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ON THE TRACE-FREE CHARACTERS

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1. Introduction

In this paper, we give a set of polynomials associated to *a Wirtinger presentation* of knot group $G(K)$ such that the locus of their zeros gives an "embedding" (one to one correspondence) of the set of *trace-free characters* of $G(K)$ in a complex space (Theorem 1.1). This set, denoted by $S_0(K)$, is an algebraic subset of *the character variety* of $G(K)$. The framework of character varieties [4] has been giving powerful tools and is now playing important roles mainly in geometry and topology. An underlying idea of character varieties is simple, though it is not easy to calculate them and thus to investigate their geometric structures in general. Let G be a finitely presented group generated by n elements $g_1, \cdots, g_n$. For a representation $\rho : G \to \mathrm{SL}_2(\mathbb{C})$, the character χ_ρ of ρ is the function on G defined by $\chi_\rho(g) := \mathrm{tr}(\rho(g))$ $(\forall g \in G)$. By [4, 7], the $\mathrm{SL}_2(\mathbb{C})$-trace identity

$$\mathrm{tr}(AB) = \mathrm{tr}(A)\mathrm{tr}(B) - \mathrm{tr}(AB^{-1}) \quad (A, B \in \mathrm{SL}_2(\mathbb{C}))$$

shows that $\mathrm{tr}(\rho(g))$ for an unspecified representation ρ and any $g \in G$ is expressed by a polynomial in $\{\mathrm{tr}(\rho(g_i))\}_{1\leq i\leq n}$, $\{\mathrm{tr}(\rho(g_ig_j))\}_{1\leq i<j\leq n}$ and $\{\mathrm{tr}(\rho(g_ig_jg_k))\}_{1\leq i<j<k\leq n}$. Then the character variety of G, denoted by $X(G)$, is basically the image of the set of characters $\mathfrak{X}(G)$ of $\mathrm{SL}_2(\mathbb{C})$-representations of G under the map

$$t : \mathfrak{X}(G) \to \mathbb{C}^{n+\binom{n}{2}+\binom{n}{3}},\ t(\chi_\rho) := (\mathrm{tr}(\rho(g_i)); \mathrm{tr}(\rho(g_ig_j)); \mathrm{tr}(\rho(g_ig_jg_k))).$$

The resulting set turns out to be a closed algebraic set (refer to [4]). By definition, the coordinates of $X(G)$ varies by *a bipolynomial map* if we change the choice of generating set of G, however, the geometric structures do not depend on the choice. Here a polynomial map $f : V \to W$ between two algebraic sets V and W in complex spaces are said to be isomorphism or bipolynomial if there exist a polynomial map $g : W \to V$ such that $g \circ f = id_V$, $f \circ g = id_W$. Hence $X(G)$ is an invariant of G up to bipolynomial map of algebraic sets. From now, we consider character varieties up to bipolynomial maps.

Now let us focus on the character varieties of knot groups. For a knot K in 3-sphere $\mathbb{S}^3$, we denote by E_K the knot exterior $\mathbb{S}^3 - N(K)$, where $N(K)$ is an open tubular neighborhood of K in $\mathbb{S}^3$, and by $G(K)$ the knot group, i.e., the fundamental group $\pi_1(E_K)$ of the knot exterior E_K. Given a knot diagram D_K with n crossings, by Wirtinger's algorithm we can always obtain so-called *the Wirtinger presentation* associated to D_K:

$$G(K) = \langle m_1, \cdots, m_n \mid r_1 = 1, \cdots, r_{n-1} = 1\rangle,$$

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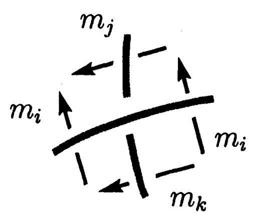
where m_i is a meridian on the ith arc of D_K, and r_j is a word in $m_1, \cdots, m_n$ associated to the jth crossing (refer to [3, 8] etc.). If the sth crossing in D_K is seen as



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we call the triple (i, j, k) a *Wirtinger triple*[1] and then $r_s = m_i m_j m_i^{-1} m_k^{-1}$ holds. By the Culler-Shalen theory mentioned above, we can construct the character variety $X(K) := X(G(K))$ associated with the above Wirtinger presentation of $G(K)$.

