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Groups of uniform homeomorphisms of metric spaces with geometric group actions

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1. Introduction

This article is a continuation of study of topological properties of spaces of uniform embeddings and groups of uniform homeomorphisms ([1, 4, 7]). Since the notion of uniform continuity depends on the choice of metrics, it is essential to select reasonable classes of metric spaces (X, d) to obtain suitable conclusions on spaces of uniform embeddings. In [1] (cf, [5, Section 5.6]) A.V. Černavskiĭ considered the case where X is the interior of a compact manifold N and the metric d is a restriction of some metric on N.

On the other hand, in the previous paper [7] we considered metric covering spaces over compact manifolds and obtained a local deformation theorem for uniform embeddings on those spaces (Theorem 2.2). In term of covering transformation groups, the metric covering spaces over compact spaces corresponds to the locally compact metric spaces with free geometric group actions. Here a group action on a metric space is called geometric if it is proper, cocompact and isometric. From our view point, it is natural to expect that the same conclusion also holds for any metric space with a geometric group action. Our key observation here is that a metric space with a geometric group action is locally a trivial metric covering space. Thus the case for any geometric group action (Theorem 3.1) follows from the one for metric covering spaces, once we show the finite additivity of deformation property for uniform embeddings. In Section 2 we recall basic definitions on uniform embeddings and the results in metric covering spaces obtained in [7] and in Section 3 we study the case of geometric group actions.

2. Spaces of uniform embeddings in metric covering spaces

2.1. **Spaces of uniform embeddings.**

First we recall basic definitions on uniform embeddings/homeomorphisms. In this article, maps between topological spaces are always assumed to be continuous.

Suppose $X = (X, d_X)$ is a metric space. For $x \in X$ and $\varepsilon > 0$ let $O_\varepsilon(x)$ ($C_\varepsilon(x)$) denote the open (closed) ε-ball in X centered at the point x. Suppose A is a subset of X. The open ε-neighborhood $O_\varepsilon(A)$ of A in X is defined by

$$O_\varepsilon(A) = \{x \in X \mid d(x,a) < \varepsilon \text{ for some } a \in A\}.$$

A neighborhood U of A in X is called a uniform neighborhood of A if $O_\varepsilon(A) \subset U$ for some $\varepsilon > 0$. We say that A is ε-discrete if $d_X(x,y) \geq \varepsilon$ for any distinct points $x, y \in A$ and that A is uniformly discrete if it is ε-discrete for some $\varepsilon > 0$. More generally a family $\{F_\lambda\}_{\lambda \in \Lambda}$ of subsets of X is said to be ε-discrete if for any $\lambda, \mu \in \Lambda$ either $d(F_\lambda, F_\mu) \geq \varepsilon$ or $F_\lambda = F_\mu$.

A map $f : X \to Y$ between metric spaces is said to be uniformly continuous if for each $\varepsilon > 0$ there is a $\delta > 0$ such that if $x, x' \in X$ and $d_X(x,x') < \delta$ then $d_Y(f(x), f(x')) < \varepsilon$. The map f is called a uniform homeomorphism if f is bijective and both f and f^{-1} are uniformly continuous. A uniform embedding is a uniform homeomorphism onto its image.

Let $\mathcal{C}^u(X,Y)$ denote the space of uniformly continuous maps $f : X \to Y$. The metric d_Y on Y induces the sup-metric d on $\mathcal{C}^u(X,Y)$ defined by

$$d(f,g) = \sup\{d_Y(f(x), g(x)) \mid x \in X\} \in [0, \infty].$$

The topology on $\mathcal{C}^u(X,Y)$ induced by this sup-metric d is called the uniform topology. Below the space $\mathcal{C}^u(X,Y)$ and its subspaces are endowed with the sup-metric d and the uniform topology, otherwise specified. The composition map

$$\mathcal{C}^u(X,Y) \times \mathcal{C}^u(Y,Z) \longrightarrow \mathcal{C}^u(X,Z)$$

is continuous.

For a subset A of X let $\mathcal{H}^u_A(X)$ denote the group of uniform homeomorphisms h of X onto itself with $h|A = \mathrm{id}_A$ (endowed with the sup-metric and the uniform topology). By $\mathcal{H}^u_A(X)_0$ we denote the connected component of the identity map id_X of X in $\mathcal{H}^u_A(X)$ and define the subgroup

$$\mathcal{H}^u_A(X)_b = \{h \in \mathcal{H}^u_A(X) \mid d(h, \mathrm{id}_X) \quad \infty\}.$$

It follows that $\mathcal{H}^u_A(X)$ is a topological group and $\mathcal{H}^u_A(X)_b$ is a clopen subgroup of $\mathcal{H}^u_A(X)$, so that $\mathcal{H}^u_A(X)_0 \subset \mathcal{H}^u_A(X)_b$. As usual, the symbol A is suppressed when it is an empty set.

