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Values on generalized reachability games

Ahmad Termimi Ab Ghani, Kojiro Higuchi and Kazuyuki Tanaka
Mathematical Institute, Tohoku University, Sendai, 980-8578, Japan
{sa9d12,sa7m24,tanaka}@math.tohoku.ac.jp

Abstract

In this study, we consider two-player (simultaneous) stochastic games on finite graphs in which each player chooses an action at every state, being unaware of the choice of the other. We will prove some interesting facts about generalized stochastic reachability games. In particular, we show that there exists a memoryless randomized optimal strategy for Player II in this game, while the same thing does not hold for Player I. Our main contribution in this paper is a proof of the existence of a memoryless ϵ-optimal strategy for Player I in any generalized reachability games. Actually, this result for reachability games was shown by Chatterjee et al. [5] in a slightly different setting. Beforehand, we show that the generalized reachability game is determinate, and give a simple expression of values for this game by defining the notion of a limit value of finite-step games.

1 Introduction

Firstly, we give a brief explanation of this game. For each round of a game, Player I and Player II choose their actions simultaneously and then the next state is determined. A finite or infinite sequence of states obtained is the result of a play. We investigate a generalized reachability game, where the goal of Player I is to force the plays to reach a specified set of target states with a higher expected value, and the objective of the opponent is to prevent it. In a reachability game, we assign value 1 to any plays that reach the target states, and value 0 otherwise. However, in generalized reachability games, we define a label function, and this can be seen as a weighted reachability payoff function which assigns to every infinite play either 0 if any target state is not visited in a play, or the reward (positive real number) of the first target state visited by the players. These can be regarded as zero-sum games, and the reachability objective is one of the most basic objectives among the Borel objectives. We are concerned with the highest expected value that Player I can achieve against any strategy of the opponent. Similarly, we also discuss the lowest expected value that the Player II can achieve against any strategy of Player I. If these two quantities are equal, we call them the values of the game and say the game is determined. An optimal strategy of Player I is a strategy that guarantees the value of the game from each position. Finally, an

ϵ-optimal strategy of Player I is a strategy that satisfies the objective within an ϵ error of the value of the game.

1.1 Related Works and Motivations

In 1953, Gale and Stewart [11] introduced the theory of infinite games, which are two-player infinite games with perfect information. This game has been investigated by many mathematicians and logicians, and until now, it is one of the major topics in game theory and mathematical logic. The determinacy result for turn-based games with Borel objectives was established as a deep result by Martin [14]. On the other hand, the determinacy for one-round simultaneous game was proved by von Neumann [15] by using his famous minimax theorem. Infinite versions of von Neumann's games were introduced by David Blackwell [1]. The determinacy for such games with Borel objectives was established by Martin [13].

Recently, Jan Krcǎl [12] studied the determinacy of stochastic turn-based games focused on some winning objectives. He mainly discussed reachability games and showed that the games are determined for both finite and infinite games. In finite reachability games, both players have memoryless and deterministic optimal strategies. In the case of infinite games with finite branching, only Player II has an optimal strategy (memoryless and deterministic). In the games with infinite branching, none of the players have an optimal strategy in general. He also considered a safety game which is a dual game of reachability games, and hence the results are dual to reachability case as well.

Inspired by his work, we try to investigate stochastic games with imperfect information and analyze the strategies of the players. Most of previous results related to this topic were obtained with use of payoff functions. Over the stochastic games on graphs, typical payoff functions are limit-average (also called mean-payoff) and discounted payoff. To know the definition of mean payoff functions, see [10], [8], and [16]; for discounted payoff, see [10] and [6]. Besides their simple definitions, these two payoff functions enjoy the property that memoryless optimal strategies always exist, especially in turn-based stochastic games. In [2], they introduced a multi-mean payoff on a turn-based stochastic parity game. Their work can be seen as an extension of [3] where mean-payoff parity games have been studied. While Chatterjee et al. [4] defined another simple payoff function, which contains both the limit-average and the discounted sum functions in two-player turn-based games on a graph.

Simultaneous stochastic games on graphs are more difficult to analyze rather than turn-based games. In simultaneous games, optimal strategies may not exist, but for every real $\epsilon > 0$, there always exists a strategy that satisfies a winning outcome with an expected value that lies within ϵ of the optimal value. The existence of memoryless ϵ-optimal strategies for simultaneous stochastic games with reachability and safety objectives for all $\epsilon > 0$ were shown in [10]. However, the proof of this fact is rather complex. A more specialized result of the existence of memoryless ϵ-optimal strategies is shown in [9] but again, the proof used deep results from analysis. Chatterjee et al. [5] showed that the existence of memoryless ϵ-optimal strategies can be established by a different way, where their proof relies on combinatorial techniques and uses properties of a Markov decision process. In particular, their proof is built upon a value-iteration scheme that converges to the value of the game. Simultaneous reachability games have been studied more specifically in [7], where they extended their study to ω-regular objectives and provided algorithms for computing the sets of winning states for each type of games.

