

## On structure of CAT(0) groups

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We introduce some results on finitely generated groups of isometries of CAT(0) spaces and CAT(0) groups in [15].

Definition and detail of CAT(0) spaces are found in [3] and [9].

Let  $X$  be a metric space and let  $\gamma$  be an isometry of  $X$ . Then the *translation length* of  $\gamma$  is defined as  $|\gamma| = \inf\{d(x, \gamma x) \mid x \in X\}$ , and the *minimal set* of  $\gamma$  is defined as  $\text{Min}(\gamma) = \{x \in X \mid d(x, \gamma x) = |\gamma|\}$ . An isometry  $\gamma$  is said to be *semi-simple* if  $\text{Min}(\gamma)$  is non-empty. Also an isometry  $\gamma$  is called

- (1) *elliptic* if  $\gamma$  has a fixed point,
- (2) *hyperbolic* if  $\gamma$  is semi-simple and not elliptic, and
- (3) *parabolic* if  $\gamma$  is not semi-simple.

(cf. [3, Chapter II.6]).

In [15], we obtained the following theorem by observing the proof of [3, Theorem II.6.12].

**Theorem 1.** *Let  $X$  be a CAT(0) space and let  $\Gamma$  be a finitely generated group acting by isometries on  $X$ . If the center of  $\Gamma$  contains a hyperbolic isometry  $\gamma_0$  of  $X$ , then there exist a normal subgroup  $\Gamma' \subset \Gamma$ , an element  $\delta_0 \in \Gamma$  and a number  $k_0 \in \mathbb{N}$  such that*

- (i)  $\Gamma = \Gamma' \rtimes \langle \delta_0 \rangle$ ,
- (ii)  $\Gamma' \rtimes \langle \delta_0^{k_0} \rangle = \Gamma' \times \langle \gamma_0 \rangle$  is a finite-index subgroup of  $\Gamma$  and
- (iii)  $\Gamma/\Gamma'$  is isomorphic to  $\mathbb{Z}$ .

A *geometric action* on a CAT(0) space is an action by isometries which is proper ([3, p.131]) and cocompact. A group  $\Gamma$  is called a *CAT(0) group*, if  $\Gamma$  acts geometrically on some CAT(0) space. We note that every CAT(0) space on

which some group acts geometrically is a proper space ([3, p.132]). Also we note that CAT(0) groups are finitely presented (cf. [3, Corollary I.8.11]).

For example, Bieberbach groups ([3, p.246], [4]), crystallographic groups ([4]), Coxeter groups and their torsion-free subgroups of finite-index ([6], [7], [19]) and fundamental groups of compact geodesic spaces of non-positive curvature ([3, p.159, p.237]) are CAT(0) groups. In particular, fundamental groups of Riemannian manifolds of non-positive sectional curvature are CAT(0) groups. Moreover, M. W. Davis [6] has constructed a closed aspherical manifold of dimension  $n \geq 5$  whose universal covering is not homeomorphic to  $\mathbb{R}^n$  ([6], [8]). The fundamental groups of these exotic manifolds are also CAT(0) groups.

On structure of CAT(0) groups, we obtained the following theorem from Theorem 1 in [15].

**Theorem 2.** *Let  $\Gamma$  be a CAT(0) group. Then there exist subgroups  $\Gamma = \Gamma_0 \supset \Gamma_1 \supset \cdots \supset \Gamma_n$ , elements  $\delta_{i+1}, \gamma_{i+1} \in \Gamma_i$  and  $k_{i+1} \in \mathbb{N}$  for  $i = 0, \dots, n-1$  such that*

- (1)  $\gamma_{i+1}$  is an element of the center of  $\Gamma_i$  with the order  $o(\gamma_{i+1}) = \infty$  for  $i = 0, \dots, n-1$ ,
- (2)  $\Gamma_i = \Gamma_{i+1} \rtimes \langle \delta_{i+1} \rangle$  for  $i = 0, \dots, n-1$ ,
- (3)  $\Gamma_{i+1} \rtimes \langle \delta_{i+1}^{k_{i+1}} \rangle = \Gamma_{i+1} \times \langle \gamma_{i+1} \rangle$  is a finite-index subgroup of  $\Gamma_i$ ,
- (4)  $\Gamma_i/\Gamma_{i+1}$  is isomorphic to  $\mathbb{Z}$  for  $i = 0, \dots, n-1$ ,
- (5)  $\Gamma = (\cdots (((\Gamma_n \rtimes \langle \delta_n \rangle) \rtimes \langle \delta_{n-1} \rangle) \rtimes \langle \delta_{n-2} \rangle) \cdots) \rtimes \langle \delta_1 \rangle$ ,
- (6)  $\Gamma_n$  has finite center, and
- (7)  $\Gamma_n \times A$  is a finite-index subgroup of  $\Gamma$  where  $A = \langle \gamma_1 \rangle \times \cdots \times \langle \gamma_n \rangle$  which is isomorphic to  $\mathbb{Z}^n$ .

Here, we introduce an easy example of a CAT(0) group.

**Example.** Let  $\Gamma = \langle a, b \mid ab^2 = b^2a \rangle$  and let  $X = \mathbb{R}^2$  the euclidean plane. We consider the action of the group  $\Gamma$  on  $X$  defined by

$$a \cdot (x, y) = (x, y + 1)$$

$$b \cdot (x, y) = (x + 1, -y)$$

for any  $(x, y) \in \mathbb{R}^2 = X$ . Then  $D = [0, 1] \times [-\frac{1}{2}, \frac{1}{2}] \subset \mathbb{R}^2$  is a fundamental domain,  $\Gamma D = X$  and  $\Gamma$  acts geometrically on  $X$ . Here we note that  $X/\Gamma$  is a

Klein bottle and the group  $\Gamma$  is a CAT(0) group which is the fundamental group of the Klein bottle.

Then  $\gamma_0 := b^2$  is a center of the CAT(0) group  $\Gamma$  and a hyperbolic isometry of  $X$ . Here we obtain that

- (i)  $\Gamma = \langle a \rangle \rtimes \langle b \rangle$ ,
- (ii)  $\langle a \rangle \rtimes \langle b^2 \rangle = \langle a \rangle \times \langle b^2 \rangle$  is a finite-index subgroup of  $\Gamma$  which is isomorphic to  $\mathbb{Z}^2$  and
- (iii)  $\Gamma/\langle a \rangle$  is isomorphic to  $\mathbb{Z}$ .

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