{ID_1})} A(t)$ and $\exists x A(x) : \simeq \bigvee_{t \in \mathcal{T}(\mathcal{L}_{ID_1})} A(t)$.

We call an $\mathcal{L}^*$-sentence $A$ a $\land$-type (conjunctive type) if $A \simeq \bigwedge_{\iota \in J} A_{\iota}$, and a $\lor$-type (disjunctive type) if $A \simeq \bigvee_{\iota \in J} A_{\iota}$, for some $A_{\iota}$. For the sake of simplicity we will write $\land_{\xi < \alpha} A_{\xi}$ instead of $\bigwedge_{\xi \in \mathcal{OT}(\mathcal{F}) \uparrow \alpha} A_{\xi}$ and write $\lor_{\xi < \alpha} A_{\xi}$ accordingly.
Lemma 4.4  
1. If either $A \simeq \bigwedge_{i \in J} A_i$ or $A \simeq \bigvee_{i \in J} A_i$, then for all $i \in J$, $k^\Pi(A_i) \subseteq \{\text{ord}(i)\} \cup k^\Omega(A)$ and $k^\Sigma(A_i) \subseteq \{\text{ord}(i)\} \cup k^\Omega(A)$.

2. For any $\alpha \in OT(F)$, if $A \simeq \bigwedge_{i < \alpha} A_i$, then $(\exists \sigma \in k^\Pi(A))(\forall \xi < \alpha)[\xi \leq \sigma]$.

3. For any $\mathcal{L}^*$-sentence $A$, $\text{rk}(A) = \omega \cdot \max k(A) + n$ for some $n \leq \text{lh}(A)$.

4. If $\text{rk}(A) = \Omega$, then either $A \equiv P_A^{<\Omega}(t)$ or $A \equiv \neg P_A^{<\Omega}(t)$.

5. If either $A \simeq \bigwedge_{i \in J} A_i$ or $A \simeq \bigvee_{i \in J} A_i$, then $N(\text{rk}(A)) \leq \max(\{N(\text{rk}(A))\} \cup \{2 \cdot N(i) + \text{lh}(A(i, *)) \mid P_A^{<\xi} \text{ or } \neg P_A^{<\xi} \text{ occurs in } A\})$ for all $i \in J$.

Proof. We only show the non-trivial property, Property 5. By Property 3, $\text{rk}(A) = \omega \cdot \max k(A) + n$ for some $n \leq \text{lh}(A)$.

CASE. $n > 0$: In this case $\text{rk}(A) = \omega \cdot \max k(A) + n_0$ for some $n_0 < n \leq \text{lh}(A)$. Hence clearly $N(\text{rk}(A_i)) \leq N(\text{rk}(A))$.

CASE. $n = 0$: In this case without loss of generality let us assume $A$ is of the form $P_A^{<\alpha}(t) \simeq \bigvee_{i < \alpha} A_i(P_A^{<\xi}, t)$ and hence $A_i \simeq A(P_A^{<\xi}, t)$. Let $\iota := \xi < \alpha$. Then $\text{rk}(A_i) = \omega \cdot \xi + n_i$ for some $n_i \leq \text{lh}(A(i, *))$. Hence $N(\text{rk}(A_i)) \leq 2 \cdot N(\xi) + \text{lh}(A(i, *))$.

Throughout this section we use the symbol $F$ to denote a weakly increasing ordinal function $F : \Omega \rightarrow \Omega$ and the symbol $f$ to denote a numerical function $f : \mathbb{N} \rightarrow \mathbb{N}$ that enjoys the following conditions.

(f.1) $f$ is a strictly increasing function such that $2m + 1 \leq f(m)$ for all $m$. Hence, in particular, $n + f(m) \leq f(n + m)$ for all $m$ and $n$.

(f.2) $2 \cdot f(m) \leq f(f(m))$ for all $m$.

We will use the notation $f[n](m)$ to abbreviate $f(n + m)$. It is easy to see that if the conditions (f.1) and (f.2) hold, then for a fixed $n$ the conditions $(f[n].1)$ and $(f[n].2)$ also hold.

Definition 4.5 Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be a numerical function. Then a function $f^\alpha : \mathbb{N} \rightarrow \mathbb{N}$ is defined by transfinite recursion on $\alpha \in OT(F)$ by

\[
\begin{align*}
  f^0(m) &= f(m), \\
  f^\alpha(m) &= \max\{f^\beta(f^\beta(m)) \mid \beta < \alpha \text{ and } N(\beta) \leq f[N(\alpha)](m)\} \quad \text{if } 0 < \alpha.
\end{align*}
\]

Corollary 4.6  
1. If $f$ is strictly increasing, then so is $f^\alpha$ for any $\alpha \in OT(F)$.

2. If $\beta < \alpha$ and $N(\beta) \leq f[N(\alpha)](m)$, then $f^\beta(m) < f^\alpha(m)$.

3. $f^\alpha(f^\alpha(m)) \leq f^{\alpha+1}(m)$.

We note that the function $f^\alpha$ is not a computable function in general even if $f$ is computable since the ordinal notation system $(OT(F), <)$ is not a computable system.
Lemma 4.7 Let \( \alpha \in \mathcal{OT}(\mathcal{F}) \) and \( F \in \mathcal{F} \). Then \( N(\alpha) \leq f^{F_\alpha}(0) \).

Lemma 4.8 Let \( \{\alpha, \beta\} \subseteq \mathcal{OT}(\mathcal{F}) | \Omega \) and \( F \in \mathcal{F} \). Then \( (f^{\alpha})^{\beta}(m) \leq f^{F^{\alpha+\beta}(0)}(m) \) for all \( m \).

Lemma 4.9
1. \( f^\alpha[n](m) \leq (f[n])^\alpha(m) \).
2. If \( n \leq m \), then \( (f[n])^\alpha(m) \leq f^\alpha[f^\alpha(f(m))](f(m)) \).

Corollary 4.10 If \( n \leq m \), then \( (f[n])^\alpha(m) \leq f^{\alpha+2}(m) \).

We define a relation \( f, F \vdash_{\rho}^\alpha \Gamma \) for a quintuple \( (f, F, \alpha, \rho, \Gamma) \) where \( \alpha < \epsilon_{\Omega+1}, \rho < \Omega + \omega \) and \( \Gamma \) is a sequent of \( \mathcal{L}^* \)-sentences. In this paper a "sequent" means a finite set of formulas. We write \( \Gamma, A \) or \( A, \Gamma \) to denote \( \Gamma \cup \{A\} \).