By following their series of studies, we present stochastic (simultaneous) games on graphs with generalized version of reachability objectives. The aim of this research is to clarify the existence of the values of games by using a weighted reachability payoff function as we stated earlier. In particular, we want to express the values of games in a more specific way. Although we know that for every general class of games (i.e. Blackwell games), the determinacy and optimal value results hold for all Borel objectives [13], our study provides a more specific proof for stochastic games and may give a deep intuitive understanding into this topic, especially games on graphs. Moreover, we investigate what type of optimal (ϵ-optimal) strategy exists for each player.

As a result, we show that a generalized reachability game is determinate. In particular, we give a simple expression of the values of this game by defining the notion of the limit values of finite-step games. Our main results show the existence of memoryless optimal strategy for Player II in this game and prove the existence of a memoryless ϵ-optimal strategy for Player I in any generalized reachability games.

2 Games

We begin our formal discussion by defining some basic concepts and descriptions of games that are necessary for understanding the argument presented in subsequent sections.

Definition 2.1. *A (two-player simultaneous infinite) stochastic game is a quadruple $\mathbb{G} = (S, A_{\mathrm{I}}, A_{\mathrm{II}}, \delta)$, where S, A_{I} and A_{II} are nonempty finite sets and δ is a function from $S \times A_{\mathrm{I}} \times A_{\mathrm{II}}$ into S. Elements of S are called states. Elements of A_{I} are called actions or moves of Player I. Similarly, elements of A_{II} are called actions or moves of Player II. δ is called a transition function.*

Definition 2.2. *A path or a play of a game $\mathbb{G} = (S, A_{\mathrm{I}}, A_{\mathrm{II}}, \delta)$ is a finite or infinite sequence $s_0 s_1 s_2 ...$ of states in S such that for all $n \in \mathbb{N}$, there exist $a_n \in A_{\mathrm{I}}$ and $b_n \in A_{\mathrm{II}}$ where $\delta(s_n, a_n, b_n) = s_{n+1}$. Infinite paths of $\mathbb{G}$ are sometimes called runs. We write $\Omega(\mathbb{G})$ for the set of all infinite plays; and $\Omega^{\mathrm{fin}}(\mathbb{G})$ for the set of all finite plays of non-zero length. Sometimes we write Ω or Ω^{fin} instead of $\Omega(\mathbb{G})$ or $\Omega^{\mathrm{fin}}(\mathbb{G})$ when $\mathbb{G}$ is clear from the context.*

Intuitively, given a game $\mathbb{G} = (S, A_{\mathrm{I}}, A_{\mathrm{II}}, \delta)$, a function $F : \Omega(\mathbb{G}) \to [0, 1]$ and a state $s \in S$, we imagine the following infinite game $\mathbb{G}_s(F)$: at stage $n \in \mathbb{N} \setminus \{0\}$, we have the finite part of a play $w \restriction n$ with $w(0) = s$, and each player selects their actions $a_{\mathrm{I}} \in A_{\mathrm{I}}$

and $a_{\mathrm{II}} \in A_{\mathrm{II}}$, simultaneously, and then, the next state $w(n) = \delta(w(n-1), a_{\mathrm{I}}, a_{\mathrm{II}})$ is determined. In this case the value of the play w is $F(w)$. We assume that Player I wants to maximize the value, whereas Player II wants to minimize. For a subset X of $\Omega(\mathbb{G})$, the infinite game $\mathbb{G}_s(X)$ is defined in the same way considering X as its characteristic function. Thus, in the case of a set X instead of a function F, Player I wants to put w into X, whereas Player II wants to avoid it.

2.1 Strategies and expected values

The notion of strategies plays an important role in infinite games. Informally, a strategy for a player in the game is a rule that specifies the next move of the player for a given finite play.

For a set A, a *probability distribution* on A is a function $\mu : A \to [0,1]$ with $\sum_{a \in A} \mu(a) = 1$. We use $\mathcal{D}(A)$ for the set of all probability distributions on A.

A randomized strategy is a rule that chosen among the pure strategies at random in various proportions (sometimes we called mixed strategy). We formalize this notion in the following definition.

Definition 2.3. *Let $\mathbb{G} = (S, A_{\mathrm{I}}, A_{\mathrm{II}}, \delta)$ be a game. A (randomized) strategy of Player I in $\mathbb{G}$ is any function $\sigma : \Omega^{\mathrm{fin}}(\mathbb{G}) \to \mathcal{D}(A_{\mathrm{I}})$. We write $\Sigma_{\mathrm{I}}^{\mathbb{G}}$ or Σ_{I} for the set of all strategies of Player I. Similary, a (randomized) strategy of Player II in $\mathbb{G}$ is any function $\tau : \Omega^{\mathrm{fin}}(\mathbb{G}) \to \mathcal{D}(A_{\mathrm{II}})$, and we write $\Sigma_{\mathrm{II}}^{\mathbb{G}}$ or Σ_{II} for the set of all strategies of Player II.*

Intuitively, for a given finite play, memoryless strategies give the next action depending on the current state rather than the finite play itself.

