

TANGLE SUMS, NORM CURVES AND CYCLIC SURGERIES

MASAHARU ISHIKAWA, THOMAS W. MATTMAN AND KOYA SHIMOKAWA

ABSTRACT. In this paper, we introduce norm curves as 1-dimensional algebraic subsets in character varieties of 3-manifolds with torus boundary and follow the arguments in [8] and [5] in this setting. We then clarify what we can say about cyclic surgeries of knots with the specific tangle presentation $N(S + T)$ as in [16]. In the second half, we focus on r -curves and give some simple observations about cyclic surgeries. We then check the factors of A -polynomials using Culler's calculations and give a few remarks and questions about A -polynomials and their factorization.

1. INTRODUCTION

A *cyclic surgery* is a Dehn surgery along a knot or link that yields a closed 3-manifold with cyclic fundamental group. The surgery slope, i.e., the image of the meridian of the surgery, is called a *cyclic slope*. A central problem in low-dimensional topology is to characterize the 3-manifolds that can be obtained by Dehn surgeries along a given knot. In the celebrated paper of Culler, Gordon, Luecke and Shalen [8], the authors prove the *Cyclic Surgery Theorem*: if a knot complement is not a Seifert fibered space then it has at most three cyclic slopes. The $SL(2, \mathbb{C})$ -representations of the fundamental group and the associated character varieties play an important role in the first chapter of their paper. Cooper, Culler, Gillet, Long and Shalen then introduced the A -polynomial as a plane curve that carries information about the character varieties [5]. Boyer and Zhang next used the precise relationship between character varieties and A -polynomials in their study of finite surgeries, i.e., Dehn surgeries that yield closed 3-manifolds with finite fundamental groups [3].

In [16], we studied the factorization of A -polynomials of knots in S^3 that have a specific tangle presentation. As usual in knot theory, we denote by $N(T)$ and $D(T)$ the numerator and denominator closures of a tangle T , respectively, and by $N(S + T)$ the knot resulting from the sum of two tangles S and T . Let $A_K(l, m)$ denote the A -polynomial of a knot K in S^3 and $A_K^\circ(l, m)$ denote the product of the factors of $A_K(l, m)$ containing the variable l . Note that $A_K(l, m)/A_K^\circ(l, m)$ is a polynomial with only one variable m . In [16, Theorem 1.2], we proved that if $N(S + T)$ and $N(T)$ are knots and $N(S)$ is a split link in S^3 then $A_{N(T)}^\circ(l, m)$ is a factor of $A_{N(S+T)}(l, m)$. For example, the knot shown in Figure 1 satisfies these conditions. As a consequence, we obtain information about Dehn surgeries of $N(S + T)$ from those of $N(T)$. In this note, we explain what we can say about cyclic surgeries of $N(T)$ and $N(S + T)$.

Suppose $N(T)$ is hyperbolic. In this case, the $SL(2, \mathbb{C})$ -character variety $X(M_{N(T)})$ of $N(T)$ has an irreducible curve component X_0 containing the character of a discrete faithful representation of $\pi_1(M_{N(T)})$, where M_K denotes the complement of a small open tubular neighborhood of a knot K in S^3 . Let $M_K(\alpha)$ denote the closed 3-manifold obtained from M_K by Dehn filling with slope $\alpha \in \mathbb{Q} \cup \{\infty\}$. Here $\mathbb{Q} \cup \{\infty\}$ is canonically identified with

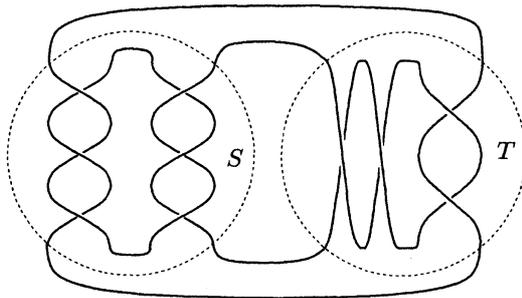


FIGURE 1. $N(T) = 4_1$, $N(S)$ is a trivial link and $N(S + T) = 9_{24}$.

the set of primitive elements of $H_1(\partial M_K; \mathbb{Z})$ modulo the sign \pm . A primitive element $\alpha \in H_1(\partial M_K; \mathbb{Z})$ is called a *boundary class* of K if there is an essential surface in M_K with boundary α , and called a *strict boundary class* if it is not a boundary class of the boundary of the fiber of any fibration in M_K over S^1 . The slope ∞ corresponds to the meridian μ as usual. For each boundary class α , define $\|\alpha\|_{X_0}$ to be the degree of a certain function $f_\alpha : \tilde{X}_0 \rightarrow \mathbb{C}P^1$ on the smooth model \tilde{X}_0 of the projective completion of X_0 , called the *Culler-Shalen norm* of α on X_0 . The function f_α and the Culler-Shalen norm will be explained in Section 2. The minimal value of $\|\alpha\|_{X_0}$ among $\alpha \in H_1(\partial M_K; \mathbb{Z}) \setminus \{0\}$ is called the *minimal norm* of $\|\cdot\|_{X_0}$. The meridian μ attains the minimal norm, i.e., $\|\mu\|_{X_0}$ is minimal, since the Dehn surgery with slope ∞ yields S^3 .

Theorem 1.1. *Suppose that $N(S + T)$ is a knot, $N(T)$ is a hyperbolic knot and $N(S)$ is a split link in S^3 . Let X_0 be the irreducible component of $X(M_{N(T)})$ containing the character of a discrete faithful representation of $\pi_1(M_{N(T)})$. If α is not a strict boundary class of $N(T)$ associated with an ideal point of X_0 and satisfies $\|\alpha\|_{X_0} > \|\mu\|_{X_0}$ then neither $\pi_1(M_{N(T)}(\alpha))$ nor $\pi_1(M_{N(S+T)}(\alpha))$ is cyclic.*

The point is that, by [16], the curve X_0 also appears in the character variety of $N(S + T)$, but this curve usually consists of the characters of non-faithful representations of $\pi_1(M_{N(S+T)})$. In particular, it generally does not contain a discrete faithful representation of $\pi_1(M_{N(S+T)})$ even if that knot is hyperbolic, and hence we need to be careful when we apply the argument in [8] to this curve. A weaker assertion still holds even if we drop the assumption that $N(T)$ is hyperbolic. This will be discussed in Section 4. Note also that this curve may not be a component of $X(M_{N(S+T)})$, i.e., may be a curve on a component of $X(M_{N(S+T)})$ of higher dimension. Because of this, we need to define norm curves as 1-dimensional algebraic subsets of character varieties. This will be done in Section 2.

As a corollary, we have a criterion for cyclic surgeries under the assumption that $N(S + T)$ is small. A knot K is said to be *small* if there is no closed essential surface in its complement. If $N(S + T)$ is small then we can use [8, Theorem 2.0.3] to argue that strict boundary slopes cannot yield cyclic fundamental groups. As mentioned, ∞ is obviously a cyclic slope since $M_K(\alpha) = S^3$. A cyclic slope is said to be *non-trivial* if it is not ∞ .

Corollary 1.2. *Suppose that $N(S + T)$ is a small knot, $N(T)$ is a hyperbolic knot and $N(S)$ is a split link in S^3 . Let X_0 be the irreducible component of $X(M_{N(T)})$ containing the character of a discrete faithful representation of $\pi_1(M_{N(T)})$. If every $\alpha \in H_1(\partial M_{N(T)}; \mathbb{Z}) \setminus \{0\}$ that is not a strict boundary class of $N(T)$ satisfies $\|\alpha\|_{X_0} > \|\mu\|_{X_0}$ then $N(S + T)$ has no non-trivial cyclic slope.*

We have the same criterion for cyclic surgeries in the context of A -polynomials. For a two variable polynomial $A(l, m)$, we define $\|\alpha\|_A$ by the integral lattice width of the two lines of slope α tangent to the Newton polygon of $A(l, m)$. More precisely, the *width* $\|\alpha\|_A$ of $\alpha \in H_1(\partial M_K; \mathbb{Z})$ is $2(\ell - 1)$, where ℓ is the number of lines of slope α that intersect the Newton polygon of $A(l, m)$ and pass through integral lattice points. Note that $\|\cdot\|_A$ is a norm if and only if the Newton polygon of $A(l, m)$ has a positive volume.