We will call the pair \( (f, F) \) operators controlling the derivation that forms \( f, F \vdash_{\rho}^\alpha \Gamma \).
In the sequel we always assume that the operator $F$ enjoys the following condition 
\[ \text{HYP}(F): \quad \eta < F(\xi) \Rightarrow F(\eta) \leq F(\xi) \quad \text{for any ordinals } \xi, \eta < \Omega. \] 

We note that the hypothesis HYP($F$) reflects the fact stated in Proposition 3.5. It is
not difficult to see that if the condition HYP($F$) holds, then the condition HYP($F[K]$)
also holds for any finite set $K < \Omega$.

Lemma 4.12 (Inversion) Assume that $A \simeq \bigwedge_{\iota \in J} A_{\iota}$. If $f, F \vdash_{\rho}^\alpha \Gamma, A$, then for all
\[ \iota \in J, \; f[N(\iota)], F[\text{ord}(\iota)] \vdash_{\rho}^\alpha \Gamma, A_{\iota}. \]

We write $f \circ g$ to denote the result of composing $f$ and $g$: $m \mapsto f(g(m))$.

Lemma 4.13 (Cut-reduction) Assume $C \simeq \bigvee_{\iota \in J} C_{\iota}$, $\text{rk}(C) = \rho \neq \Omega$, $\max\{|N(\sigma)| \mid \sigma \in k_\Omega(C)\} \leq f(g(0))$, and $k_\Omega(C) < F(0)$. If $f, F \vdash_{\rho}^\beta \Gamma, C$ and $g, F \vdash_{\rho}^\beta \Gamma, \neg C$, then $f \circ g, F \vdash_{\rho}^{\alpha+\beta} \Gamma$. For a sequent $\Gamma$ we write $k_\Omega^\Pi(\Gamma)$ to denote the set $\bigcup_{B \in \Gamma} k_\Omega^\Pi(B)$.

Lemma 4.14 (First Cut-elimination) Let $k < \omega$. If $f, F \vdash_{\Omega+k+2}^\alpha \Gamma$, then $f^{F^{\alpha}(0)+1}, F \vdash_{\Omega+k+1}^\alpha \Gamma$.

Lemma 4.15 (Predicative Cut-elimination) Assume that $\{\alpha, \beta, \gamma\} < \Omega$, $N(\alpha) \leq f(0)$ and $K_\Omega \alpha < F(0)$. If $f, F \vdash_{\rho+\omega+\alpha}^\beta \Gamma$, then $f^{F^{\alpha+\beta}(0)+1}, F \vdash_{\rho}^{\alpha+\beta} \Gamma$.

Definition 4.16 For each $\mathcal{L}^*$-formula $B$ let $B^\alpha$ be the result of replacing in $B$ every occurrence of $P_\alpha^{\lt \Omega}$ by $P_\alpha^{\lneq \alpha}$.

Lemma 4.17 (Boundedness) Assume that $f, F \vdash_{\rho}^\alpha \Gamma, A$. Then for all $\xi$ if $\alpha \leq \xi \leq F(0)$, then $f, F \vdash_{\rho}^\xi \Gamma, A^\xi$.

We will write $f, F \vdash_{\rho}^\alpha \Gamma$ instead of $f, F \vdash_{\rho}^\alpha \Gamma$.

Lemma 4.18 (Impredicative Cut-elimination) If $f, F \vdash_{\rho+1}^\alpha \Gamma$, then $f^{F^{\alpha}(0)+1}, F^{\alpha+1} \vdash_{\rho}^{F^{\alpha}(0)} \Gamma$.

Lemma 4.19 (Witnessing) For each $j < l$ let $B_j(x)$ be a $\Delta_0^{\lt \Omega}$-$\mathcal{L}_{PA}$-formula such that
\[ \text{FV}(B_j(x)) = \{x\}. \] Let $\Gamma \equiv \exists x_0 B_0(x_0), \ldots, \exists x_{l-1} B_{l-1}(x_{l-1})$. If $f, F \vdash_{\rho}^\alpha \Gamma$ for some $\alpha \in \mathcal{O}^*(F)$, then there exists a sequence $m_0, \ldots, m_{l-1}$ of naturals such that $\max\{m_j \mid j < l\} \leq f(0)$ and $B_0(m_0) \lor \cdots \lor B_{l-1}(m_{l-1})$ is true in the standard model $\mathbb{N}$ of PA.
5 Embedding ID$_1$ into ID$_1^\infty$

In this section we embed the system ID$_1$ into the infinitary system ID$_1^\infty$. Following conventions in the previous section we use the symbol $f$ to denote a strict increasing function $f : \mathbb{N} \to \mathbb{N}$ that enjoys the conditions (f.1) and (f.2) (p. 8). Let us recall that the function symbol $E \in \mathcal{F}$ denotes the function $E : \Omega \to \Omega$ such that $E(\alpha) = \min\{\xi < \Omega \mid \omega^\xi = \xi\text{ and } \alpha < \xi\}$. It is easy to see that the condition HYP($E$) holds since $E(\xi) = \varepsilon_0 \leq E(0)$ for all $\xi < E(0) = \varepsilon_0$.

Lemma 5.1 (Tautology lemma) Let $s,t \in \mathcal{T}(\mathcal{L}_{ID_1})$, $\Gamma$ be a sequent of $\mathcal{L}^*$-formulas, and $A(x)$ be an $\mathcal{L}^*$-formula such that $\text{FV}(A) = \{x\}$. If $\text{val}(s) = \text{val}(t)$, then

$$f[n], E[k_\Omega(A)] \vdash_{0}^{r(k(A))} \Gamma, \neg A(s), A(t),$$

where $n := \max\{\{N(rk(A))\} \cup \{2 \cdot N(\sigma) + \text{lh}(A(\cdot, *)) \mid \sigma \in k_\Omega(A) \text{ and } P_{A}^{<\xi} \text{ or } \neg P_{A}^{<\xi}\text{ occurs in } A\}$.