Definition 2.4. *Let $\mathbb{G} = (S, A_{\mathrm{I}}, A_{\mathrm{II}}, \delta)$ be a game. A strategy σ of Player I is called memoryless if $\sigma(p) = \sigma(q)$ holds whenever $p, q \in \Omega^{\mathrm{fin}}(\mathbb{G})$ satisfy $p(|p|-1) = q(|q|-1)$. A memoryless strategy of Player II is defined similarly. We write $\Sigma_{\mathrm{I}}^{\mathrm{M}}$ and $\Sigma_{\mathrm{II}}^{\mathrm{M}}$ for the set of all memoryless strategies of Player I and Player II, respectively.*

Clearly, given a memoryless strategy $\sigma \in \Sigma_{\mathrm{I}}^{\mathrm{M}}$, there exists *the* function $\sigma' : S \to \mathcal{D}(A_{\mathrm{I}})$ such that $\sigma(ps) = \sigma'(s)$ holds for any $ps \in \Omega^{\mathrm{fin}}(\mathbb{G})$ with $s \in S$. We sometimes identify σ with σ'. Similar identification will be used for Player II.

A pair $(\sigma, \tau) \in \Sigma_{\mathrm{I}} \times \Sigma_{\mathrm{II}}$ and a state $s \in S$ determine a probability measure $P_s^{\sigma,\tau}$ on $\Omega_s = \{w \in \Omega : w(0) = s\}$ as follows.

Definition 2.5. *Let $\mathbb{G} = (S, A_{\mathrm{I}}, A_{\mathrm{II}}, \delta)$ be a game. For a pair $(\sigma, \tau) \in \Sigma_{\mathrm{I}}^{\mathbb{G}} \times \Sigma_{\mathrm{II}}^{\mathbb{G}}$ of strategies and a state $s \in S$, $P_s^{\sigma,\tau}$ denotes the probability measure on Ω_s determined by*

$$P_s^{\sigma,\tau}([p]) = \prod_{n \in \{1, \cdots, |p|-1\}} \sum_{a,b \in A_{\mathrm{I}} \times A_{\mathrm{II}}} \{\sigma(p \upharpoonright n)(a)\tau(p \upharpoonright n)(b) : (p(n-1), a, b) \in \delta^{-1}(p(n))\}$$

for any $p \in \Omega_s^{\mathrm{fin}}$, where $[p] = \{w \in \Omega : p \subset w\}$.

Intuitively, for a function $F : \Omega \to [0,1]$ with $P_s^{\sigma,\tau}(F) = \int_{\Omega_s} F dP_s^{\sigma,\tau}$ exists, $P_s^{\sigma,\tau}(F)$ means the expectated value of an infinite game $\mathbb{G}(F)$ from s when Player I and Player II use the strategy σ and τ, respectively. In the case of a subset X of Ω instead of F, $P_s^{\sigma,\tau}(X)$ means the probability that the infinite play in Ω_s belongs to X when Player I and Player II use the corresponding strategies.

2.2 Values and optimal strategies

Let $\mathbb{G} = (S, A_{\mathrm{I}}, A_{\mathrm{II}}, \delta)$ be a game, and let $F : \Omega(\mathbb{G}) \to [0,1]$ satisfy that $P_s^{\sigma,\tau}(F)$ exists for any $\sigma \in \Sigma_{\mathrm{I}}^{\mathbb{G}}$, $\tau \in \Sigma_{\mathrm{II}}^{\mathbb{G}}$ and $s \in S$. We call such a function F a *payoff function* of $\mathbb{G}$. (In the game $\mathbb{G}(X)$, the set X with such a property is called a *winning set* of $\mathbb{G}$.) The value of Player I in a game $\mathbb{G}_s(F)$ for a state s is the supremum of expected value which Player I can ensure. Formally, it is $\sup_{\sigma \in \Sigma_{\mathrm{I}}^{\mathbb{G}}} \inf_{\tau \in \Sigma_{\mathrm{II}}^{\mathbb{G}}} P_s^{\sigma,\tau}(F)$. Let $\neg F$ be a function defined by $\neg F(w) = 1 - F(w)$. The value of Player II is defined as $\sup_{\sigma \in \Sigma_{\mathrm{I}}^{\mathbb{G}}} \inf_{\tau \in \Sigma_{\mathrm{II}}^{\mathbb{G}}} P_s^{\sigma,\tau}(\neg F)$. This value is equal to $1 - \inf_{\tau \in \Sigma_{\mathrm{II}}^{\mathbb{G}}} \sup_{\sigma \in \Sigma_{\mathrm{I}}^{\mathbb{G}}} P_s^{\sigma,\tau}(F)$. We say that the game $\mathbb{G}(F)$ is *determinate* if