Let $A_K(l, m)$ denote the A -polynomial of a knot K in S^3 . Each norm curve X_1 of $X(M_K)$ corresponds to the product of some factors of the A -polynomial with suitable multiplicities \mathbf{d} , which we denote by $A_1^{\mathbf{d}}(l, m)$. This correspondence appears in [3], in which they proved that the two norms $\|\cdot\|_{A_1^{\mathbf{d}}}$ and $\|\cdot\|_{X_1}$ coincide. Therefore we can replace the inequality $\|\alpha\|_{X_0} > \|\mu\|_{X_0}$ in Theorem 1.1 and Corollary 1.2 by $\|\alpha\|_{A_1^{\mathbf{d}}} > \|\mu\|_{A_1^{\mathbf{d}}}$.

This paper is organized as follows. In Section 2, we define norm curves and explain their properties. The statements in this section appear originally in [8] and [2]. In Section 3, we state the corresponding statements in the context of A -polynomials. Section 4 deals with knots in S^3 of the type $N(S + T)$. In that section, we show theorems concerning norm curves and cyclic surgeries, and then give the proofs of Theorem 1.1 and Corollary 1.2. In Section 5, we focus on r -curves and give some simple observations about cyclic surgeries. In the last section, we check the factors of A -polynomials using Culler's calculations and give a few remarks and questions about A -polynomials and their factorization.

The first author would like to thank Profs. T. Kitano, T. Morifuji and Y. Yamashita for inviting him to the RIMS seminar in Hakone and giving him an opportunity to discuss this topic.

2. NORM CURVES AND CYCLIC SURGERIES

Let M be a compact, connected, irreducible and ∂ -irreducible 3-manifold whose boundary ∂M is a torus. We denote by $R(M) = \text{Hom}(\pi_1(M), SL(2, \mathbb{C}))$ the set of representations of $\pi_1(M)$ into $SL(2, \mathbb{C})$, called the representation variety of M . Let $t : R(M) \rightarrow X(M)$ be the map which sends a representation ρ to its character χ_ρ . The image $X(M)$ is called the *character variety* of M . Note that $R(M)$ and $X(M)$ are complex affine algebraic sets. For each $\gamma \in \pi_1(M)$, denote by $I_\gamma : X(M) \rightarrow \mathbb{C}$ the regular function defined by $I_\gamma(\chi_\rho) = \chi_\rho(\gamma)$ and by $f_\gamma : X(M) \rightarrow \mathbb{C}$ the function given by $f_\gamma = I_\gamma^2 - 4$.

In [8], the authors study cyclic surgeries using character varieties. In the first chapter of their paper, M is assumed to be hyperbolic since they focused on the irreducible components of the character variety containing the character of a discrete faithful representation of $\pi_1(M)$. Their results generalize to norm curve components, introduced by Boyer and Zhang in [3] in the proof of the finite filling conjecture. Here a norm curve component is a 1-dimensional irreducible component of the character variety on which I_α is not constant for every $\alpha \in H_1(\partial M; \mathbb{Z}) \setminus \{0\}$.

In this note, we relax two words in this definition; *irreducible* and *component*.

Definition 2.1. A 1-dimensional algebraic subset X_1 of $X(M)$ is called a *norm curve* if f_α is non-constant on X_1 for every $\alpha \in H_1(\partial M; \mathbb{Z}) \setminus \{0\}$. Remark that X_1 need not be irreducible.

Let X_1 be a norm curve with irreducible components $X_1^{(1)}, X_1^{(2)}, \dots, X_1^{(k)}$ and \tilde{X}_1 be the smooth model of the projective completion of X_1 with smooth model $\tilde{X}_1^{(i)}$ of $X_1^{(i)}$ for each $i = 1, \dots, k$. The function f_α is regarded as a function on \tilde{X}_1 canonically. For

each $i = 1, \dots, k$, let $R_1^{(i)}$ denote the algebraic subset of the representation variety of M whose image is $X_1^{(i)}$. The birational equivalence from \tilde{X}_1 to X_1 is regular except at a finite number of points of \tilde{X}_1 . A non-regular point on \tilde{X}_1 is called an *ideal point* of \tilde{X}_1 . For each ideal point x , $f_\gamma : \tilde{X}_1 \rightarrow \mathbb{C}\mathbb{P}^1$ may have a pole, in which case we denote the order of the pole by $\Pi_x(f_\gamma)$. If it has no pole at x then we set $\Pi_x(f_\gamma) = 0$.

Lemma 2.2 ([8], cf. [2] Proposition 4.7). *For each ideal point x of $\tilde{X}_1^{(i)}$ there is a linear function $\phi_x : H_1(\partial M; \mathbb{Z}) \rightarrow \mathbb{Z}$ satisfying the following:*

- (1) $\Pi_x(f_\alpha) = |\phi_x(\alpha)|$ for each $\alpha \in H_1(\partial M; \mathbb{Z})$.
- (2) If $\phi_x \equiv 0$ then there is a closed essential surface in M associated to x .
- (3) If $\phi_x \not\equiv 0$ then there is a unique boundary class α with $\phi_x(\alpha) \equiv 0$.

Proof. This is proved in [8] for an irreducible component X_0 of $X(M)$ containing the character of a discrete faithful representation. As is mentioned in [2, Proposition 4.7] in the context of $PSL(2, \mathbb{C})$ -character varieties, this assertion holds for any curve in $X(M)$. We here explain the proof in our setting. Let $P : \pi_1(M) \rightarrow SL(2, F)$ be the tautological representation of $R_1^{(i)}$, where $F = \mathbb{C}(R_1^{(i)})$. Let $\{\alpha_1, \alpha_2\}$ be a basis of $H_1(\partial M; \mathbb{Z})$. Suppose that $P(\alpha_1)$ and $P(\alpha_2)$ are diagonalizable over some extension E of the field F . In this case, we set

$$P(\alpha_i) = \begin{pmatrix} a_i & 0 \\ 0 & a_i^{-1} \end{pmatrix} \in SL(2, E)$$

for $i = 1, 2$. Then for any element $\alpha = m\alpha_1 + n\alpha_2$ in $H_1(\partial M; \mathbb{Z})$, we have

$$\Pi_x(f_\alpha) = \frac{2}{d} |mv_x(a_1) + nv_x(a_2)|,$$

where v_x is the valuation induced by the ideal point x and the extension, and d is some positive integer depending on the extension. Then we set $\phi_x(\alpha) = (2/d)(mv_x(a_1) + nv_x(a_2))$. This is the linear function in the assertion.

Suppose that $P(\alpha_1)$ is not diagonalizable. Up to conjugation, we may assume that

$$P(\alpha_1) = \begin{pmatrix} a_1 & 1 \\ 0 & a_1^{-1} \end{pmatrix} \in SL(2, E'), \quad a_1 = \pm 1,$$

where E' is an extension of the field F . Since $P(\alpha_1)$ and $P(\alpha_2)$ commute, $P(\alpha_2)$ must be an upper triangular matrix with diagonal, say, $\{a_2, a_2^{-1}\}$. We then set $\phi_x(\alpha) = (2/d')(mv_x(a_1) + nv_x(a_2))$, where d' is some integer depending on the extension.

If $v_x(a_1) = v_x(a_2) = 0$ then $\phi_x \equiv 0$, in which case a closed essential surface is detected by the ideal point x because any $\alpha \in \pi_1(\partial M)$ must be disjoint from the surface. Otherwise,

$$\alpha = \frac{1}{\gcd(v_x(a_2), v_x(a_1))} (v_x(a_2)\alpha_1 - v_x(a_1)\alpha_2)$$

is the unique boundary class with $\phi_x(\alpha) \equiv 0$, and the valuation at x detects an essential surface with boundary class α as in [9, 8]. \square

For each $\alpha \in H_1(\partial M; \mathbb{Z})$, define $\|\cdot\|_{X_1}$ by

$$\|\alpha\|_{X_1} = \sum_{i=1}^k (\text{the degree of } f_\alpha \text{ on } \tilde{X}_1^{(i)}).$$

Lemma 2.3. $\|\cdot\|_{X_1}$ is a norm.