Proof. By induction on $rk(A)$. Let $n := \max\{\{N(rk(A))\} \cup \{2 \cdot N(\sigma) + \text{lh}(A(\cdot, *)) \mid \sigma \in k_\Omega(A) \text{ and } P_{A}^{<\xi} \text{ or } \neg P_{A}^{<\xi}\text{ occurs in } A\}$). From Lemma 4.4.3 one can check that the condition HYP$(f[n]; E(k_\Omega(A)); rk(A) \cdot 2)$ holds. If $rk(A) = 0$, then $A$ is an $\mathcal{L}_{ID_1}$-literal, and hence (1) is an instance of (Ax1). Suppose that $rk(A) > 0$. Without loss of generality we can assume that $A \simeq \bigvee_{i \in J} A_i$. Let $i \in J$. By Lemma 4.4.5 we observe that $N(rk(A_i)) \leq f(n) = f[n]([N(\iota)](0))$ since $2m + 1 \leq f(m)$ for all $m$ by the condition (f.1). Further by Lemma 4.4.1 $k_\Omega(rk(A_i) \cdot 2) \subseteq k_\Omega(A) \cup \{\text{ord}(\iota)\} \leq E[k_\Omega(A)][\text{ord}(\iota)]$. Summing up, we have the condition

$$\text{HYP}(f[n][N(\iota)]; E[k_\Omega(A)][\text{ord}(\iota)]; rk(A_i) \cdot 2).$$

Hence by IH we can obtain the sequent

$$f[n][N(\iota)], E[k_\Omega(A)][\text{ord}(\iota)] \vdash_{0}^{r(k(A_i))} \Gamma, \neg A_i(s), A_i(t).$$

(2)

It is not difficult to see $\text{ord}(\iota) \leq rk(A_i) < rk(A_i) \cdot 2 + 1$ and $N(rk(A_i) \cdot 2 + 1) = N(rk(A_i) \cdot 2) + 1 \leq f[n][N(\iota)](0)$. This allows us to apply (V) to the sequent (2) yielding

$$f[n][N(\iota)], E[k_\Omega(A)][\text{ord}(\iota)] \vdash_{0}^{r(k(A_i))} \Gamma, \neg A_i(s), A_i(t).$$

We can see that $rk(A_i) \cdot 2 + 1 < rk(A) \cdot 2$, max$\{N(\sigma) \mid \sigma \in k^0_\Omega(A)\} \leq f[n](0)$ and $k^0_\Omega(A) < E[k_\Omega(A)]$. Hence we can apply (A) concluding (1). \hfill \Box

Lemma 5.2 Let $B_j$ be an $\mathcal{L}_{ID_1}$-sentence for each $j = 0, \ldots, l - 1$. Suppose that $B_0 \vee \cdots \vee B_{l-1}$ is a logical consequence in the first order predicate logic with equality. Then there exists a natural $k < \omega$ such that $f[m+k], E \vdash_{0}^{r(2+k)} \Gamma, B_0, \ldots, B_{l-1}$, where $m = \max\{\{N(rk(B_j)) \mid 0 \leq j \leq l-1\} \cup \{\text{lh}(A(\cdot, *)) \mid P_{A}^{<\xi} \text{ or } \neg P_{A}^{<\xi}\text{ occurs in } B_j \text{ for some } j\}$.
Proof. Let $B_j$ be an $\mathcal{L}_{ID_1}$-sentence for each $j = 0, \ldots, l - 1$ and suppose that $B_0 \lor \cdots \lor B_{l-1}$ is a logical consequence in the first order predicate logic with equality. Then we can find a cut-free proof of the sequent $\Gamma, B_0, \ldots, B_{l-1}$ in an LK-style sequent calculus. More precisely we can find a cut-free proof $P$ of $\Gamma, B_0, \ldots, B_{l-1}$ in the sequent calculus that is known as $G_3m$. Let $h$ denote the tree height of the cut-free proof $P$. Then by induction on $h$ one can find a witnessing natural $k$ such that $f[m+k], E \vdash_{0}^{\alpha} \Gamma, B_0, \ldots, B_{l-1}$ for all $\alpha \geq \Omega + k$. In case $h = 0$ Tautology lemma (Lemma 5.1) can be applied since for any $\mathcal{L}_{ID_1}$-sentence $A$, $\text{rk}(A) \in \omega \cup \{\Omega + k | k < \omega\}$ and $k(A) \subseteq \{0, \Omega\}$, and hence $k_0(A) = \{0\}$ and max\{N(\sigma)| \sigma \in k_0(A)\} = 0.

Lemma 5.3 Let $m \in \mathbb{N}$ and $A(x)$ be an $\mathcal{L}_{ID_1}$-formula such that $\text{FV}(A(x)) = \{x\}$. Then for any $t \in T(\mathcal{L}_{ID_1})$ and for any sequent $\Gamma$ of $\mathcal{L}_{ID_1}$-sentences, if val($t$) = m, then

$$f[n + m], E \vdash_{0}^{\text{rk}(A)+m+2} \Gamma, \neg A(0), \forall x(A(x) \rightarrow A(S(x))), A(t),$$

where $n := \max(\{N(\text{rk}(A))\} \cup \{\text{lh}(A(\cdot, *)) | P_{\text{A}}^{<\xi} \text{ or } \neg P_{\text{A}}^{<\xi} \text{ occurs in } A\})$.

Proof. By induction on $m$. The base case val($t$) = m = 0 follows from Tautology lemma (Lemma 5.1). For the induction step suppose val($t$) = m + 1. Fix a sequent $\Gamma$ of $\mathcal{L}_{ID_1}$-sentences. Then (3) holds by IH. On the other hand again by Tautology lemma,

$$f[n], E \vdash_{0}^{\text{rk}(A)-2} \Gamma, \neg A(0), \exists x(A(x) \land \neg A(S(x))), A(m), \neg A(m).$$

(4)

An application of (\lor) to the two sequents (3) and (4) yields

$$f[n + m], E \vdash_{0}^{\alpha+2+1} \Gamma, \neg A(0), \exists x(A(x) \land \neg A(S(x))), A(t), A(m) \land \neg A(m),$$

The final application of (\lor) yields

$$f[n + m + 1], F \vdash_{0}^{\text{rk}(A)+m+1+2} \Gamma, \neg A(0), \exists x(A(x) \land \neg A(S(x))), A(t).$$