$$\sup_{\sigma \in \Sigma_{\mathrm{I}}^{\mathbb{G}}} \inf_{\tau \in \Sigma_{\mathrm{II}}^{\mathbb{G}}} P_s^{\sigma,\tau}(F) + \sup_{\tau \in \Sigma_{\mathrm{II}}^{\mathbb{G}}} \inf_{\sigma \in \Sigma_{\mathrm{I}}^{\mathbb{G}}} P_s^{\tau,\sigma}(\neg F) = 1$$

holds for any $s \in S$. Or equivalently, the game $\mathbb{G}(F)$ is determinate if and only if

$$\sup_{\sigma \in \Sigma_{\mathrm{I}}^{\mathbb{G}}} \inf_{\tau \in \Sigma_{\mathrm{II}}^{\mathbb{G}}} P_s^{\sigma,\tau}(F) = \inf_{\tau \in \Sigma_{\mathrm{II}}^{\mathbb{G}}} \sup_{\sigma \in \Sigma_{\mathrm{I}}^{\mathbb{G}}} P_s^{\sigma,\tau}(F)$$

holds for any $s \in S$. In this case, we write $\mathrm{val}_s^{\mathbb{G}}(F)$ or $\mathrm{val}_s(F)$ instead of $\sup_{\sigma \in \Sigma_{\mathrm{I}}^{\mathbb{G}}} \inf_{\tau \in \Sigma_{\mathrm{II}}^{\mathbb{G}}} P_s^{\sigma,\tau}(F)$, and call it the *value* at s in the game $\mathbb{G}(F)$.

The following is a well-known theorem obtained by Martin.

Theorem 2.6 (Martin [13]). *Let $\mathbb{G}$ be a game and let $F : \Omega(\mathbb{G}) \to [0,1]$ a Borel measurable function. Then the game $\mathbb{G}(F)$ is determinate.* □

Definition 2.7. *Let $\mathbb{G} = (S, A_{\mathrm{I}}, A_{\mathrm{II}}, \delta)$, $F : \Omega \to [0,1]$ and $\epsilon \in [0,1]$. Suppose that $\mathbb{G}(F)$ is determinate. A strategy $\sigma \in \Sigma_{\mathrm{I}}$ of Player I is ϵ-optimal if $\inf_{\tau \in \Sigma_{\mathrm{II}}^{\mathbb{G}}} P_s^{\sigma,\tau}(F) \geq \mathrm{val}_s(F) - \epsilon$ holds for any $s \in S$. Similarly, a strategy $\tau \in \Sigma_{\mathrm{II}}$ of Player II is ϵ-optimal if $\sup_{\sigma \in \Sigma_{\mathrm{I}}^{\mathbb{G}}} P_s^{\sigma,\tau}(F) \leq \mathrm{val}_s(F) + \epsilon$ holds for any $s \in S$. Optimal strategies are 0-optimal strategies.*

By the definition, a strategy $\sigma \in \Sigma_{\mathrm{I}}$ of Player I is optimal if and only if

$$\inf_{\tau \in \Sigma_{\mathrm{II}}} P_s^{\sigma,\tau}(F) = \mathrm{val}_s(F)$$

holds for all $s \in S$, and $\tau \in \Sigma_{\mathrm{II}}$ is optimal if and only if

$$\sup_{\sigma \in \Sigma_{\mathrm{I}}} P_s^{\sigma,\tau}(F) = \mathrm{val}_s(F)$$

holds for all $s \in S$.

When $\mathbb{G}(F)$ is determinate and ε is a positive real number, then ε-optimal strategies of Player I and Player II always exist by the definition. However, there are some cases that Player I or Player II has no optimal strategy.

Let $\mathbb{G} = (S, A_{\mathrm{I}}, A_{\mathrm{II}}, \delta)$ be a game and let $V : S \to [0,1]$. We define $F_V : \Omega(\mathbb{G}) \to [0,1]$ by $F_V(w) = V(w(1))$. Games of the form $\mathbb{G}(F_V)$ are called *one-step games.* We write $\mathbb{G}(V)$ meaning $\mathbb{G}(F_V)$, and we write $\mathrm{val}_s(V)$ for $s \in S$ instead of $\mathrm{val}_s(F_V)$. In one-step games optimal strategies always exist for each player. This theorem is well-known as von Neumann's minimax theorem.

Theorem 2.8 (von Neumann [15]). *In any one-step game, both players have their optimal strategies.* □

3 Generalized Reachability Games

Reachability games are in some respect the simplest infinite games. In this section, we will prove some basic facts on a generalized version of reachability games. In particular, we will describe the value of generalized reachability games as a limit value of finite-step games. We will see that Player II has a memoryless optimal strategy, and Player I has a memoryless ε-optimal strategy in any generalized reachability games for every positive real number ε. However, in general it is known that, even in a reachability game, Player I may not have an optimal strategy.