Proof. Since $\|\cdot\|_{X_1}$ is a sum of absolute values of linear functions, positive homogeneity and sub-additivity hold. Hence, it is enough to show that $\|\alpha\|_{X_1} > 0$ for $\alpha \neq 0$. From the definition of norm curves, f_α is non-constant on X_1 for every $\alpha \in H_1(\partial M; \mathbb{Z}) \setminus \{0\}$. Hence $\|\alpha\|_{X_1} > 0$. \square

Lemma 2.4. *The unit ball for $\|\cdot\|_{X_1}$ is a finite-sided polygon whose vertices are rational multiples of boundary classes of M .*

Proof. Let $\Phi_x : \mathbb{R}^2 \rightarrow \mathbb{R}$ be the linear function obtained by extending $\phi_x : H_1(\partial M; \mathbb{Z}) \rightarrow \mathbb{Z}$ linearly. Each vertex on the boundary of the unit ball corresponds to a boundary class α with $\Phi_x(\alpha) = 0$ for some ideal point x , i.e., $\Pi_x(f_\alpha) = 0$. Hence α is a boundary slope by Lemma 2.2 (3). \square

The minimal value of $\|\alpha\|_{X_1}$ among $\alpha \in H_1(\partial M; \mathbb{Z}) \setminus \{0\}$ is called the *minimal norm* of $\|\cdot\|_{X_1}$. We denote it by s_{X_1} . For a primitive element α in $H_1(\partial M; \mathbb{Z})$, let $M(\alpha)$ denote the closed 3-manifold obtained from M by Dehn filling with slope α .

Proposition 2.5 (cf. [8] Proposition 1.1.3). *Let M be a compact, connected, irreducible and ∂ -irreducible 3-manifold with torus boundary ∂M and X_1 be a norm curve of M . Assume that each irreducible component of X_1 contains the character of an irreducible representation of $\pi_1(M)$. If α is not a boundary class associated with an ideal point of X_1 and satisfies $\|\alpha\|_{X_1} > s_{X_1}$ then $\pi_1(M(\alpha))$ is not cyclic.*

Proof. The assertion corresponds to [8, Proposition 1.1.3] and the main part of the proof is written in Section 1.5 and 1.6 of that paper. The proof goes through for the norm curve X_1 except for Proposition 1.5.5. In the proof of Proposition 1.5.5, they use the property that the discrete faithful representation of the hyperbolic manifold M is irreducible. In our case, the representation is irreducible by assumption.

We also need to be careful about the difference in hypotheses. In [8, Proposition 1.1.3], it's assumed α is not a strict boundary class. In our statement, we assume it is not a boundary class associated with an ideal point of X_1 . In the proof of [8, Proposition 1.3.9], their hypothesis is used to conclude that " α is a strict boundary class", and we replace it by " α is a boundary class associated with an ideal point" so that we do not use [8, Proposition 1.2.7]. The hypothesis is used again in 1.6.2 and in page 264. In both parts, we can easily verify that the proofs work in our case. \square

3. A-POLYNOMIALS AND CYCLIC SURGERIES

We introduce the A -polynomial of a norm curve following [3]. We need to modify the setting slightly since X_1 may not be irreducible. Fix a basis $\{\mu, \lambda\}$ of $\pi_1(\partial M)$. Let $\iota^* : X(M) \rightarrow X(\partial M)$ be the regular map induced by the homomorphism $\iota_\# : \pi_1(\partial M) \rightarrow \pi_1(M)$, Λ be the set of diagonal representations of $\pi_1(\partial M)$, which is an algebraic subset of $R(\partial M)$ and $t|_\Lambda : \Lambda \rightarrow X(\partial M)$, which is surjective and of degree two. We then define $p : \Lambda \rightarrow \mathbb{C}^* \times \mathbb{C}^*$ by taking the top-left entries of $\rho(\mu)$ and $\rho(\lambda)$. Now let X_1 be a norm curve of M with irreducible components $X_1^{(i)}$ for $i = 1, \dots, k$. For each i , the algebraic closure $Y_1^{(i)}$ of $\iota^*(X_1^{(i)})$ in $X(\partial M)$ is an irreducible curve in $X(\partial M)$. We then define $D_1^{(i)}$ to be the algebraic closure of $p(t|_\Lambda^{-1}(Y_1^{(i)}))$ in \mathbb{C}^2 . Let $A_1^{(i)}(l, m)$ denote the defining equation of $D_1^{(i)}$ and $d_1^{(i)}$ denote the degree of the map $\iota^*|_{X_1^{(i)}} : X_1^{(i)} \rightarrow X(\partial M)$. Then the

A -polynomial $A_1^d(l, \mathbf{m})$ of X_1 is defined as

$$A_1^d(l, \mathbf{m}) = \prod_{i=1}^k A_1^{(i)}(l, \mathbf{m})^{d_1^{(i)}}.$$

By construction, the variables \mathbf{m} and l correspond to the eigenvalues of $\rho(\mu)$ and $\rho(\lambda)$ respectively. Let $A_1(l, \mathbf{m}) = \prod_{i=1}^k A_1^{(i)}(l, \mathbf{m})$, denote the A -polynomial without multiplicities.

For the A -polynomial $A_1^d(l, \mathbf{m})$, the *width* $\|\alpha\|_{A_1^d}$ of $\alpha \in H_1(\partial M_K; \mathbb{Z})$ is defined by $2(\ell - 1)$, where ℓ is the number of lines of slope α that intersect the Newton polygon of $A_1^d(l, \mathbf{m})$ and pass through integral lattice points. Since X_1 is a norm curve, the Newton polygon of $A_1^d(l, \mathbf{m})$ has a positive volume, otherwise there is an $\alpha \in H_1(\partial M; \mathbb{Z})$ with f_α constant. Therefore $\|\cdot\|_{A_1^d}$ is a norm. It is proved in [3] that the two norms $\|\cdot\|_{A_1^d}$ and $\|\cdot\|_{X_1}$ coincide. In particular, the assertion in Proposition 2.5 can also be stated in terms of A -polynomials by replacing the inequality $\|\alpha\|_{X_1} > s_{X_1}$ by $\|\alpha\|_{A_1^d} > s_{A_1^d}$, where $s_{A_1^d}$ is the minimal norm of $\|\cdot\|_{A_1^d}$.

If M is a knot complement in a homology 3-sphere, we usually set the basis $\{\mu, \lambda\}$ to be the meridian and longitude pair of the knot. In this case, as mentioned in [5, Section 2.5], if $X_1^{(i)}$ contains the character of an irreducible representation then the curve $A_1^{(i)}(\mathbf{m}, l) = 0$ is different from the line $l = 1$. In particular, Proposition 2.5 can be restated as follows.

Proposition 3.1. *Let M be a knot complement in a homology 3-sphere, X_1 a norm curve of M and $A_1^d(l, \mathbf{m})$ be the A -polynomial corresponding to X_1 . Suppose that $A_1^d(l, \mathbf{m})$ has no factor $1 - l$. If α is not a boundary class associated with an ideal point of X_1 and satisfies $\|\alpha\|_{A_1^d} > s_{A_1^d}$ then $\pi_1(M(\alpha))$ is not cyclic.*

Proof. Since the line $l = 1$ is excluded from $A_1^d(l, \mathbf{m}) = 0$, as mentioned in [5, Section 2.5], each irreducible component $X_1^{(i)}$ contains the character of an irreducible representation of $\pi_1(M)$. Hence the assertion follows from Proposition 2.5 and the coincidence of $\|\cdot\|_{A_1^d}$ and $\|\cdot\|_{X_1}$. \square

Remark 3.2. The boundary class associated with an ideal point of X_1 corresponds to a slope of the boundary of the Newton polygon of $A_1(l, \mathbf{m})$. More precisely, if the boundary class is $\alpha = pm + ql$ then the boundary of the Newton polygon has a segment with the slope p/q , see [5, Theorem 3.4].