Lemma 5.4 Let $\xi \leq \Omega$, $F(x)$ be an $\mathcal{L}_{ID_1}$-formula such that $\text{FV}(F(x)) = \{x\}$ and $B(X)$ be an X-positive $\mathcal{L}_{PA}(X)$-formula such that $\text{FV}(B) = \emptyset$. Then

$$f[n], E[K_{\text{BF}}\xi] \vdash_{0}^{\sigma+\alpha+1} \Gamma, \neg A(F(x) \rightarrow F(x)), \neg B(P_{\text{A}}^{<\xi}), B(F),$$

where $\sigma := \text{rk}(F)$, $\alpha := \text{rk}(B(P_{\text{A}}^{<\xi}))$ and $n := \max(\{N(\sigma + \alpha + 1)\} \cup \{\text{lh}(B) | P_{B}^{<\gamma} \text{ or } \neg P_{B}^{<\gamma} \text{ occurs in } F\}).$

Proof. By main induction on $\xi$ and side induction on $\text{rk}(B(P_{\text{A}}^{<\xi}))$. Let $\text{Cl}_{\text{A}}(F)$ denote $\neg \forall x(A(F(x) \rightarrow F(x)))$. Then $\neg \text{Cl}_{\text{A}}(F) \equiv \exists x(A(F(x) \land \neg F(x)))$. The argument splits into several cases depending on the shape of the formula $B(X)$. 

CASE. $B(X)$ is an $\mathcal{L}_{PA}$-literal: In this case $B$ does not contain the set free variable $X$, and hence Tautology lemma (Lemma 5.1) can be applied. Note that the operator form $B$ does not occur in $B$.

CASE. $B \equiv X(t)$ for some $t \in \mathcal{T}(\mathcal{L}_{ID_1})$: In this case $\neg B(P_{\mathcal{A}}^{<\xi}) \equiv \bigwedge_{\eta<\xi} \neg \mathcal{A}(P_{\mathcal{A}}^{<\eta}, t)$.

CASE. $B(X) \equiv \forall y B_0(X, y)$ for some $\mathcal{L}_{PA}$-formula $B_0(X, y)$: Let $\alpha_0$ denote the ordinal $\text{rk}(B_0(P_{\mathcal{A}}^{<\xi}, Q))$. Then $\alpha = \alpha_0 + 1$. By the definition of the rank function $\text{rk}$, $\alpha_0 = \text{rk}(B_0(P_{\mathcal{A}}^{<\xi}, t))$ for all $t \in \mathcal{T}(\mathcal{L}_{ID_1})$. Fix a closed term $t \in \mathcal{T}(\mathcal{L}_{ID_1})$. Then from SIH we have the sequent

$$f[n], E[K_{\Omega\xi}] \vdash_{0}^{\sigma+\alpha_0 \cdot 2+1} \Gamma, \neg \forall y B_0(P_{\mathcal{A}}^{<\xi}, y), B_0(P_{\mathcal{A}}^{<\xi}, t).$$

An application of $(\wedge)$ allows us to conclude

$$f[n], E[K_{\Omega\xi}] \vdash_{0}^{\sigma+\alpha_0 \cdot 2+1} \Gamma, \neg \forall y B_0(P_{\mathcal{A}}^{<\xi}, y), B_0(P_{\mathcal{A}}^{<\xi}, t).$$

The other cases can be treated in similar ways. \hfill \square

Lemma 5.5 1. $f[n], E[\llbracket (\mathcal{A}(P_{\mathcal{A}}^{<\Omega}, x) \rightarrow P_{\mathcal{A}}^{<\Omega}(x)) \rrbracket_{0}^{\Omega \cdot 2+\omega}, \forall x(A(P_{\mathcal{A}}^{<\Omega}, x) \rightarrow P_{\mathcal{A}}^{<\Omega}(x))$, where $n := \max\{N(\text{rk}(\mathcal{A}(P_{\mathcal{A}}^{<\Omega}, Q))), \text{lh}(\mathcal{A}(P_{\mathcal{A}}^{<\Omega}, Q))\}$
2. $f[3 + l], E \vdash _0^{\Omega \cdot 2 + \omega} \Gamma, \forall \vec{y} \forall x\{\mathcal{A}(F(\cdot, \vec{y}), x) \rightarrow F(x, \vec{y})\} \rightarrow \forall x\{P_{A_{\vec{y}}}^{\Omega}(x) \rightarrow F(x, \vec{y})\}$, where $\vec{y} = y_0, \ldots, y_{l-1}$.

**Proof.** 1. Let $\alpha = \text{rk}(A(P_{A_{\vec{y}}}^{\Omega}, 0))$ and $t \in \mathcal{T}(\mathcal{L}_{ID_1})$. By the definition of $\text{rk}$ we can find a natural $k \leq \text{lh}(A(P_{A_{\vec{y}}}^{\Omega}, 0))$ such that $\alpha = \text{rk}(A(P_{A_{\vec{y}}}^{\Omega}, t)) = \Omega + k$. This implies $k(A(P_{A_{\vec{y}}}^{\Omega}, t)) = \{0, \Omega\}$ and hence $k_0(A(P_{A_{\vec{y}}}^{\Omega}, t)) = \{0\} < E(0)$. By Tautology lemma (Lemma 5.1),

$$f[n], E \vdash _0^{\Omega \cdot 2 + k} \Gamma, P_{A_{\vec{y}}}^{\Omega}(t), \neg A(P_{A_{\vec{y}}}^{\Omega}, t), A(P_{A_{\vec{y}}}^{\Omega}, t).$$

Since $\Omega < \Omega \cdot 2 + k + 1$, we can apply the closure rule (C1) obtaining the sequent

$$f[n], E \vdash _0^{\Omega \cdot 2 + k + 1} \Gamma, \neg A(P_{A_{\vec{y}}}^{\Omega}, t), P_{A_{\vec{y}}}^{\Omega}(t).$$

An application of $(\Lambda)$ followed by an application of $(\vee)$ enables us to conclude

$$f[n], E \vdash _0^{\Omega \cdot 2 + \omega} \Gamma, \forall x\{A(P_{A_{\vec{y}}}^{\Omega}, x) \rightarrow P_{A_{\vec{y}}}^{\Omega}(x)\}.$$  