Similarly, let $\mathcal{E}^u_A(X,Y)$ denote the space uniform embeddings $f : X \to Y$ with $f|_A = \mathrm{id}_A$ (with the sup-metric and the uniform topology). When $X \subset Y$, for a subset C of Y we use the symbol $\mathcal{E}^u(X,Y;C)$ to denote $\mathcal{E}^u_{X\cap C}(X,Y)$. When Y is a topological n-manifold possibly with boundary and $X \subset Y$, an embedding $f : X \to Y$ is said to be proper if $f^{-1}(\partial Y) = X \cap \partial Y$. Let $\mathcal{E}^u_*(X,Y;C)$ denote the subspace of $\mathcal{E}^u(X,Y;C)$ consisting of proper embeddings.

2.2. **Metric covering projections.**

In [7] we introduced the notion of metric covering projections as the C^0-version of Riemannian coverings in the smooth category. For the basics on covering spaces, we refer to

[6, Chapter 2, Section 1]. Note that if $p : M \to N$ is a covering projection and N is a topological n-manifold possibly with boundary, then so is M and $\partial M = \pi^{-1}(\partial N)$.

Definition 2.1. A map $\pi : X \to Y$ between metric spaces is called a metric covering projection if it satisfies the following conditions:

$(*)_1$ There exists an open cover $\mathcal{U}$ of Y such that for each $U \in \mathcal{U}$ the inverse $\pi^{-1}(U)$ is the disjoint union of open subsets of X each of which is mapped isometrically onto U by π.

$(*)_2$ For each $y \in Y$ the fiber $\pi^{-1}(y)$ is uniformly discrete in X.

$(*)_3$ $d_Y(\pi(x), \pi(x')) \leq d_X(x, x')$ for any $x, x' \in X$.

2.3. **Edwards - Kirby's local deformation theorem for embeddings of compact subsets.**

In [3, Theorem 5.1] R. D. Edwards and R. C. Kirby obtained a fundamental local deformation theorem for embeddings of a compact subspace in a manifold.

Theorem 2.1. Suppose M is a topological n-manifold possibly with boundary, C is a compact subset of M, $K \subset L$ are compact neighborhoods of C in M and $D \subset E$ are two closed subsets of M such that $D \subset \mathrm{Int}_M E$. Then there exists a neighborhood $\mathcal{U}$ of the inclusion map $i_L : L \to M$ in $\mathcal{E}^u_*(L, M; E)$ and a homotopy $\varphi : \mathcal{U} \times [0,1] \longrightarrow \mathcal{E}^u_*(L, M; D)$ such that

(1) for each $f \in \mathcal{U}$,

(i) $\varphi_0(f) = f$, (ii) $\varphi_1(f) = \mathrm{id}$ on C, (iii) $\varphi_t(f) = f$ on $L - K$ $(t \in [0,1])$,

(iv) if $f = \mathrm{id}$ on $L \cap \partial M$, then $\varphi_t(f) = \mathrm{id}$ on $L \cap \partial M$ $(t \in [0,1])$,

(2) $\varphi_t(i_L) = i_L$ $(t \in [0,1])$.

2.4. **Deformation theorem for uniform embeddings.**

In [7, Theorem 1.1] from Edwards - Kirby's deformation theorem we deduced a local deformation theorem for uniform embeddings in any metric covering space over a compact manifold. There, the Arzela-Ascoli theorem (cf. [2, Theorem 6.4]) played an essential role in order to pass from the compact case to the uniform case.

Theorem 2.2. Suppose $\pi : (M, d) \to (N, \varrho)$ is a metric covering projection, N is a compact n-manifold possibly with boundary, X is a subset of M, $W' \subset W$ are uniform neighborhoods of X in (M, d) and Z, Y are subsets of M such that Y is a uniform neighborhood of Z. Then there exists a neighborhood $\mathcal{W}$ of the inclusion map $i_W : W \subset M$ in $\mathcal{E}^u_*(W, M; Y)$ and a homotopy $\varphi : \mathcal{W} \times [0,1] \longrightarrow \mathcal{E}^u_*(W, M; Z)$ such that

(1) for each $h \in \mathcal{W}$

(i) $\varphi_0(h) = h$, (ii) $\varphi_1(h) = \mathrm{id}$ on X,

(iii) $\varphi_t(h) = h$ on $W - W'$ and $\varphi_t(h)(W) = h(W)$ $(t \in [0,1])$,

(iv) if $h = \mathrm{id}$ on $W \cap \partial M$, then $\varphi_t(h) = \mathrm{id}$ on $W \cap \partial M$ $(t \in [0,1])$,

(2) $\varphi_t(i_W) = i_W$ $(t \in [0,1])$.