We close this section by introducing a simple sufficient condition for the inequality $\|\alpha\|_{A_1^d} > s_{A_1^d}$ in the case of the complement of a knot in S^3 .

Lemma 3.3. *Let M_K be the complement of a knot K in S^3 , X_1 a norm curve of M_K and $A_1^d(l, \mathbf{m})$ the A -polynomial corresponding to X_1 . Suppose that $A_1^d(l, \mathbf{m})$ has no factor $1 - l$. Let d_l be the degree of $A_1^d(l, \mathbf{m})$ as a polynomial in l and (x_0, y_0) be the center of the Newton polygon of $A_1^d(l, \mathbf{m})$, where (x, y) are the coordinates on which we describe the Newton polygon. If the point $(x_0, y_0 - d_l/2)$ is contained in the interior of the Newton polygon then any $\alpha \in H_1(\partial M_K; \mathbb{Z}) \setminus \{0\}$ except for μ satisfies $\|\alpha\|_{A_1^d} > s_{A_1^d}$ ($= \|\mu\|_{A_1^d}$). In particular, if the knot K is small then K has no non-trivial cyclic slopes.*

Proof. Let ℓ_μ be the number of lines with slope ∞ which pass through the integral points and intersect the Newton polygon of $A_1^d(l, \mathbf{m})$. Suppose that d_l is even. Then, since $(x_0, y_0 - d_l/2)$ is an interior point of the Newton polygon, we can easily see that there

are at least ℓ_μ integral points on the line $x = x_0$ in the interior of the Newton polygon. If d_l is odd then there are at least $\ell_\mu - 1$ integral points on the line $x = x_0 + 1/2$ in the interior of the Newton polygon and we can find at least one another integral point on the line $x = x_0 - 1/2$ in the interior of the polygon. Therefore $\|\alpha\|_{A_1^d} > \|\mu\|_{A_1^d}$ holds for any $\alpha \neq \mu$. \square

Example 3.4. The A -polynomial of 4_1 is calculated in [8] as

$$A_{4_1}(l, m) = m^4 + l(-1 + m^2 + 2m^4 + m^6 - m^8) + l^2m^4.$$

The Newton polygon of this polynomial is shown in Figure 2. As mentioned in Remark 3.2, the slopes ± 4 on the boundary of this polygon correspond to the boundary slopes of essential surfaces in M_{4_1} . Since $A_{4_1}(l, m) = 0$ is irreducible, $\|\alpha\|_{A_{4_1}^d} > \|\mu\|_{A_{4_1}^d}$ if and only if $\|\alpha\|_{A_{4_1}} > \|\mu\|_{A_{4_1}}$.

Now we use Lemma 3.3. The center of the polygon is $(1, 4)$ and $d_l = 2$, hence $(x_0, y_0 - d_l/2) = (1, 3)$, which is a point in the interior of the polygon. Thus, by Lemma 3.3, $\|\alpha\|_{A_1^d} > s_{A_1^d} = \|\mu\|_{A_1^d}$ for any $\alpha \neq \mu$. Since 4_1 is small, it has no non-trivial cyclic slopes.

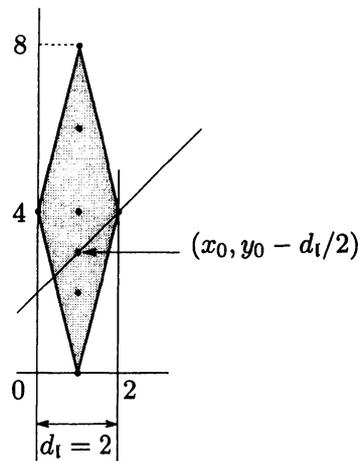


FIGURE 2. The Newton polygon of $A_{4_1}(l, m)$.

4. KNOTS OF TYPE $N(S + T)$ AND CYCLIC SURGERIES

In this section, we focus on the knots $N(T)$ and $N(S+T)$ explained in the introduction. Recall that the main theorems are stated under the assumption that $N(S+T)$ and $N(T)$ are knots and $N(S)$ is a split link in S^3 . We will assume this throughout this section. We denote by M_K the complement of a knot K in S^3 . A *norm curve* of K means a norm curve of M_K in $X(M_K)$.

Lemma 4.1. *Let X_1 be a norm curve of $N(T)$ and suppose that f_μ is not constant on any irreducible component of X_1 . Then X_1 appears in $X(M_{N(S+T)})$ and is a norm curve of $N(S+T)$.*

Proof. In [16], it is proved that a $SL(2, \mathbb{C})$ -representation of $\pi_1(M_{N(T)})$ extends to one of $\pi_1(M_{N(S+T)})$ if f_μ is not locally constant in $R(M_{N(T)})$. Therefore X_1 appears in

$X(M_{N(S+T)})$, where we denote it as \hat{X}_1 . In the construction of the extension, $\pi_1(\partial M_{N(S+T)})$ is identified with $\pi_1(\partial M_{N(T)})$ by sending the generators of the Wirtinger presentation of $\pi_1(\partial M_{N(S+T)})$ in the tangle S to suitable elements in $\pi_1(\partial M_{N(S+T)})$ so that they are globally canceled, while the generators in the tangle T are unchanged. Therefore, we have a canonical identification of $H_1(\partial M_{N(S+T)}; \mathbb{Z})$ and $H_1(\partial M_{N(T)}; \mathbb{Z})$. In particular, for each $\alpha \in H_1(\partial M_{N(T)}; \mathbb{Z}) = H_1(\partial M_{N(S+T)}; \mathbb{Z})$, we have $f_\alpha = \hat{f}_\alpha$, where \hat{f} is the corresponding function on the character variety of $N(S+T)$. Hence \hat{X}_1 is a norm curve of $N(S+T)$. \square

Theorem 4.2. *Suppose that $N(S+T)$ and $N(T)$ are knots and $N(S)$ is a split link in S^3 . Let X_1 be a norm curve of $N(T)$. Suppose that each irreducible component of X_1 contains the character of an irreducible representation and that f_μ is not constant on any irreducible component of X_1 . If α is not a boundary class of $N(T)$ associated with an ideal point of X_1 and satisfies $\|\alpha\|_{X_1} > \|\mu\|_{X_1}$ then neither $\pi_1(M_{N(T)}(\alpha))$ nor $\pi_1(M_{N(S+T)}(\alpha))$ is cyclic.*

Proof. The conclusion for $\pi_1(M_{N(T)}(\alpha))$ follows directly from Proposition 2.5. To show the assertion for $\pi_1(M_{N(S+T)}(\alpha))$ note that, by Lemma 4.1, the norm curve X_1 also appears in $X(M_{N(S+T)})$ as a norm curve of $N(S+T)$. \square

Theorem 4.3. *Suppose that $N(S+T)$ and $N(T)$ are knots and $N(S)$ is a split link in S^3 . Let $A_1^d(l, \mathbf{m})$ be the A -polynomial corresponding to a norm curve X_1 of $N(T)$. Suppose that $A_1^d(l, \mathbf{m})$ does not contain the factor $1-l$ nor factors that depend only on the variable \mathbf{m} . If α is not a boundary slope of $N(T)$ associated with an ideal point of X_1 and satisfies $\|\alpha\|_{A_1^d} > \|\mu\|_{A_1^d}$ then neither $\pi_1(M_{N(T)}(\alpha))$ nor $\pi_1(M_{N(S+T)}(\alpha))$ is cyclic.*

Proof. Since $A_1^d(l, \mathbf{m})$ does not contain factors that depend only on \mathbf{m} , f_μ is not constant. Hence the norm curve X_1 appears in $X(M_{N(S+T)})$ as a norm curve of $N(S+T)$ by Lemma 4.1, denoted it by \hat{X}_1 . We can easily see that $\iota^* : X_1 \rightarrow X(\partial M_{N(T)})$ and $\hat{\iota}^* : \hat{X}_1 \rightarrow X(\partial M_{N(S+T)})$ have the same degree. Therefore, the multiplicities of the factors of the A -polynomial of \hat{X}_1 are same as those of $A_1^d(l, \mathbf{m})$. Hence the assertion follows from Theorem 4.2 and the coincidence of the norms $\|\cdot\|_{X_1}$ and $\|\cdot\|_{A_1^d}$, \square

Proof of Theorem 1.1. As in Lemma 4.1, there exists a norm curve X_1 of $N(S+T)$ which is exactly same as the irreducible component X_0 of $X(M_{N(T)})$ containing the character of a discrete faithful representation. Let $P : \pi_1(M_{N(S+T)}) \rightarrow SL(2, F)$ be the tautological representation of X_1 . Since X_0 contains the character of a discrete faithful representation, P is irreducible.