2. By definition $\text{rk}(A(P_{A_{\vec{y}}}^{\Omega})) = \omega \cdot \Omega = \Omega$, On the other hand $\text{rk}(F) < \omega$ and hence $(\text{rk}(F) + \text{rk}(P_{A_{\vec{y}}}^{\Omega}) + 1) \cdot \omega = \Omega \cdot 2 + 2$. Let $s, t = s, t_0, \ldots, t_{l-1} \in \mathcal{T}(\mathcal{L}_{ID_1})$. Then by the previous lemma (Lemma 5.4)

$$f[2], E \vdash _0^{\Omega \cdot 2 + 1} \neg \forall x\{A(F(\cdot, \vec{t}), x) \rightarrow F(x, \vec{t})\}, \neg P_{A_{\vec{y}}}^{\Omega}(t), F(s, \vec{t})$$

since $N(\Omega + 1) = 2$. It is not difficult to see that applications of $(\forall)$, $(\Lambda)$ and $(\vee)$ in this order yield the sequent

$$f[3], E \vdash _0^{\Omega \cdot 2 + 5} \forall x\{A(F(\cdot, \vec{t}), x) \rightarrow F(x, \vec{t})\} \rightarrow \forall x\{P_{A_{\vec{y}}}^{\Omega}(x) \rightarrow F(x, \vec{t})\}$$

Finally, $l$-fold application of $(\Lambda)$ allows us to conclude.  

Let us recall that $s$ denotes the numerical successor $m \mapsto m + 1$.

**Theorem 5.6** Let $A \equiv \forall \vec{z} \exists y B(\vec{z}, y)$ be a $\Pi^0_2$-sentence for a $\Delta^0_0$-formula $B(\vec{z}, y)$ such that $\text{FV}(B(\vec{z}, y)) = \{\vec{z}, y\}$. If $\text{ID}_1 \vdash A$, then we can find an ordinal term $\alpha \in OT(\mathcal{F}) \uparrow \Omega$ built up without the Veblen function symbol $\varphi$ such that for all $\overline{m} = m_0, \ldots, m_{l-1} \in \mathbb{N}$ there exists $n \leq s^\alpha(m_0 + \cdots + m_{l-1})$ such that $B(\overline{m}, n)$ is true in the standard model $\mathbb{N}$ of $PA$.

**Proof.** Assume $\text{ID}_1 \vdash A$. Then there exist $\text{ID}_1$-axioms $A_0, \ldots, A_{k-1}$ such that $(\neg A_0) \lor \cdots \lor (\neg A_{k-1}) \lor A$ is a logical consequence in the first order predicate logic with equality. Hence by Lemma 5.2,

$$f[c_0], E \vdash _0^{\Omega^3} \neg A_0, \ldots, \neg A_{k-1}, A$$

for some constant $c_0 < \omega$ depending on $N(\text{rk}(A_0)), \ldots, N(\text{rk}(A_{k-1})), N(\text{rk}(A))$ and max$\{\text{lh}(A(\cdot, *)) \mid P_{A_{\vec{y}}}^{\xi} \text{ or } P_{A_{\vec{y}}}^{\xi} \text{ occurs in } A_j \text{ or } A\}$, and depending also on the tree height of a cut-free $\mathcal{L}_K$-derivation of the sequent $\neg A_0, \ldots, \neg A_{k-1}, A$. By Lemma 5.3 and 5.5, for each $j \leq k - 1$, there exists a constant $c_j$ depending on $\text{rk}(A_j)$ such that $f[c_j], E \vdash _0^{\Omega \cdot 2 + \omega} A_j$. Hence $k$-fold application of $(\text{Cut})$ yields $f[c], E \vdash _0^{\Omega^3} A$.  

\[\square\]
where \( c := \max(\{ k \} \cup \{ c_j \mid j \leq k - 1 \} \cup \{ \text{lh}(A_j) \mid j \leq k - 1 \}) \) and \( d := \max(\{ \Omega, \text{rk}(A_0), \ldots, \text{rk}(A_{k-1}) \}) \).

For each \( n \in \mathbb{N} \) and \( \alpha \in \mathcal{O}(\mathcal{F}) \) let us define ordinal \( \Omega_n(\alpha) \) and \( \gamma_n \) by

\[
\Omega_0(\alpha) = \alpha, \quad \gamma_0 = \Omega \cdot 3, \\
\Omega_{n+1}(\alpha) = \Omega^{\Omega_n(\alpha)}, \quad \gamma_{n+1} = E^{\gamma_n}(0) + 1.
\]

Then \( d \)-fold iteration of Cut-reduction lemma (Lemma 4.13) yields the sequent \( f[c]^\gamma, E \vdash_{\Omega+1}^{\Omega_d(\Omega \cdot 3)} A \). Hence Impredicative cut-elimination lemma (Lemma 4.18) yields \( (f[c]^\gamma)^{E^{\Omega_d(\Omega \cdot 3)}(0)}, E^{\Omega_d(\Omega \cdot 3)+1} \vdash_{0}^{E^{\Omega_d(\Omega \cdot 3)}(0)} A \).

Let \( F := E^{\Omega_d(\Omega \cdot 3)+1} \) and \( \beta := E^{\Omega_d(\Omega \cdot 3)}(0) \). Then \( (f[c]^\gamma)^{\beta}, F \vdash_{\omega^\beta}^{\beta} \).

Thanks to Lemma 4.8 we can find an ordinal \( \alpha \in \mathcal{O}(\mathcal{F}) \vdash \Omega \) built up without the Veblen function symbol \( \varphi \) such that

\[
((s^{\omega+c+1})^\gamma)^{F^{\Omega_d(\varphi\beta\beta)(+1)}(0)} \leq s^\alpha(0).
\]

This together with \( (l\text{-fold application of}) \) Inversion lemma (Lemma 4.12) yields the sequent

\[
s^\alpha[m_0] \cdots [m_{l-1}], F \vdash_{0}^{\varphi\beta\beta} \exists y B(\vec{m}, y),
\]

where \( \vec{m} = m_0, \ldots, m_{l-1} \). By Witnessing lemma (Lemma 4.19) we can find a natural \( n \leq s^\alpha[m_0] \cdots [m_{l-1}](0) = s^\alpha(m_0 + \cdots + m_{l-1}) \) such that \( B(\vec{m}, n) \) is true in the standard model \( \mathbb{N} \) of PA.