In [1] it is shown that $\mathcal{H}^u(M,d)$ is locally contractible in the case where M is the interior of a compact manifold N and the metric d is a restriction of some metric on N. The next corollary is a direct consequence of Theorem 2.2.

Corollary 2.1. Suppose $\pi : (M,d) \to (N,\varrho)$ is a metric covering projection onto a compact n-manifold N possibly with boundary. Then $\mathcal{H}^u(M,d)$ is locally contractible.

The Euclidean space $\mathbb{R}^n$ with the standard Euclidean metric admits the canonical Riemannian covering projection $\pi : \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$ onto the flat torus. Therefore we can apply Theorem 2.2 to uniform embeddings in $\mathbb{R}^n$. The important feature of $\mathbb{R}^n$ is that it admits similarity transformations

$$k_\gamma : \mathbb{R}^n \approx \mathbb{R}^n : \ k_\gamma(x) = \gamma x \qquad (\gamma > 0).$$

This enables us to deduce a global deformation of uniform embeddings directly from a local one. A similar argument is also applied to the Euclidean end $\mathbb{R}^n_r = \mathbb{R}^n - O(r)$ $(r > 0)$, where $O(r)$ is the round open r-ball in $\mathbb{R}^n$ centered at the origin. Since the deformation property for uniform embeddings is preserved by bi-Lipschitz homeomorphisms, we can pass from the model space $\mathbb{R}^n$ to any metric spaces with finitely many bi-Lipschitz Euclidean ends. For the precise statement, we refer to [7, Theorem 1.2]. For example, in the case of $\mathbb{R}^n$ itself we can construct a strong deformation retraction of $\mathcal{H}^u(\mathbb{R}^n)_b$ onto $\mathcal{H}^u_{\mathbb{R}^n_3}(\mathbb{R}^n)$. Since the latter is contractible by Alexander's trick, we have the following conclusion.

Corollary 2.2. $\mathcal{H}^u(\mathbb{R}^n)_b$ is contractible for every $n \geq 0$.

In [4] we studied the topological type of $\mathcal{H}^u(\mathbb{R})_b$ as an infinite-dimensional manifold and showed that it is homeomorphic to ℓ_∞.

3. Spaces of uniform embeddings in metric spaces with geometric group actions

3.1. **Locally geometric group actions.**

First we fix some symbols and recall some related notions. When a group G acts on a set S, for a subset $F \subset S$ let $GF = \{gx \mid g \in G, x \in F\} \subset S$ and $G_F = \{g \in G \mid gF = F\} < G$.

Suppose X is a locally compact separable metric space. An action of a group G on X is called a geometric group action if it is proper, cocompact and isometric. More generally, we call it a locally geometric group action if it is proper, cocompact and "locally isometric" in the following sense:

($\natural$) For any $x \in X$ there exists $\varepsilon_x > 0$ such that each $g \in G$ maps $O_{\varepsilon_x}(x)$ isometrically onto $O_{\varepsilon_x}(gx)$.

Definition 3.1. For $\varepsilon > 0$ we say that a point $x \in X$ satisfies the (G, ε)-discrete condition if the following holds:

(i) $C_\varepsilon(x)$ is compact, (ii) each $g \in G$ induces an isometry $g : O_\varepsilon(x) \cong O_\varepsilon(gx)$,
(iii) the subset $Gx \subset M$ is 3ε-discrete, so that the family $\{O_\varepsilon(gx) \mid g \in G\}$ is ε-discrete.

As for the condition (iii) note that $O_\varepsilon(gx) = O_\varepsilon(hx)$ if $\overline{g} = \overline{h}$ in G/G_x.

Lemma 3.1. If a group G acts on X locally geometrically, then the following holds:

(1) G is a countable group.
(2) The family $\{gC \mid g \in G\}$ is locally finite and G_C is finite for any compact subset C of M. In particular, Gx is a discrete subset of M.
(3) $M = GK$ for some compact subset K of M.
(4) Each point $x \in X$ satisfies the (G, ε)-discrete condition for some $\varepsilon > 0$.

3.2. **Additivity of deformation property for uniform embeddings.**

Suppose (M, d) is a topological n-manifold possibly with boundary with a fixed metric d.