Let \mathcal{T} be the tree associated with an ideal point x of X_1 and \mathcal{O} the induced valuation ring. Suppose that there is a non-trivial normal subgroup of $\pi_1(M_{N(S+T)})$ which fixes a vertex of \mathcal{T} and let N denote its image by the representation P . Assume that N is contained in the center of $SL(2, F)$, i.e. in \pm the identity matrix. If the trace of N is not constant, N is diagonalizable over an extension E of the field F . We can easily check that this diagonal matrix and an element of $SL(2, F)$ commute only when it is diagonal. Hence $SL(2, F)$ must be reducible. If the trace of N is constant then N is conjugate to

$$\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix} \in SL(2, E'), \quad a = \pm 1,$$

where E' is an extension of F . Then an element of $SL(2, F)$ commutes with this matrix only when it is upper-triangular. Hence $SL(2, F)$ is again reducible. Since P is irreducible,

we conclude that N cannot be contained in the center of $SL(2, F)$. Since N fixes a vertex of \mathcal{T} , N is contained in a conjugate of $SL(2, \mathcal{O})$.

By [8, Lemma 1.2.8], either every element of $P(\pi_1(M_{N(S+T)}))$ has a trace in \mathcal{O} or $P(\pi_1(M_{N(S+T)}))$ has an abelian subgroup of index at most 2. The first possibility implies that for any element $\gamma \in \pi_1(M_{N(T)})$, f_γ is constant, but this is impossible. The second possibility implies that, since P is faithful for $N(T)$, $\pi_1(M_{N(T)})$ has an abelian subgroup of index at most 2, contradicting the assumption that $N(T)$ is hyperbolic. This proves that there is no non-trivial normal subgroup of $\pi_1(M_{N(S+T)})$ which fixes a vertex of \mathcal{T} . In particular, α is not the boundary class of a fiber of any fibration of $N(T)$ over S^1 . As consequence, we conclude that α is a strict boundary class in the proof of Proposition 2.5. This completes the proof of Theorem 1.1. \square

Proof of Corollary 1.2. If α is not a boundary class associated with an ideal point of X_0 then the assertion follows from Theorem 1.1. If α is, then it is a boundary class of $N(S+T)$ because X_0 also appears in $X(M_{N(S+T)})$. In particular, α is a strict boundary class. Since $N(S+T)$ is small, by [8, Theorem 2.0.3 and Addendum 2.0.4], $\pi_1(M_{N(S+T)}(\alpha))$ is not cyclic. Therefore $N(S+T)$ has no non-trivial cyclic slope. \square

Example 4.4. In [16], there is a table of factorization of A -polynomials of RTF knots. For example, 9_{24} has a knot diagram as in Figure 1, shown in the introduction, i.e., $9_{24} = N((1/3 + (-1/3)) + 5/2)$. Since $N(5/2) = 4_1$, [16, Theorem 1.2] implies that the A -polynomial of 4_1 appears in the A -polynomial of 9_{24} as a factor. As explained in Example 3.4, the Newton polygon of $A_{4_1}(l, m)$ ensures that $\|\alpha\|_{A_{4_1}^d} > \|\mu\|_{A_{4_1}^d}$ for any $\alpha \in H_1(\partial M_{4_1}; \mathbb{Z}) \setminus \{0\}$ with $\alpha \neq \infty$. Since 9_{24} is small, Corollary 1.2 and the coincidence of the norms $\|\cdot\|_{X_{4_1}}$ and $\|\cdot\|_{A_{4_1}^d}$ show that 9_{24} has no non-trivial cyclic surgery. Note that cyclic surgeries of Montesinos knots, such as 9_{24} , have been classified by Ichihara and Jong [15] (cf. [10]).

Remark 4.5. Even if $N(S+T)$ is not small (i.e., is large), Theorem 4.2 and 4.3 leave only a few slopes of $N(S+T)$ as candidates for a cyclic slope. For example, suppose that we already know the A -polynomial $A_1^d(l, m)$ of a norm curve X_1 of $M_{N(T)}$ from which we can deduce that $N(T)$ has no cyclic slope. Then, Theorem 4.3 shows that $\pi_1(M_{N(S+T)}(\alpha))$ is not cyclic for any $\alpha \in H_1(M_{N(S+T)}) \setminus \{0\}$ except for the boundary classes of $N(T)$ associated with ideal points of X_1 . Therefore, if we want to determine the cyclic surgeries of $N(S+T)$, we need only check the boundary classes of $N(T)$ appearing as the slopes of the boundary of the Newton polygon of $A_1^d(l, m)$.

5. r -CURVES AND CYCLIC SURGERIES

Let M be a compact, connected, irreducible and ∂ -irreducible 3-manifold whose boundary ∂M is a torus. By definition, an irreducible component $X_1^{(i)}$ of a norm curve X_1 need not be a norm curve. Let Y be a 1-dimensional algebraic subset Y of $X(M)$; for instance $Y = X_1^{(i)}$. For each $\alpha \in H_1(\partial M; \mathbb{Z})$, define $\|\cdot\|_Y$ by

$$\|\alpha\|_Y = \text{the degree of } f_\alpha \text{ on } \tilde{Y},$$

where \tilde{Y} is the smooth model of the projective completion of Y .

Definition 5.1 (cf. [2], p.769). A 1-dimensional algebraic subset Y of $X(M)$ is called an r -curve if $\|\cdot\|_Y$ is non-zero, not a norm curve and $\|\alpha\|_Y = 0$ only when $\alpha = r$.

Lemma 5.2. *Let X_1 be an algebraic curve in $X(M)$ whose irreducible components are r -curves. Suppose that X_1 has two r -curves with different slopes, i.e., an r_1 -curve and an r_2 -curve with $r_1 \neq r_2$. Then X_1 is a norm curve.*

Proof. For each $i = 1, 2$, let $f_\alpha^{(i)}$ be the function f_α on the r_i -curve. If $f_\alpha^{(1)}$ and $f_\alpha^{(2)}$ are both constant then we have $\alpha = r_1$ and $\alpha = r_2$, which is a contradiction. \square

Proposition 5.3. *Suppose that $X(M)$ contains an algebraic curve X_1 consisting of two r -curves with different slopes, $r_1 \neq r_2$, each of which contains the character of an irreducible representation. If $\alpha \neq r_1, r_2$ satisfies $\|\alpha\|_{X_1} > s_{X_1}$ then $\pi_1(M(\alpha))$ is not cyclic.*

Proof. The assertion follows from Lemma 5.2 and Proposition 2.5. \square

Remark 5.4. In the case where M is a knot complement in a homology 3-sphere, if Y consists entirely of characters of reducible representations, then $A_Y(l, \mathbf{m}) = 1 - l$ [5, Section 2.5]. Therefore, if $r \neq 0$ then the r -curve contains the character of an irreducible representation.

In [2], the character variety of $PSL(2, \mathbb{C})$ -representations of $\pi_1(M)$ is used in the study of finite surgeries. We denote it by $X^{PSL}(M)$. The r -curve appears in their paper in the context of $PSL(2, \mathbb{C})$ -representations. We here introduce some of their results related to r -curves. For two slopes $\alpha_i = p_i \mathbf{m} + q_i l$, $i = 1, 2$, we denote $\Delta(\alpha_1, \alpha_2) = \Delta(p_1/q_1, p_2/q_2) = |p_1 q_2 - p_2 q_1|$.