Definition 3.1. *Let $\mathbb{G} = (S, A_{\mathrm{I}}, A_{\mathrm{II}}, \delta)$ be a game. A function ℓ is called a label on S if $\mathrm{dom}(\ell) \subset S$ and $\ell(s) \in [0,1]$ for any $s \in \mathrm{dom}(\ell)$. We define $\mathcal{R}^{\mathbb{G},\ell} : \Omega(\mathbb{G}) \to [0,1]$ by*

$$\mathcal{R}^{\mathbb{G},\ell}(w) = \begin{cases} \ell(w(N_w)) & \text{if } (\exists N \in \mathbb{N})[w(N) \in \mathrm{dom}(\ell)], \\ 0 & \text{otherwise,} \end{cases}$$

where N_w is the least natural number N such that $w(N) \in \mathrm{dom}(\ell)$. A game of the form $\mathbb{G}(\mathcal{R}^{\mathbb{G},\ell})$ is called a generalized reachability game.

For a subset T of S, let $\mathcal{R}^{\mathbb{G},T} = \mathcal{R}^{\mathbb{G},\ell_T}$, where $\ell_T : T \to \{1\}$. Games of the form $\mathbb{G}(\mathcal{R}^{\mathbb{G},T})$ are called *reachability games.*

Definition 3.2. *Let $\mathbb{G} = (S, A_{\mathrm{I}}, A_{\mathrm{II}}, \delta)$ be a game and let ℓ a label on S. For every state $s \in S$ and $n \in \mathbb{N}$, we define $V_n^{\mathbb{G},\ell} : S \to [0,1]$ inductively by*

$$V_0^{\mathbb{G},\ell}(s) = \begin{cases} \ell(s) & \text{if } s \in \mathrm{dom}(\ell), \\ 0 & \text{otherwise,} \end{cases} \qquad V_{n+1}^{\mathbb{G},\ell}(s) = \begin{cases} \ell(s) & \text{if } s \in \mathrm{dom}(\ell), \\ \mathrm{val}_s(V_n^{\mathbb{G},\ell}) & \text{otherwise.} \end{cases}$$

We let $V^{\mathbb{G},\ell}(s) = \lim_{n\to\infty} V_n^{\mathbb{G},\ell}(s)$ for any state s, and we call it the limit value at s.

3.1 Determinacy and optimal strategy

For a label ℓ on S and $n \in \mathbb{N}$, we define $\mathcal{R}_n^{\mathbb{G},\ell} : \Omega(\mathbb{G}) \to [0,1]$ by $\mathcal{R}_n^{\mathbb{G},\ell}(w) = s_w$ if there exists $m \le n$ with $w(m) \in \mathrm{dom}(\ell)$ and $\mathcal{R}_n^{\mathbb{G},\ell}(w) = 0$ otherwise.

Theorem 3.3. *Let $\mathbb{G} = (S, A_{\mathrm{I}}, A_{\mathrm{II}}, \delta)$ be a game and let ℓ a label on S. For any $n \in \mathbb{N}$, both players have their optimal strategies in the game $\mathbb{G}(\mathcal{R}_n^{\mathbb{G},\ell})$, and the equality $V_n^{\mathbb{G},\ell}(s) = \mathrm{val}_s(\mathcal{R}_n^{\mathbb{G},\ell})$ holds for all $s \in S$.*

Proof. We define σ_n^* and τ_n^* inductively. Let σ_0^* and τ_0^* be any strategies. Now suppose that we have constructed σ_n^* and τ_n^*. Choose σ and τ as optimal strategies of Player I and II respectively in the one-step game $\mathbb{G}(V_n^{\mathbb{G},\ell})$. Define σ_{n+1}^* by $\sigma_{n+1}^*(s) = \sigma(s)$ and $\sigma_{n+1}^*(s\rho) = \sigma_n^*(\rho)$ for any $s \in S$ and any $\rho \neq \emptyset$ with $s\rho \in \Omega^{\mathrm{fin}}$. Similarly, define τ_{n+1}^* by $\tau_{n+1}^*(s) = \tau(s)$ and $\tau_{n+1}^*(s\rho) = \tau_n^*(\rho)$ for any $s \in S$ and any $\rho \neq \emptyset$ with $s\rho \in \Omega^{\mathrm{fin}}$. It is easy to see by induction on n that σ_n^* and τ_n^* satisfy the equalities $V_n^{\mathbb{G},\ell}(s) = \inf_{\tau \in \Sigma_{\mathrm{II}}} P_s^{\sigma_n^*,\tau}(\mathcal{R}_n^{\mathbb{G},\ell}) = \sup_{\sigma \in \Sigma_{\mathrm{I}}} P_s^{\sigma,\tau_n^*}(\mathcal{R}_n^{\mathbb{G},\ell})$. This equalities imply that the σ_n^* and τ_n^* are optimal strategies in the game $\mathbb{G}(\mathcal{R}_n^{\mathbb{G},\ell})$ and that $V_n^{\mathbb{G},\ell}(s) = \mathrm{val}_s(\mathcal{R}_n^{\mathbb{G},\ell})$ holds. □

Now we verify the value $\mathrm{val}_s(\mathcal{R}^{\mathbb{G},\ell})$ is equivalent to the limit value $V^{\mathbb{G},\ell}(s)$.