We say a function \( f \) is elementary (in another function \( g \)) if \( f \) is definable explicitly from the successor \( s \), projection, zero \( 0 \), addition \( + \), multiplication \( \cdot \), cut-off subtraction \( \vdash \) (and \( g \)), using composition, bounded sums and bounded products.

**Corollary 5.7** Every function provably computable in \( \text{ID}_1 \) is elementary in \( \{ s^\alpha \mid \alpha \in \mathcal{O}(\mathcal{F}) \vdash \Omega \} \).
6 A computable ordinal notation system $\mathcal{O}(\Omega)$

In order to obtain a precise characterisation of the provably computable functions of ID$_1$, we introduce a computable ordinal notation system $(\mathcal{O}(\Omega), <)$. Essentially $\mathcal{O}(\Omega)$ is a subsystem of $\mathcal{OT}(\mathcal{F})$.

**Definition 6.1** We define three sets $\mathcal{SC} \subseteq \mathbb{H} \subseteq \mathcal{O}(\Omega)$ of ordinal terms simultaneously. Let $0$, $\Omega$, $S$, and $+$ be distinct symbols.

1. $0 \in \mathcal{O}(\Omega)$ and $\Omega \in \mathcal{SC}$.
2. If $\alpha \in \mathcal{OT}(\mathcal{F}) \upharpoonright \Omega$, then $S(\alpha) \in \mathcal{O}(\Omega)$.
3. If $\{\alpha_1, \ldots, \alpha_l\} \subseteq \mathbb{H}$ and $\alpha_1 \geq \cdots \geq \alpha_l$, then $\alpha_1 + \cdots + \alpha_l \in \mathcal{O}(\Omega)$.
4. If $\alpha \in \mathcal{O}(\Omega)$, then $\omega^\alpha \in \mathbb{H}$.
5. If $\alpha \in \mathcal{O}(\Omega)$ and $\xi \in \mathcal{O}(\Omega) \upharpoonright \Omega$, then $\Omega^\alpha \cdot \xi \in \mathbb{H}$.
6. If $\alpha \in \mathcal{O}(\Omega)$ and $\xi \in \mathcal{O}(\Omega) \upharpoonright \Omega$, then $S^\alpha(\xi) \in \mathcal{SC}$.

The relation $<$ on $\mathcal{O}(\Omega)$ is defined in the obvious way. One will see that $\mathcal{O}(\Omega)$ is indeed a computable ordinal notation system. Let us define the norm $N(\omega^\alpha)$ of $\omega^\alpha$ in the most natural way, i.e., $N(\omega^\alpha) = N(\alpha) + 1$.

**Lemma 6.2** Let $\alpha$ denote an ordinal term built up in $\mathcal{OT}(\mathcal{F})$ without the Veblen function symbol $\varphi$. Then there exists an ordinal term $\alpha' \in \mathcal{O}(\Omega)$ such that $\alpha \leq \alpha'$ and $N(\alpha) \leq N(\alpha')$.

**Proof.** By induction over the term construction of $\alpha \in \mathcal{OT}(\mathcal{F})$. In the base case let us observe that $E(\alpha) \leq S^1(\alpha)$ for all $\alpha < \Omega$ and that $N(E(\alpha)) = N(\alpha) + 1 < N(S(\alpha)) + 1 = N(S^1(\alpha))$. In the induction case we employ Lemma 3.11.

**Lemma 6.3** For any ordinal term $\alpha \in \mathcal{OT}(\mathcal{F})$ built up without the Veblen function symbol $\varphi$ there exists an ordinal term $\alpha' \in \mathcal{O}(\Omega)$ such that $s^\alpha(m) \leq s^{\alpha'}(m)$ for all $m$.

**Corollary 6.4** A function is provably computable in ID$_1$ if and only if it is elementary in $\{s^\alpha \mid \alpha \in \mathcal{O}(\Omega) \upharpoonright \Omega\}$.

The “only if” direction follows from Corollary 5.7 and Lemma 6.3. The “if” direction can be seen as follows. One can show that for each $\alpha \in \mathcal{O}(\Omega) \upharpoonright \Omega$ the system ID$_1$ proves that the initial segment $(\mathcal{O}(\Omega) \upharpoonright \alpha, <)$ of $(\mathcal{O}(\Omega), <)$ is a well-ordering. For the full proof, we kindly refer the readers to, e.g., Pohlers [11, §29]. From this one can show that for each $\alpha \in \mathcal{O}(\Omega) \upharpoonright \Omega$ the function $s^\alpha$ is provably computable in ID$_1$, and hence the assertion.
7 A quick proof-theoretic analysis of ID\(_1\)

In the final section we show that the collapsing function \(F : \Omega_1 \times \varepsilon_{\Omega_1} \rightarrow \Omega_1\); \((\xi, \alpha) \mapsto F^{\varepsilon}(\xi)\) can be used for a smooth proof-theoretic analysis of ID\(_1\). Suppose a positive operator form \(A\). Let \(\Phi_A : \mathcal{P}(N) \rightarrow \mathcal{P}(N)\) denote the operator induced by the operator form \(A\). Namely \(\Phi_A(X) = \{n \in N \mid N \models A(X, n)\}\) if \(X \subseteq N\). By positiveness of \(A\) the operator \(\Phi_A\) is monotone, i.e., \(X \subseteq Y \Rightarrow \Phi_A(X) \subseteq \Phi_A(Y)\), and hence \(\Phi_A\) has the least fixed point \(I_{\Phi_A}\) that corresponds to the predicate \(P_A\). Further, for an ordinal \(\alpha\), let \(I_{\Phi_A}^\alpha\) denote the \(\alpha\)-th stage of iterating \(\Phi_A\). More precisely, corresponding to the predicate \(P_A^\alpha\), \(I_{\Phi_A}^\alpha\) is defined by \(I_{\Phi_A}^0 = \emptyset\) and \(I_{\Phi_A}^\alpha = \Phi_A(\bigcup_{\xi<\alpha} I_{\Phi_A}^\xi)\) (\(0 < \alpha\)). Recall that \(\Omega_1\) denotes the least non-computable ordinal \(\omega^\text{CK}\). From an elementary fact in generalised recursion theory, it is known that \(I_{\Phi_A}^\alpha\) is consumed at \(\alpha = \Omega_1\), i.e., \(I_{\Phi_A}^{\Omega_1} = I_{\Phi_A}\). The norm \(|n|_{\Phi_A}\) of a natural number \(n\) is defined by \(|n|_{\Phi_A} = \min\{\alpha \in \text{Ord} \mid n \in I_{\Phi_A}^\alpha\}\). It is natural to ask what can be said about the norm \(|n|_{\Phi_A}\) in case that \(ID_1 \vdash P_A(\bar{n})\) holds.