Theorem 5.5 ([2] Corollary 6.5 (1) with Remark 6.3 (i) and (iii)). *Suppose that $X^{PSL}(M)$ has an r -curve which contains the character of an irreducible representation.*

- (1) *If α is not a boundary slope and $\pi_1(M(\alpha))$ is cyclic then $\Delta(r, \alpha) \leq 1$.*
- (2) *If there is no closed essential surface in M which remains incompressible in $M(r)$ (for instance when M is small) and $\pi_1(M(\alpha))$ is cyclic then $\Delta(r, \alpha) \leq 1$.*

The next corollary is the $SL(2, \mathbb{C})$ -version of [3, Corollary 6.7], and the proof is also analogous.

Corollary 5.6. *Let K be a knot in S^3 . Suppose that the meridian is not a boundary slope of an essential surface (for instance when K is small). If $X(M_K)$ has an r -curve then $r \in \mathbb{Z}$.*

Proof. Let Y be an r -curve in $X(M_K)$. If Y consists of the characters of reducible representations, then $r = 0 \in \mathbb{Z}$. Suppose that Y contains the character of an irreducible representation. There exists an r -curve in $X^{PSL}(M_K)$ with the same r . Since $1/0$ is not a boundary slope, $r \neq \infty$. Moreover, since $M(1/0) = S^3$, $\alpha = 1/0$ is a cyclic slope. Hence we have $\Delta(p/q, 1/0) \leq 1$ by Theorem 5.5 (1), which is satisfied only when $q = 1$. \square

By an easy observation, we obtain the following simple sufficient condition for the non-existence of cyclic slopes using r -curves.

Corollary 5.7. *Let K be a small knot in S^3 . Suppose that $X(M_K)$ has an r_1 -curve and an r_2 -curve with $r_i \neq 0$ for $i = 1, 2$ and $|r_1 - r_2| > 2$. Then K has no non-trivial cyclic slope.*

Proof. As before, there exist an r_1 -curve and an r_2 -curve in $X^{PSL}(M_K)$. By Corollary 5.6, $r_1, r_2 \in \mathbb{Z}$. Since $r_i \neq 0$, each r_i -curve contains the character of an irreducible representation. Assume that there exists a cyclic slope p/q . Since K is small, Theorem 5.5 (2)

implies that $\Delta(r_i, p/q) = |qr_i - p| \leq 1$ for each $i = 1, 2$. In particular, we have

$$\left| r_i - \frac{p}{q} \right| \leq \frac{1}{q} \leq 1.$$

Hence, if $|r_1 - r_2| > 2$ then such a p/q does not exist. \square

Example 5.8. The following is the list of r -curves of knots based on the calculation of A -polynomials by Marc Culler, which can be found in [4]. Torus knots are not included in the list. Note that the A -polynomial of a (p, q) -torus knot has $1 + \text{lm}^{pq}$ as a factor, see [5, Proposition 2.7].

$8_{11} : \text{l} + \text{m}^6$	$9_{48} : \text{l} - \text{m}^4$	$10_{144} : \text{l} - \text{m}^{12}$
$8_{21} : \text{l} + \text{m}^2$	$10_{61} : 1 - \text{lm}^{12}$	$10_{152} : (\text{l} + \text{m}^{11})(\text{l} - \text{m}^{11})$
$9_{23} : \text{l} + \text{m}^{18}$	$10_{139} : 1 - \text{lm}^{20}$	$10_{155} : \text{l} + \text{m}^2$
$9_{37} : \text{l} - \text{m}^4$	$10_{140} : 1 - \text{l}$	$12n_{475} : \text{l} - \text{m}^4$
$9_{38} : (1 - \text{m})^2(1 + \text{m})^2$	$10_{141} : (\text{l} - \text{m}^4)(1 + \text{lm}^2)$	$12n_{582} : 1 - \text{l}$
$9_{41} : 1 + \text{lm}^2$	$10_{142} : 1 - \text{lm}^{12}$	$12n_{725} : (1 + \text{lm}^{15})(1 - \text{lm}^{15})$
$9_{46} : 1 + \text{lm}^2$	$10_{143} : \text{l} - \text{m}^8$	

Note that the A -polynomial of the mirror image K^{mir} of a knot K is given by $A_{K^{\text{mir}}}(\text{l}, \text{m}) = \text{m}^d A_K(\text{l}, 1/\text{m})$, where d is a suitable positive integer so that the right hand side becomes a polynomial. In particular, if K is amphichiral then we have $A_K(\text{l}, \text{m}) = \text{m}^d A_K(\text{l}, 1/\text{m})$. It is well-known that 4_1 is amphichiral and this is why the Newton polygon in Figure 2 is so symmetric.

Among the knots listed above, only 10_{141} has two r -curves with $r = -4$ and 2 . Since $10_{141} = N(1/4 + 2/3 + (-1/3))$ is small, it has no cyclic slope by Corollary 5.7. As mentioned above, this was previously shown in [15]. Remark that we cannot use Lemma 3.3 directly in this case because we need to know the multiplicities of these r -curves.

Remark 5.9. From the table of known RTH knots in [16], we can conclude that the A -polynomials of $8_{10}, 8_{11}, 10_{40}, 10_{59}, 10_{98}, 10_{99}, 10_{143}$ and 10_{147} have the factor $1 + \text{lm}^6$, and the A -polynomials of 10_{21} and 10_{62} have the factor $1 + \text{lm}^{10}$. Among these, the A -polynomials of $8_{10}, 8_{11}, 10_{21}, 10_{143}$ and 10_{147} are calculated by Culler, although the r -curve factors are missing except in the case of 8_{11} . Note that $9_{24} = N(1/3 + 5/2 + (-1/3))$ and $9_{37} = N(1/3 + 5/3 + (-1/3))$ with $N(5/2) = N(5/3) = 4_1$, and we can verify in Culler's calculation that the A -polynomial of 4_1 appears in their A -polynomials as a factor.

6. SOME REMARKS AND QUESTIONS

In Culler's calculations, the following knots have A -polynomials containing at least two factors other than r -curves or $A_{4_1}(\text{l}, \text{m})$:

$$7_4, 7_7, 8_5, 8_{15}, 8_{16}, 8_{18}, 9_{10}, 9_{16}, 9_{17}, 9_{23}, 9_{28}, 9_{29}, 9_{31}, 9_{35}, 9_{37}, 9_{38}, 9_{41}, 9_{47}, 9_{48}, 9_{49}, \\ 10_{61}, 10_{64}, 10_{136}, 10_{138}, 10_{142}, 10_{144}, 10_{145}, 10_{146}, 10_{147}, 10_{154}, 10_{155}, 10_{160}, 10_{163}.$$

Sometimes we can find factors with a small number of monomials. We here list the knots with such small factors. The left hand side of each column indicates a knot type and the

right hand side is the small factor of its A -polynomial:

$$\begin{aligned}
7_4 &: 1 + l(-1 + m^2 + 2m^4 + m^6 - m^8) + l^2 m^8 \\
7_7 &: m^4 + l(-1 + 2m^2 + 2m^4 - m^8 + m^{10}) + l^2(1 - m^2 + 2m^6 + 2m^8 - m^{10}) + l^3 m^6 \\
8_5 &: 1 + l(m^2 - m^4 + 2m^8 + 2m^{10} - m^{12}) + l^2(-m^{10} + 2m^{12} + 2m^{14} - m^{18} + m^{20}) + l^3 m^{22} \\
8_{18} &: (m^2 + l(1 - 2m^2 - 3m^4 + 2m^6 + 6m^8 + 2m^{10} - 3m^{12} - 2m^{14} + m^{16}) + l^2 m^{14}) \\
&\quad \times (m^{14} + l(1 - 2m^2 - 3m^4 + 2m^6 + 6m^8 + 2m^{10} - 3m^{12} - 2m^{14} + m^{16}) + l^2 m^2) \\
9_{49} &: 1 + l(-m^4 + m^6 + 2m^8 + m^{10} - m^{12}) + l^2 m^{16} \\
10_{136} &: 1 + l(-1 + 2m^2 + m^4) + l^2(-m^6 + m^8) + l^3(-m^{10} - 2m^{12} + m^{14}) - l^4 m^{14} \\
10_{142} &: 1 + l(-m^8 + m^{10} + 2m^{12} + m^{14} - m^{16}) + l^2 m^{24} \\
10_{145} &: m^6 + l(1 - m^2 - 2m^4 - m^6 + m^8) + l^2 m^2 \\
10_{146} &: m^{12} + l(m^2 - m^4 - 2m^6 - m^8 + m^{10}) + l^2 \\
10_{147} &: m^2 + l(1 - m^2 - 2m^4 - m^6 + m^8) + l^2 m^6 \\
10_{155} &: m^{10} + l(1 - 2m^2 - 3m^4 + 2m^6 + 6m^8 + 2m^{10} - 3m^{12} - 2m^{14} + m^{16}) + l^2 m^6 \\
10_{160} &: 1 + l(-m^8 + 2m^{10} + 2m^{12} - m^{16} + m^{18}) + l^2(m^{20} - m^{22} + 2m^{26} + 2m^{28} - m^{30}) + l^3 m^{38} \\
10_{163} &: m^{12} + l(-m^4 + 2m^6 + 2m^8 - m^{12} + m^{14}) + l^2(1 - m^2 + 2m^6 + 2m^8 - m^{10}) + l^3 m^2
\end{aligned}$$

The A -polynomial of 8_{18} has a pair of small factors, one of which is obtained from the other by substituting $1/m$ for m and multiplying a positive power of m . This is because 8_{18} is an amphichiral knot. Another remark is that the small factors of 10_{145} and 10_{147} coincide up to the substitution of $1/m$ for m and a multiplication of a positive power of m . It would be interesting to understand why they have the same factor.

Question 6.1. Why do the A -polynomials of 10_{145} and 10_{147} have the same factor?

Next, we would like to point out that Culler's program sometimes misses factors coming from an epimorphism, not always but in many cases. It is shown in [14] that if there is an epimorphism $\phi : \pi_1(M_{K_1}) \rightarrow \pi_1(M_{K_2})$ for two knots K_1 and K_2 which preserves peripheral structure then their A -polynomials have the factorization

$$(1) \quad A_{K_2}(l, m) \mid (l^d - 1)A_{K_1}(l^d, m),$$

where d is an integer given by $\phi(\lambda_{K_1}) = \lambda_{K_2}^d$. Here λ_K represents the longitude of a knot K . In [18, 19], the existence of such epimorphisms is determined for any pairs of knots in Rolfsen's knot table. In particular, they conclude that there is an epimorphism $\pi_1(M_K) \rightarrow \pi_1(M_{4_1})$ which preserves peripheral structure if and only if K is in the following list:

$$8_{18}, 9_{37}, 9_{40}, 10_{58}, 10_{59}, 10_{60}, 10_{122}, 10_{136}, 10_{137}, 10_{138}.$$

Among them, we can find the A -polynomials of $8_{18}, 9_{37}, 10_{136}, 10_{137}, 10_{138}$ in Culler's calculation.

As studied in [19], the epimorphism $\phi : \pi_1(M_{8_{18}}) \rightarrow \pi_1(M_{4_1})$ satisfies $\phi(\lambda_{8_{18}}) = \lambda_{4_1}^2$. Therefore, factorization (1) ensures that $A_{8_{18}}(l^2, m)$ must contain $A_{4_1}(l, m)$ as a factor. However we cannot find it in Culler's calculation. The knots 10_{136} and 10_{138} are in the same situation. The knot 9_{37} is an RTF knot with $N(T) = 4_1$ and we can verify the factorization $A_{4_1}(l, m) \mid A_{9_{37}}(l, m)$ in his calculation as already mentioned in Remark 5.9.

The epimorphism for 10_{137} maps the longitude $\lambda_{10_{137}}$ to the trivial element in $\pi_1(M_{4_1})$ and hence factorization (1) does not work in this case.

By the way, we can see in Culler's calculation that the A -polynomial of 10_{139} is quite small:

$$A_{10_{139}}(l, m) = (1 - lm^{20}) \\ \times (1 + l(-m^{12} + 7m^{14} - 3m^{16} + m^{18}) + 6l^2m^{28} + l^3(m^{38} - 3m^{40} + 7m^{42} - m^{44}) + l^4m^{56}).$$

Another example of a knot with small A -polynomial is the $(-2, 3, 7)$ -pretzel knot. Its A -polynomial is

$$A_{(-2, 3, 7)\text{-pretzel}}(l, m) = 1 + l(-m^{16} + 2m^{18} - m^{20}) + l^2(-2m^{36} - m^{38}) + l^4(m^{72} + 2m^{74}) \\ + l^5(m^{90} - 2m^{92} + m^{94}) - l^6m^{110},$$

as calculated in [5]. We can easily check that all strict boundary classes of these knots appear as the slopes of the Newton polygons of the above A -polynomials, which are the slopes 12, 13, 18, 20 for 10_{139} and the slopes 16, $37/2$, 20 for the $(-2, 3, 7)$ -pretzel knot.

Both of them are fibered knots of slalom divides introduced by A'Campo in [1], which are the slalom curves shown in Figure 3. There are several way to obtain a knot diagram from a divide [12, 7, 11] and we can easily check that the knots of divides in Figure 3 are 10_{139} and the $(-2, 3, 7)$ -pretzel knot respectively. It is known that the knot of a divide is strongly invertible and fibered, and its monodromy is represented as a product of right-handed Dehn twists. Moreover, in the case of 10_{139} and the $(-2, 3, 7)$ -pretzel knot, the core curves of the Dehn twists lie on the fiber surface such that their intersection diagram is a tree as shown in Figure 3. These knots have such particular properties. Remark that the Alexander polynomial of the $(-2, 3, 7)$ -pretzel knot is a famous polynomial in Lehmer's problem.

Question 6.2. Why are the A -polynomials of 10_{139} and the $(-2, 3, 7)$ -pretzel knot so small?

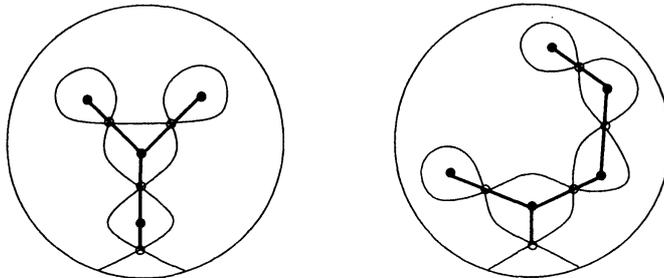


FIGURE 3. Slalom divides and their intersection diagrams of the core curves of positive Dehn twists. The knot of the divide on the left is 10_{139} and the one on the right is the $(-2, 3, 7)$ -pretzel knot.

Finally, we return to the factorization of A -polynomials as in [16]. In [16, Theorem 1.2], we proved that if $N(S + T)$ and $N(T)$ are knots and $N(S)$ is a split link in S^3 then $A_{N(T)}^\circ(l, m)$ is a factor of $A_{N(S+T)}(l, m)$, where $A_{N(T)}^\circ(l, m)$ is the product of factors of $A_{N(S+T)}(l, m)$ containing the variable l . The next theorem shows that we can produce infinitely many prime knots which have the A -polynomial of a given knot as a factor.

Theorem 6.3. *For any prime knot K , there exist infinitely many prime knots K_n of different knot types such that $A_K^\circ(l, m) \mid A_{K_n}(l, m)$.*

Proof. A marked tangle of an oriented link is a tangle whose four ends have specific orientations as shown on the left in Figure 4. The sum of two marked tangles S and T is a marked tangle obtained as shown on the right, denote by $S \dot{+} T$. Let $\Delta_K(t)$ denote the Alexander polynomial of a link K . As mentioned in [16], if $N(S)$ is split then the Alexander polynomial of the knot $N(S \dot{+} T)$ has the factorization $\Delta_{N(S \dot{+} T)}(t) = \Delta_{N(T)}(t)\Delta_{D(S)}(t)$.