Theorem 3.4. *For any state $s \in S$, the equation $V^{\mathbb{G},\ell}(s) = \mathrm{val}_s(\mathcal{R}^{\mathbb{G},\ell})$ holds.*

Proof. It is enough to show that the following inequalities:

$$\inf_{\tau \in \Sigma_{\mathrm{II}}} \sup_{\sigma \in \Sigma_{\mathrm{I}}} P_s^{\sigma,\tau}(\mathcal{R}^{\mathbb{G},\ell}) \le V^{\mathbb{G},\ell}(s) \le \sup_{\sigma \in \Sigma_{\mathrm{I}}} \inf_{\tau \in \Sigma_{\mathrm{II}}} P_s^{\sigma,\tau}(\mathcal{R}^{\mathbb{G},\ell}).$$

To show the first inequality, choose an optimal strategy τ^* of Player II in the one-step game $\mathbb{G}(V^{\mathbb{G},\ell})$. We may see τ^* as a memoryless strategy of Player II in the generalized reachability game $\mathbb{G}(\mathcal{R}^{\mathbb{G},\ell})$. We show that τ^* satisfies the inequality $\sup_{\sigma \in \Sigma_{\mathrm{I}}} P_s^{\sigma,\tau^*}(\mathcal{R}^{\mathbb{G},\ell}) \le V^{\mathbb{G},\ell}(s)$ for any $s \in S$. (Thus, if we prove the second inequality, then we can say this τ^* is, in fact, an optimal strategy of Player II in the game $\mathbb{G}(\mathcal{R}^{\mathbb{G},\ell})$.) It is enough to show that $\sup_\sigma P_s^{\sigma,\tau^*}(\mathcal{R}_n^{\mathbb{G},\ell}) \le V^{\mathbb{G},\ell}(s)$ for any $s \in S$ and $n \in \mathbb{N}$. We show this by induction on n. The case $n = 0$ is clear. Suppose that $\sup_\sigma P_s^{\sigma,\tau^*}(\mathcal{R}_n^{\mathbb{G},\ell}) \le V^{\mathbb{G},\ell}(s)$ holds for any $s \in S$ as an induction hypothesis. Fix $s \in S$. If $s \in \mathrm{dom}(\ell)$, then it is obvious that the inequality holds for s. Otherwise, we have the equality $P_s^{\sigma,\tau^*}(\mathcal{R}_{n+1}^{\mathbb{G},\ell}) = \sum_{s' \in S} P_s^{\sigma,\tau^*}([ss'])P_{s'}^{\sigma,\tau^*}(\mathcal{R}_n^{\mathbb{G},\ell})$ for any $\sigma \in \Sigma_{\mathrm{I}}$. By the induction hypothesis, we know that $P_s^{\sigma,\tau^*}(\mathcal{R}_{n+1}^{\mathbb{G},\ell}) \le \sum_{s' \in S} P_s^{\sigma,\tau^*}([ss'])V^{\mathbb{G},\ell}(s')$. Hence the equalities

$$\sup_\sigma P_s^{\sigma,\tau^*}(\mathcal{R}_{n+1}^{\mathbb{G},\ell}) \le \sup_\sigma \sum_{s' \in S} P_s^{\sigma,\tau^*}([ss'])V^{\mathbb{G},\ell}(s') = V^{\mathbb{G},\ell}(s)$$

hold by the optimality of τ^* in the one-step game. Let us now show the second inequality. We have $P_s^{\sigma,\tau}(\mathcal{R}_n^{\mathbb{G},\ell}) \le P_s^{\sigma,\tau}(\mathcal{R}^{\mathbb{G},\ell})$ since $\mathcal{R}_n^{\mathbb{G},\ell}(w) \le \mathcal{R}^{\mathbb{G},\ell}(w)$ for any $w \in \Omega$. Hence $\sup_\sigma \inf_\tau P_s^{\sigma,\tau}(\mathcal{R}_n^{\mathbb{G},\ell}) \le \sup_\sigma \inf_\tau P_s^{\sigma,\tau}(\mathcal{R}^{\mathbb{G},\ell})$ holds. By Theorem 3.3, $V_n^{\mathbb{G},\ell}(s) = \mathrm{val}_s(\mathcal{R}_n^{\mathbb{G},\ell}) = \sup_\sigma \inf_\tau P_s^{\sigma,\tau}(\mathcal{R}_n^{\mathbb{G},\ell})$ holds. Thus the second inequality holds. □

Corollary 3.5. *Player II has a memoryless optimal strategy in any generalized reachability game.* □

Contrary to the case of Player II, Player I has no even optimal strategy in some reachability games. We give such an example below.