Definition 3.2. We say that a subset U of M satisfies the condition (LD) if the following holds:

(#) Suppose X is a subset of U, $W' \subset W$ are uniform neighborhoods of X in M and Z, Y are subsets of M such that Y is a uniform neighborhood of Z. Then there exists a neighborhood $\mathcal{W}$ of the inclusion map $i_W : W \subset M$ in $\mathcal{E}^u_*(W, M; Y)$ and a homotopy $\varphi : \mathcal{W} \times [0, 1] \longrightarrow \mathcal{E}^u_*(W, M; Z)$ such that

(1) for each $h \in \mathcal{W}$

(i) $\varphi_0(h) = h$, (ii) $\varphi_1(h) = \mathrm{id}$ on X,
(iii) $\varphi_t(h) = h$ on $W - W'$ and $\varphi_t(h)(W) = h(W)$ $(t \in [0, 1])$,
(iv) if $h = \mathrm{id}$ on $W \cap \partial M$, then $\varphi_t(h) = \mathrm{id}$ on $W \cap \partial M$ $(t \in [0, 1])$,

(2) $\varphi_t(i_W) = i_W$ $(t \in [0, 1])$.

The condition (LD) has the following properties:

Lemma 3.2. (1) If $U \subset V \subset M$ and V satisfies the condition (LD), then so does U.

(2) Suppose $K, L \subset M$ and $\varepsilon > 0$. If both $O_\varepsilon(K)$ and $O_\varepsilon(L)$ satisfy the condition (LD), then so does $O_\delta(K \cup L)$ for any $\delta \in (0, \varepsilon)$.

3.3. **Deformation theorem for uniform embeddings.**

Our goal is to show the following theorem.

Theorem 3.1. Suppose (M, d) is a topological n-manifold possibly with boundary with a fixed metric d and it admits a locally geometric action of a group G. Suppose X is a subset of M, $W' \subset W$ are uniform neighborhoods of X in M and Z, Y are subsets of M such that Y is a uniform neighborhood of Z. Then there exists a neighborhood $\mathcal{W}$ of the inclusion map $i_W : W \subset M$ in $\mathcal{E}^u_*(W, M; Y)$ and a homotopy $\varphi : \mathcal{W} \times [0,1] \longrightarrow \mathcal{E}^u_*(W, M; Z)$ such that

(1) for each $h \in \mathcal{W}$

(i) $\varphi_0(h) = h$, (ii) $\varphi_1(h) = \mathrm{id}$ on X,

(iii) $\varphi_t(h) = h$ on $W - W'$ and $\varphi_t(h)(W) = h(W)$ $(t \in [0,1])$,

(iv) if $h = \mathrm{id}$ on $W \cap \partial M$, then $\varphi_t(h) = \mathrm{id}$ on $W \cap \partial M$ $(t \in [0,1])$,

(2) $\varphi_t(i_W) = i_W$ $(t \in [0,1])$.

Corollary 3.1. Suppose (M, d) is a topological n-manifold possibly with boundary with a fixed metric d which admits a locally geometric action of a group G. Then $\mathcal{H}^u(M, d)$ is locally contractible.

Sketch of Proof of Theorem 3.1.

[1] First we show that each point $x \in M$ admits a G-invariant open neighborhood U_x in M and $\delta_x > 0$ such that $O_{\delta_x}(U_x)$ satisfies the condition (LD). For this purpose, take any point $x \in M$ and let Λ be a complete set of representatives of cosets in G/G_x.

(1) The point x satisfies the $(G, 2\varepsilon)$-discrete condition for some $\varepsilon > 0$. Since $O_\varepsilon(Gx)$ is the disjoint union of open subsets $O_\varepsilon(gx)$ $(g \in \Lambda)$, we have the map

$$\pi : (O_\varepsilon(Gx), d) \to (O_\varepsilon(x), d) : \ \pi(y) = g^{-1}y \ \ (g \in \Lambda,\ y \in O_\varepsilon(gx)).$$

The map π is shown to be a metric covering projection.

(2) Take a closed n-ball neighborhood N of x in $O_\varepsilon(x)$ and let $F = \pi^{-1}(N)$. Then F is an n-submanifold (with boundary) of M which is closed in M, and the restriction $\pi : (F, d) \to (N, d)$ is also a metric covering projection. Therefore, by Theorem 2.2 F satisfies the condition (LD) in (F, d) itself.

(3) Take a $\delta = \delta_x \in (0, \varepsilon)$ such that $O_{4\delta}(x) \subset N$. Then $V = O_{2\delta}(Gx)$ is an open subset of M with $O_{2\delta}(V) \subset F$, so that V satisfies the condition (LD) in (M, d). Hence $U_x = O_\delta(Gx)$ is a G-invariant open neighborhood of x in M and $O_\delta(U_x)$ also satisfies the condition (LD).

[2] There exists a compact subset K of M such that $GK = M$. Then there exist finitely many points $x_1, \cdots, x_m \in K$ such that $\{U_{x_i}\}_{i=1}^m$ covers K. Since each U_{x_i} is G-invariant, $\{U_{x_i}\}_{i=1}^m$ also covers M. Since each $O_{\delta_{x_i}}(U_{x_i})$ satisfies the condition (LD), by Lemma 3.2 (2) so does $M = \cup_i U_{x_i}$. This completes the proof. $\square$

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