An elegant proof-theoretic way to answer this question can be found in lecture notes [4] by W. Buchholz. (See [4, Theorem 9.19].) By slightly modifying the exposition in [4] we present an alternative simplified way to answer this question.

In contrast to the infinitary system ID\(_1^\infty\) we investigate the associated semiformal system ID\(_1^*\) which is modelled following the lecture notes [4]. As until the previous section we will identify each element \(\alpha \in \mathcal{O}\mathcal{T}(\mathcal{F})\) with its value \([\alpha] \in \text{Ord}\), e.g., \(\Omega \in \mathcal{O}\mathcal{T}(\mathcal{F})\) with \(\Omega_1 \in \text{Ord}\). We also follow a convention that \(F : \Omega \rightarrow \Omega\) denotes a weakly increasing function such that \(\xi < F(\xi)\) for all \(\xi \in \Omega\). Further in this section we use an additional convention that \(\omega^F(\xi) = F(\xi)\), and hence \(E(\xi) \leq F(\xi)\) for all \(\xi \in \Omega\). (Recall \(E(\xi) = \min\{\xi \in \text{Ord} \mid \omega^F = \xi \land \alpha < \xi\}\) and \(\omega^F(\xi) = F(\xi)\) for all \(\xi < \Omega\).) Let us recall that for a sequent \(\Gamma\), \(k^\Omega_{\mathcal{I}}(\Gamma)\) denotes the set \(\bigcup_{B \in \Gamma} k^\Omega_{\mathcal{I}}(B)\).

**Definition 7.1** \(F \vdash^\rho \Gamma\) if \(k^\Omega_{\mathcal{I}}(\Gamma) \cup K_G \alpha < F(0)\) and one of the following holds.

(Ax1) \(\exists A(x)\): an \(\mathcal{L}_{ID_1}\)-literal, \(\exists s, t \in \mathcal{T}(\mathcal{L}_{ID_1})\) s.t. \(FV(A) = \{x\}\), \(\text{val}(s) = \text{val}(t)\) and \(\{\neg A(s), A(t)\} \subseteq \Gamma\).

(Ax2) \(\Gamma \cap \text{TRUE}_\rho \neq \emptyset\).

(V) \(\exists A \simeq \bigvee_{v \in J} A_v \in \Gamma\), \(\exists \alpha_0 < \alpha, \exists \iota_0 \in J\) s.t. \(\text{ord}(\iota_0) < F(0)\), and \(F \vdash^\alpha_0 \Gamma, A_{\iota_0}\).

(\wedge) \(\exists A \simeq \bigwedge_{v \in J} A_v \in \Gamma\) s.t. \((\forall t \in J) (\exists \alpha < \alpha) F[\text{ord}(t)] \vdash^\alpha_0 \Gamma, A_{t}\).

(Cl) \(\exists t \in \mathcal{T}(\mathcal{L}_{ID_1})\), \(\exists \alpha_0 < \alpha\) s.t. \(P_A^{<\Omega}(t) \in \Gamma\) and \(F \vdash^\alpha_0 \Gamma, A(P_{\mathcal{A}}^{<\Omega}, t)\).

(Cut) \(\exists C\): an \(\mathcal{L}^*\)-sentence of \(\mathcal{V}\)-type, \(\exists \alpha_0 < \alpha\) s.t. \(\text{rk}(C) < \rho, F \vdash^\alpha_0 \Gamma, C, \) and \(F \vdash^\alpha_0 \Gamma, \neg C\).

**Lemma 7.2 (Inversion)** Assume that \(A \simeq \bigwedge_{v \in J} A_v\). If \(F \vdash^\alpha_0 \Gamma, A\), then \(F[\text{ord}(t)] \vdash^\alpha_0 \Gamma, A_{t}\) for all \(t \in J\).

**Proof.** By induction on \(\alpha\). \(\square\)
Lemma 7.3 (Cut-reduction) Assume that $C \simeq \bigvee_{i \in J} C_i$ and $\text{rk}(C) = \Omega + k + 1$. If $F \vdash_{\Omega + k + 1}^\alpha \Gamma, \neg C$ and $F \vdash_{\Omega + k + 1}^\beta \Gamma, C$, then $F \vdash_{\Omega + k + 1}^{\alpha + \beta} \Gamma$.

Proof. By induction on $\beta$. □

Lemma 7.4 (Cut-elimination) Let $k < \omega$. If $F \vdash_{\Omega + k + 2}^\alpha \Gamma$, then $F \vdash_{\Omega + k + 1}^\alpha \Gamma$.

Lemma 7.5 $F[\xi]^\alpha(\xi) \leq F^\alpha(\xi)$.

Proof. By induction on $\alpha$. □

Lemma 7.6 If $\eta < \xi$ and $\alpha_\eta < \alpha$ and $K\alpha_\eta < F[\eta](0)$ then $F[\eta]^\alpha_\eta(\xi) \leq F^\alpha(\xi)$.

Lemma 7.7 If $\eta < F(0)$ and $\alpha_\eta < \alpha$ and $K\alpha_\eta < F[\eta](0)$ then $F[\eta]^\alpha_\eta(\xi) \leq F^\alpha(\xi)$.

Definition 7.8 For each $\mathcal{L}^*$-formula $B$ let $B^{\alpha,\beta}$ denote the result of replacing in $B$ every negative occurrence of $P_A^\alpha$ by $P_A^\alpha$ and every positive occurrence of $P_A^{<\Omega}$ by $P_A^{<\beta}$. For each sequent $\Gamma$ consisting of $\mathcal{L}^*$-formulas let $\Gamma^{\alpha,\beta} := \{B^{\alpha,\beta} \mid B \in \Gamma\}$. It is known that, viewing $\text{ID}_1$ as a subsystem of set theory in a standard way, $L_\Omega \models \text{ID}_1$ holds for the $\Omega$th stage $L_\Omega$ of the constructible hierarchy $(L_\alpha)_{\alpha \in \text{Ord}}$. We will just write $\models B$ ($B$ is an $\mathcal{L}^*$ sentence) or $\models \Gamma$ ($\Gamma$ is an $\mathcal{L}^*$ sequent) to refer to this relation if no confusion arises.