Example 1. *Consider the following simultaneous reachability game as shown in Figure 1. For a game* $\mathbb{G} = (S, A_{\mathrm{I}}, A_{\mathrm{II}}, \delta)$*, let* $S = \{s_0, s_1, s_2\}$*,* $A_{\mathrm{I}} = \{x_1, x_2\}$ *and* $A_{\mathrm{II}} = \{y_1, y_2\}$*. Define a transition function* δ *by* $\delta(s_0, x_1, y_1) = s_0$*,* $\delta(s_0, x_2, y_2) = s_2$*,* $\delta(s_0, x_1, y_2) = \delta(s_0, x_2, y_1) = s_1$ *and* $\delta(s_i, x, y) = s_i$ *for any* $i \in \{1, 2\}$ *and* $(x, y) \in A_{\mathrm{I}} \times A_{\mathrm{II}}$*. Now consider the reachability game* $\mathbb{G}(\mathcal{R}^{\mathrm{G},T})$ *with the target state* $T = \{s_1\}$*.*

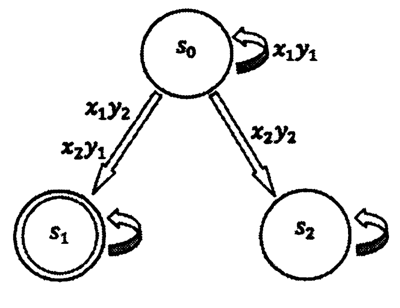

Fig. 1. An illustration of reachability game

One can prove that $\mathrm{val}_{s_0}(\mathcal{R}^{\mathrm{G},\{s_1\}}) = 1$*. We show that Player I has no optimal strategy in the reachability game* $\mathbb{G}(\mathcal{R}^{\mathrm{G},\{s_1\}})$*.*

Proof. Fix a strategy $\sigma \in \Sigma_{\mathrm{I}}$. We construct $\tau \in \Sigma_{\mathrm{II}}$ such that $P_{s_0}^{\sigma,\tau}(\mathcal{R}_{\{s_1\}}) < 1$. For $\rho \in \Omega^{\mathrm{fin}}(\mathbb{G})$, define $\tau(\rho)(y_1) = 1$ if $\sigma(\rho)(x_1) = 1$, and define $\tau(\rho)(y_2) = 1$ otherwise. It is clear that $P_{s_0}^{\sigma,\tau}(\mathcal{R}_{\{s_1\}}) < 1$ by the definitions of $\mathbb{G}$ and τ. □

The next theorem says that, given a generalized reachability game, Player I always has a memoryless ε-optimal strategy in this game for any positive real number ε. In fact, this result for reachability games was shown by Chatterjee et al. [5] in a slightly different setting. We essentially use their method to prove our theorem.

Theorem 3.6. *In every generalized reachability game* $\mathbb{G}(\mathcal{R}^{\mathrm{G},\ell})$*, there exists an* ε*-optimal memoryless strategy of Player I for any* $\varepsilon > 0$*.*

Proof. Let $\mathbb{G} = (S, A_{\mathrm{I}}, A_{\mathrm{II}}, \delta)$ be a game and let ℓ a label on S. Without loss of generality, we may assume that if $s \in \mathrm{dom}(\ell)$ or $\mathrm{val}_s(\mathcal{R}^{\mathrm{G},\ell}) = 0$, then $\delta(s, x, y) = s$ holds for any $(x, y) \in A_{\mathrm{I}} \times A_{\mathrm{II}}$.

Fix a positive real $\varepsilon > 0$. Choose $n \in \mathbb{N}$ such that for any $s \in S$, the inequality $V_{n-1}^{\mathrm{G},\ell}(s) \geq \mathrm{val}_s(\mathcal{R}^{\mathrm{G},\ell}) - \varepsilon$ holds, and $\mathrm{val}_s(\mathcal{R}^{\mathrm{G},\ell}) > 0$ implies $V_{n-1}^{\mathrm{G},\ell}(s) > 0$. For $m \leq n$, choose $\sigma_m \in \Sigma_{\mathrm{I}}^{\mathrm{M}}$ such that σ_m is an optimal strategy of Player I in the one-step game $\mathbb{G}(V_{m-1}^{\mathrm{G},\ell})$. We define a strategy $\sigma^* \in \Sigma_{\mathrm{I}}^{\mathrm{M}}$ by $\sigma^*(s) = \sigma_{m_s}(s)$ for any $s \in S$, where m_s is the least number $m \leq n$ such that $V_m^{\mathrm{G},\ell}(s) = V_n^{\mathrm{G},\ell}(s)$. By the definition,

$V^{\mathbb{G},\ell}_{m_s}(s) = \inf_{\tau\in\Sigma^{\mathrm{M}}_{\mathrm{II}}} P^{\sigma^*,\tau}_s(V^{\mathbb{G},\ell}_{m_s-1})$ holds for any $s \in S \setminus \mathrm{dom}(\ell)$. Now choose a strategy $\tau^* \in \Sigma^{\mathrm{M}}_{\mathrm{II}}$ such that $P^{\sigma^*,\tau^*}_s(\mathcal{R}^{\mathbb{G},\ell}) = \inf_\tau P^{\sigma^*,\tau}_s(\mathcal{R}^{\mathbb{G},\ell})$ for all $s \in S$.