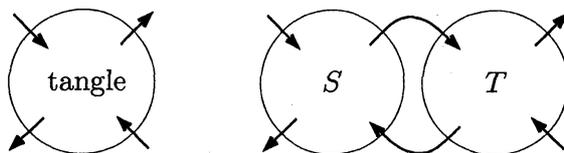


FIGURE 4. A marked tangle and the sum of marked tangles.

Let T be a marked tangle such that $N(T)$ is isotopic to K . Take S_n to be the sum of rational marked tangles with slope $1/(2n+1)$ and $-1/(2n+1)$. Note that $N(S_n)$ is a trivial link with two components, which is split. By [16, Theorem 1.2], we have $A_{N(T)}^\circ(l, m) \mid A_{N(S_n+T)}(l, m)$. We set $K_n = N(S_n + T)$ and check that these knots are prime and not isotopic to each other.

The knots K and the K_n 's are not isotopic because their Alexander polynomials are

$$\Delta_{K_n}(t) = \Delta_K(t)\Delta_{K_{2,2n+1}}(t)^2,$$

where $K_{2,2n+1}$ represents the $(2, 2n+1)$ -torus knot. The Alexander polynomial of $K_{2,2n+1}$ is given by the formula

$$\Delta_{K_{2,2n+1}}(t) = \frac{(t^{2(2n+1)} - 1)(t - 1)}{(t^{2n+1} - 1)(t^2 - 1)}$$

and we can easily check that $\Delta_K(t) \neq \Delta_{K_n}(t)$ for any n and that $\Delta_{K_n}(t) \neq \Delta_{K_m}(t)$ for any n and m with $n \neq m$.

To complete the proof, it is enough to check the primeness of K_n . Since K is prime, the tangle T is also prime. Here a tangle is said to be prime if it is non-split, locally trivial, indivisible, and it is not a trivial 1-string tangle, see [17, Definition 3.5.6]. We now check that the tangle S is also prime. The tangle S is not split, otherwise $D(S)$ is split. If it is not locally trivial then, since $D(S)$ is the connected sum of $K_{2,2n+1}$ and its mirror image, it must have $K_{2,2n+1}$ as a factor of the prime decomposition, which implies that $N(S)$ has $K_{2,2n+1}$ as a factor. However, $N(S)$ is a trivial link and cannot have $K_{2,2n+1}$ as a factor, which is a contradiction. Therefore S is locally trivial. Suppose that there exists a properly embedded disk D in the ball B^3 of the tangle such that $\partial D \subset \partial B^3$ and D intersects S at one point transversely. We can easily observe that one of the two tangles obtained from B^3 by the division by D is a trivial tangle with only one trivial strand. This means that S is indivisible. Finally it is obvious that S is not a trivial 1-string tangle. Thus we conclude that S is prime. It is shown in [20] that a link obtained from two prime tangles by any tangle sum is prime. This implies that $N(S + T)$ is a prime knot. \square

Remark 6.4. If we don't require primeness in the theorem above, we can easily produce such examples by connected sums. Note that for any knots K_1 and K_2 , $A_{K_1}(l, m)$ and

$A_{K_2}(l, m)$ are factors of $A_{K_1 \# K_2}(l, m)$, where $K_1 \# K_2$ represents the connected sum of K_1 and K_2 . See [6, Proposition 4.3].

We conclude this paper with a question concerning factorization of A -polynomials.

Question 6.5. For any positive integer n , does there exist a prime knot whose A -polynomial has at least n factors?

In the RIMS seminar, L. Paoluzzi gave a talk about non-standard components of the character varieties of Montesinos knots. Her talk is based on joint work with J. Porti [21], where they proved that for any positive integer n , there exists a Montesinos knot whose character variety has at least n irreducible components. Question 6.5 is the same question for A -polynomials, but it is more difficult since we need to prove that the irreducible components of the character variety correspond to distinct algebraic curves of the A -polynomial. If we could generalize the $N(S + T)$ construction to many tangles, then we might have such examples.

REFERENCES

- [1] N. A'Campo, *Planar trees, slalom curves and hyperbolic knots*, Publ. Math. de l'I.H.E.S. **88** (1998), 171–180.
- [2] S. Boyer and X. Zhang, *On Culler-Shalen seminorms and Dehn filling*, Ann. of Math. **148** (1998), 737–801.
- [3] S. Boyer and X. Zhang, *A proof of the finite filling conjecture*, J. Diff. Geom. **59** (2001), 87–176.
- [4] J. C. Cha and C. Livingston, KnotInfo: Table of Knot Invariants, <http://www.indiana.edu/~knotinfo/> (2 May, 2012).
- [5] D. Cooper, M. Culler, H. Gillet, D.D. Long and P.B. Shalen, *Plane curves associated to character varieties of 3-manifolds*, Invent. Math. **118** (1994), 47–84.
- [6] D. Cooper and D.D. Long *Remarks on the A-polynomial of a knot*, J. Knot Theory Ramifications **5** (1996), 609–628.
- [7] O. Couture and B. Perron, *Braid for links associated to plane immersed curves*, J. Knot Theory Ramifications **9** (2000), 1–30.
- [8] M. Culler, C.McA. Gordon, J. Luecke and P.B. Shalen, *Dehn surgery on knots*, Ann. of Math. **125** (1987), 237–300.
- [9] M. Culler and P.B. Shalen, *Varieties of group representations and splittings of 3-manifolds*, Ann. of Math. **117** (1983), 109–146.
- [10] D. Futer, M. Ishikawa, Y. Kabaya, T. Mattman and K. Shimokawa, *Finite surgeries on three-tangle pretzel knots*, Algebraic & Geometric Topology **9** (2009) 743–771.
- [11] W. Gibson and M. Ishikawa, *Links of oriented divides and fibrations in link exteriors*, Osaka J. Math. **39** (2002), 681–703.
- [12] M. Hirasawa, *Visualization of A'Campo's fibered links and unknotting operation*, Topology Appl. **121** (2002), 287–304.
- [13] C. Hodgson, *Degeneration and Regeneration of Hyperbolic Structures on Three-Manifolds*, Thesis, Princeton University, 1986.
- [14] J. Hoste and P.D. Shanahan, *Epimorphisms and boundary slopes of 2-bridge knots*, Algebraic & Geometric Topology **10** (2010), 1221–1244.
- [15] K. Ichihara and I.D. Jong, *Cyclic and finite surgeries on Montesinos knots*, Algebraic & Geometric Topology **9** (2009), 731–742.
- [16] M. Ishikawa, T.W. Mattman and K. Shimokawa, *Tangle sums and factorization of A-polynomials*, preprint, arXiv:mathGT/1107.2640.
- [17] A. Kawauchi, *A survey of knot theory*, Birkhäuser, Basel, 1996.
- [18] T. Kitano and M. Suzuki, *A partial order in the knot table*, Experiment. Math. **14** (2005), 385–390.
- [19] T. Kitano and M. Suzuki, *A partial order in the knot table. II*, Acta Math. Sin. **24** (2008), 1801–1816.
- [20] Y. Nakanishi, *Primeness of links*, Math. Sem. Notes Kobe Univ. **9** (1981), 415–440.

- [21] L. Paoluzzi and J. Porti, *Non-standard components of the character variety for a family of Montesinos knots*, preprint, arXiv:mathGT/1111.0933.

MATHEMATICAL INSTITUTE, TOHOKU UNIVERSITY, SENDAI 980-8578, JAPAN
E-mail address: `ishikawa@math.tohoku.ac.jp`

DEPARTMENT OF MATHEMATICS AND STATISTICS, CALIFORNIA STATE UNIVERSITY, CHICO, CHICO
CA 95929-0525, USA
E-mail address: `TMattman@CSUChico.edu`

DEPARTMENT OF MATHEMATICS, SAITAMA UNIVERSITY, 255 SHIMO-OKUBO SAITAMA-SHI, 338-
8570, JAPAN
E-mail address: `kshimoka@rimath.saitama-u.ac.jp`