Theorem 7.9 (Witnessing) If $F \vdash_{\Omega + 1}^\alpha \Gamma$, then $\models \Gamma^{\Omega,\text{rk}(\xi)}$ for all $\xi < \Omega$.

Proof. By induction on $\text{rk}(\alpha)$. □

In embedding $\text{ID}_1$ into $\text{ID}_1^*$, we follow (very closely) the exposition in the lecture notes [4] and indicate how the operators can be adapted accordingly. As in case of embedding $\text{ID}_1$ into $\text{ID}_1^\infty$, the condition $\text{HYP}(E)$ on page 10 holds.

Lemma 7.10 (Tautology lemma) Let $s, t \in \mathcal{T}(\mathcal{L}_{\text{ID}_1})$, $\Gamma$ a sequent of $\mathcal{L}^*$-sentences, and $A(x)$ be an $\mathcal{L}^*$-formula such that $\text{FV}(A) = \{x\}$. If $\text{val}(s) = \text{val}(t)$, then $F \vdash_{0}^{\text{rk}(A)^2} \Gamma, \neg A(s), A(t)$, provided $k^{\Pi}_{11}(\Gamma) \cup k^{\Pi}_{11}(A) < F(0)$.

Proof. By induction on $\text{rk}(A)$. □

Lemma 7.11 Let $B_j$ be an $\mathcal{L}_{\text{ID}_1}$-sentence for each $j = 0, \ldots, l - 1$. Suppose that $B_0 \vee \cdots \vee B_{l-1}$ is a logical consequence in the first order predicate logic with equality. Then there exists a natural $k < \omega$ such that $F \vdash_{0}^{\Omega + 2 + k} \Gamma, B_0, \ldots, B_{l-1}$, provided $k^{\Pi}_{11}(\Gamma) < F(0)$.

This can be shown like Lemma 5.2.
Lemma 7.12 Let $m \in \mathbb{N}$ and $A(x)$ be an $\mathcal{L}_{ID_{1}}$-formula such that $FV(A(x)) = \{x\}$. Then for any $t \in T(\mathcal{L}_{ID_{1}})$ and for any sequent $\Gamma$ of $\mathcal{L}_{ID_{1}}$-sentences

$$F \vdash_{0}^{(rk(A)+val(t)-2)} \Gamma, \neg A(0), \forall x(A(x) \rightarrow A(S(x))), A(t),$$

provided $k_{\Omega}^{\Pi}(\Gamma) \cup k_{\Omega}^{\Pi}(A) < F(0)$.

**Proof.** By induction on val$(t)$.

Lemma 7.13 Let $\xi \leq \Omega$, $A(x)$ be an $\mathcal{L}_{ID_{1}}$-formula such that $FV(A(x)) = \{x\}$ and $B(X)$ be an $X$-positive $\mathcal{L}_{PA}(X)$-formula such that $FV(A) = \emptyset$. Then

$$F \vdash_{0}^{(rk(A)+\alpha+1)-2} \Gamma, \forall x(A(A, x) \rightarrow A(x)), \neg B(P_{\mathcal{A}}^{<\xi}), B(A),$$

provided $k_{\Omega}^{\Pi}(\Gamma) \cup k_{\Omega}^{\Pi}(A) \cup \{ord(\xi)\} < F(0)$ where $\alpha := rk(B(P_{\mathcal{A}}^{<\xi}))$.

**Proof.** By induction on rk$(B(P_{\mathcal{A}}^{<\xi}))$.

Lemma 7.14

1. $F \vdash_{0}^{\Omega+\omega} \Gamma, \forall x(A(P_{\mathcal{A}}^{\Omega}, x) \rightarrow P_{\mathcal{A}}^{\Omega}(x))$, provided $k_{\Omega}^{\Pi}(\Gamma) < F(0)$.

2. $F \vdash_{0}^{\Omega+2+\omega} \Gamma, \forall \vec{y}[\forall x(A(B(\cdot, \vec{y}), x) \rightarrow B(x, \vec{y})] \rightarrow \forall x(P_{\mathcal{A}}^{\Omega}(x) \rightarrow B(x, \vec{y}))$, provided $k_{\Omega}^{\Pi}(\Gamma) \cup k_{\Omega}^{\Pi}(B) < F(0)$.

Let us recall that $S$ denotes the ordinal successor.

Theorem 7.15 Let $n \in \mathbb{N}$. If $ID_{1} \vdash P_{\mathcal{A}}(n)$, then there exists an ordinal $\alpha < \varepsilon_{\Omega+1}$ such that $|n|_{\mathcal{A}} < S^{\alpha}(0)$.

Note that the latter bound is sharp in the sense that for each $\alpha < S^{\varepsilon_{\Omega+1}}(0) := \sup\{S^{\varepsilon_{m}(\Omega+1)}(0) \mid m < \omega\}$ there exists an operator form $\mathcal{A}$ and a natural number $n$ such that $ID_{1} \vdash P_{\mathcal{A}}(n)$ and $\alpha \leq |n|_{\mathcal{A}}$.

8 Conclusion

In [13] the second author has started a new approach to provably total computable functions, providing a streamlined characterisation of those functions provably computable in PA. In this work we extend this approach to those functions provably computable in the system $ID_{1}$ of non-iterated inductive definitions. The approach introduced in this work should be extended to stronger impredicative systems. The obvious next step is to extension to the system $ID_{2}$ of an iterated inductive definitions. This extension seems to be made possible by employing an additional ordinal operator, i.e., $f, F_{0}, F_{1} \vdash_{\rho}^{\Omega} \Gamma$ where $F_{0}$ is an ordinal function $F_{0} : \Omega_{1} \rightarrow \Omega_{1}$, $F_{1}$ is another ordinal function $F_{1} : \Omega_{2} \rightarrow \Omega_{2}$, and $\Omega_{2}$ denotes the least recursively regular ordinal above $\Omega_{1}$.
References