Fix an $s \in S\setminus\mathrm{dom}(\ell)$ with $V^{\mathbb{G},\ell}_{m_s}(s) > 0$. Suppose that $V_n(s) \geq V_n(s')$ holds for any $s' \in S$ with $P^{\sigma^*,\tau^*}_s([ss']) > 0$. We have $V^{\mathbb{G},\ell}_{m_s}(s) = V^{\mathbb{G},\ell}_{m_s-1}(s') = V^{\mathbb{G},\ell}_n(s')$ for any $s' \in S$ with $P^{\sigma^*,\tau^*}_s([ss']) > 0$ since $V^{\mathbb{G},\ell}_n(s) = V^{\mathbb{G},\ell}_{m_s}(s)$, $V^{\mathbb{G},\ell}_{m_s-1}(s') \leq V^{\mathbb{G},\ell}_n(s')$ and $V^{\mathbb{G},\ell}_{m_s}(s) \leq P^{\sigma^*,\tau}_s(V^{\mathbb{G},\ell}_{m_s-1})$ hold. Therefore, if $s' \in S$ satisfies $P^{\sigma^*,\tau^*}_s([ss'])$, then $m_s > m_{s-1}$. As a result, we know that for any $s \in S \setminus \mathrm{dom}(\ell)$ there exists s' with $P^{\sigma^*,\tau^*}_s([ss']) > 0$ such that

$$V^{\mathbb{G},\ell}_n(s) < V^{\mathbb{G},\ell}_n(s') \text{ or } m_s > m_{s-1}.$$

Note that $\{V^{\mathbb{G},\ell}_n(s) : s \in S \setminus \mathrm{dom}(\ell)\}$ is finite, and $m_s = 0$ implies $s \in \mathrm{dom}(\ell)$ or $V^{\mathbb{G},\ell}_n(s) = 0$. Here $V^{\mathbb{G},\ell}_n(s) = 0$ implies $\mathrm{val}_s(\mathcal{R}^{\mathbb{G},\ell}) = 0$. Hence for any $s \in S$ there exists $\rho \in \Omega^{\mathrm{fin}}_s$ such that $P^{\sigma^*,\tau^*}_s([\rho]) > 0$ and

$$\rho(|\rho| - 1) \in \mathrm{dom}(\ell) \text{ or } \mathrm{val}_{\rho(|\rho|-1)}(\mathcal{R}^{\mathbb{G},\ell}) = 0.$$

As a conclusion, we have $P^{\sigma^*,\tau^*}_s(A) = 0$ for any $s \in S$, where $A = \{w \in \Omega : (\forall n \in \mathbb{N})[w(n) \in \mathrm{dom}(\ell) \;\&\; \mathrm{val}_{w(n)}(\mathcal{R}^{\mathbb{G},\ell}) > 0]\}$. Thus, the sum

$$\sum \{V^{\mathbb{G},\ell}_n(\rho(|\rho| - 1))P^{\sigma^*,\tau^*}_s([\rho]) : \rho \in \Omega^{\mathrm{fin}}_s \;\&\; |\rho| = k\}$$

tends to $P^{\sigma^*,\tau^*}_s(\mathcal{R}^{\mathbb{G},\ell})$ as k to ∞. It is easy to see by induction on $k \in \mathbb{N}$ that

$$\sum \{V^{\mathbb{G},\ell}_n(\rho(|\rho| - 1))P^{\sigma^*,\tau^*}_s([\rho]) : \rho \in \Omega^{\mathrm{fin}}_s \;\&\; |\rho| = k\} \geq V^{\mathbb{G},\ell}_{n-1}(s)$$

holds for any $k \in \mathbb{N}$. Hence we have $P^{\sigma^*,\tau^*}_s(\mathcal{R}^{\mathbb{G},\ell}) \geq V^{\mathbb{G},\ell}_{n-1}(s) \geq \mathrm{val}_s(\mathcal{R}^{\mathbb{G},\ell}) - \varepsilon$. □

4 Concluding Remarks

In this work, we contributed some results on generalized stochastic reachability games. We proved the game has a value by defining a limit value and show such a value is equal to the value of the game. We also showed that there exists a memoryless optimal strategy for Player II in any generalized reachability games, while Player I must settle for ϵ-optimality (memoryless).

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