This paper focuses on a special class of representations of a knot group, called *trace-free representations*. Let μ be a meridian of $G(K)$. Then a representation $\rho : G(K) \to \mathrm{SL}_2(\mathbb{C})$ is said to be trace-free if $\mathrm{tr}(\rho(\mu)) = 0$ holds. Then we call its character χ_ρ a trace-free character. The set of trace-free characters gives us a subset of the set of characters $\mathfrak{X}(K)$, denoted by $\mathfrak{S}_0(K)$:

$$\mathfrak{S}_0(K) = \{\chi_\rho \in \mathfrak{X}(K) \mid \chi_\rho(\mu) = 0\}.$$

Again, by the Culler-Shalen theory, $\mathfrak{S}_0(K)$ can be realized as an algebraic subset of the character variety $X(K)$, which is denoted by $S_0(K)$. By definition, the subset $S_0(K)$ of $X(K)$ can be thought of as a slice by the hyperplane $\mathrm{tr}(\rho(\mu)) = 0$. Since any meridians are conjugate, $\mathrm{tr}(\rho(\mu)) = 0$ means $\mathrm{tr}(\rho(m_i)) = 0$ for any $1 \leq i \leq n$. So we have

$$S_0(K) = \left\{ (0, \cdots, 0; \mathrm{tr}(\rho(m_i m_j)); \mathrm{tr}(\rho(m_i m_j m_k))) \in \mathbb{C}^{n+\binom{n}{2}+\binom{n}{3}} \,\middle|\, \chi_\rho \in \mathfrak{S}_0(K) \right\}.$$

We call this slice *the trace-free slice*. In general, the map t in Section 1 is not injective and thus the character variety $X(G)$ is not a genuine embedding (not a one-to-one correspondence) of $\chi(G)$, however, for the characters of irreducible representations t is injective. Since the knot determinant $|\Delta_{-1}(K)|$ is non-zero, by [2, 5] there does not exist reducible non-abelian trace-free representations. So the trace-free slice $S_0(K)$ turns out to be a genuine embedding of the trace-free characters $\mathfrak{S}_0(K)$.

The trace-free slices have several interesting properties, for example, a relationship to *the knot signature* [9], a 2-fold branched covering structure coming from *metabelian representations* [9, 14] etc. However, it is also not easy to calculate them in general. The main result in the present paper is to give a set of polynomials whose common zeros coincide with the trace-free slice $S_0(K)$ of the character variety $X(K)$ of a knot K.

Theorem 1.1. *Let $G(K) = \langle m_1, \cdots, m_n \mid r_1 = 1, \cdots, r_{n-1} = 1 \rangle$ be a Wirtinger presentation. Then $S_0(K)$ is given as the following algebraic set in $\mathbb{C}^{\binom{n}{2}+\binom{n}{3}}$:*

$$\left\{ (x_{12}, \cdots, x_{nn-1}; x_{123}, \cdots, x_{n-2n-1n}) \in \mathbb{C}^{\binom{n}{2}+\binom{n}{3}} \,\middle|\, (1),\ (2),\ (3) \right\},$$

where (1), (2) *and* (3) *are the equations defined as follows:*

[1]The positive integers i, j and k in (i, j, k) are ordered such that the meridian m_i is on the overarc and the others m_j and m_k are on the underarcs, respectively.

Type (1): *the fundamental relations* (F)

$$x_{ka} = x_{ij}x_{ia} - x_{ja} \text{ (F2)}, \quad x_{kab} = x_{ij}x_{iab} - x_{jab} \text{ (F3)},$$
$$(1 \le a, b \le n, \ (i,j,k) : \textit{any Wirtinger triple}),$$

Type (2): *the hexagon relations* (H)

$$x_{i_1i_2i_3} \cdot x_{j_1j_2j_3} = \frac{1}{2}\begin{vmatrix} x_{i_1j_1} & x_{i_1j_2} & x_{i_1j_3} \\ x_{i_2j_1} & x_{i_2j_2} & x_{i_2j_3} \\ x_{i_3j_1} & x_{i_3j_2} & x_{i_3j_3} \end{vmatrix},$$
$$(1 \le i_1 < i_2 < i_3 \le n, \ 1 \le j_1 < j_2 < j_3 \le n),$$

Type (3): *the rectangle relations* (R)

$$\begin{vmatrix} 2 & x_{12} & x_{1a} & x_{1b} \\ x_{21} & 2 & x_{2a} & x_{2b} \\ x_{a1} & x_{a2} & 2 & x_{ab} \\ x_{b1} & x_{b2} & x_{ba} & 2 \end{vmatrix} = 0 \quad (3 \le a < b \le n).$$

In fact, the coordinates x_{ij} and x_{ijk} correspond to $-\mathrm{tr}(\rho(m_im_j))$ and $-\mathrm{tr}(\rho(m_im_jm_k))$ for an unspecified representation $\rho : G(K) \to \mathrm{SL}_2(\mathbb{C})$. So we have $x_{ii} = 2$, $x_{ji} = x_{ij}$ and $x_{i_{\sigma(1)}i_{\sigma(2)}i_{\sigma(3)}} = \mathrm{sign}(\sigma)x_{i_1i_2i_3}$, where σ is an element in the symmetric group of degree three.

2. Backgrounds and Motivations

2.1. **Backgrounds and Motivations of Theorem 1.1.** We first observe the slice $S_0(K_m)$ for twist knots K_m shown in Figure 1.

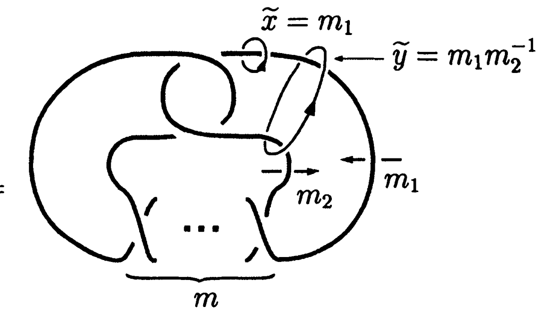
$K_m =$ 

Figure 1. Twist knot K_m and loops $\widetilde{x}$ and $\widetilde{y}$ parametrizing $X(K_m)$

Let m_1 and m_2 be meridians shown in Figure 1, respectively. These loops give a presentation $G(K_m) = \langle m_1, m_2 \mid w(m_1, m_2) = 1\rangle$, where $w(m_1, m_2)$ be a word in m_1 and m_2 associated to this diagram. Let x and y be the following trace functions:

$$x := -\mathrm{tr}\left(\rho(\widetilde{x})\right) = -\mathrm{tr}\left(\rho(m_1)\right), \ y := -\mathrm{tr}\left(\rho(\widetilde{y})\right) = -\mathrm{tr}\left(\rho(m_1m_2^{-1})\right).$$

These functions give the parameters of the character variety $X(K_m)$ as follows. Let $S_n(z)$ be the Chebyshev polynomial of the second kind defined recursively by

$$S_{n+2}(z) = zS_{n+1}(z) - S_n(z), \ S_1(z) = z, \ S_0(z) = 1$$

for any integers n. Then we define a polynomial

$$R_m(x,y) := (y+2)\left(S_m(y) - S_{m-1}(y) + x^2 \sum_{i=0}^{m-1} S_i(y)\right). \tag{1}$$

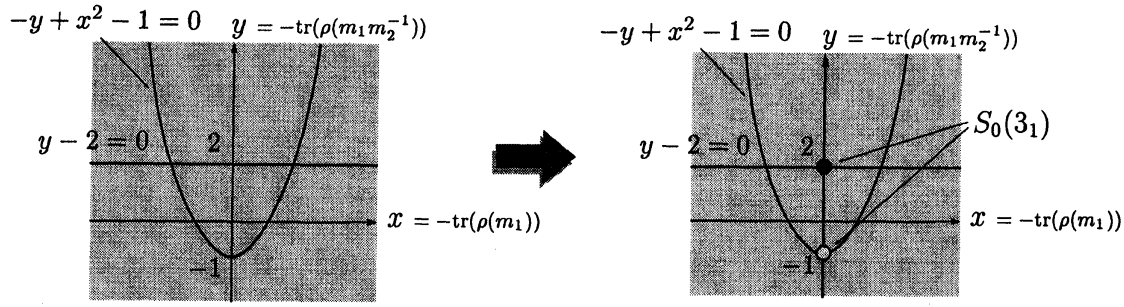
In fact, this gives us the trace-free slice $S_0(K_m)$ for any positive integer m.

Theorem 2.1 ([6, 11]). *For any positive integer*[2] *m, the character variety $X(K_m)$ is given by the algebraic set*[3] $\{(x,y) \in \mathbb{C}^2 \mid R_m(-x,-y) = 0\}$.

Note that the expression in (1) is the prime decomposition over the complex number field $\mathbb{C}$. (See [12] for the case where $2m+1$ is prime. For general cases, refer to [13].) So $R_m(-x,-y) = 0$ gives the prime decomposition of $X(K_m)$. In the case of trefoil knot 3_1, which is K_1, we have $R_1(-x,-y) = (y-2)(-y+x^2-1)$ and thus it follows from Theorem 2.1 that

$$X(3_1) = \{(x,y) \in \mathbb{C}^2 \mid (y-2)(-y+x^2-1) = 0\}.$$

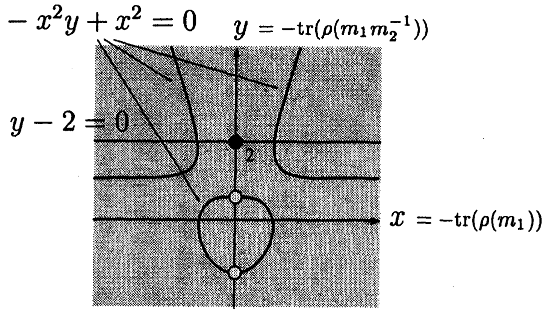
Hence we obtain $S_0(3_1) = \{2, -1\}$.



For the figure-8 knot 4_1, which is K_2, we have $R_2(-x,-y) = (y-2)(y^2 - x^2 y + y + x^2 - 1)$ and thus it follows from Theorem 2.1 that

$$X(4_1) = \{(x,y) \in \mathbb{C}^2 \mid (y-2)(y^2+y-1-x^2y+x^2) = 0\}.$$

Hence we obtain $S_0(4_1) = \{2, (-1 \pm \sqrt{5})/2\}$.

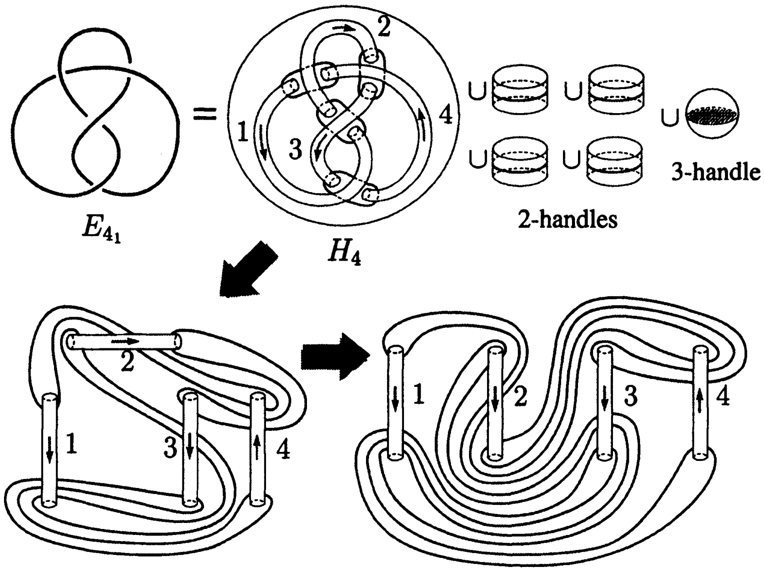


These calculations can be done because we have the defining polynomial of $X(K_m)$. We want to calculate $S_0(K)$ directly without the calculation of $X(K)$. Theorem 1.1 gives us a way to do it.

[2]For a negative integer $-m$ ($m > 1$), taking the mirror image of K_{-m} and arranging it, we can obtain $X(K_{-m}) = X(K_{m-1})$ and thus a similar result to Theorem 2.1. In that case, $R_{-m}(x,y)$ will shift to $R_{m-1}(x,y)$.

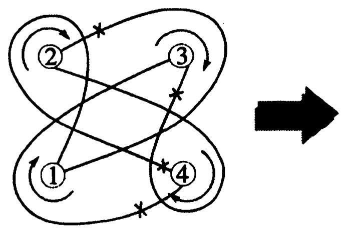
[3]We can replace $R_m(-x,-y)$ with $R_m(x,y)$. The negative signs are just for a convention.

2.2. **An observation of Theorem 1.1.** Theorem 1.1 uses a handle decomposition of the exterior E_K. Let H_n be a handlebody of genus n. Then the exterior E_K can be decomposed into 2-handles and a 3-handle and a handlebody H_n. In the case of 4_1, that decomposition can be seen as below.



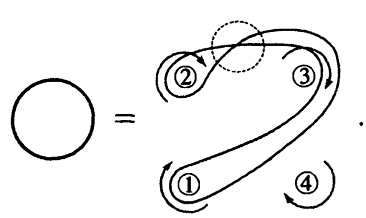
From now on, we use this example to discuss the mechanism of Theorem 1.1. First, we isotope the handlebody H_4 to the product of a 4-punctured disk D_4 and an interval $[0,1]$. Along with this isotopy, the attaching curves of 2-handles on the boundary of H_4 can be seen as curves on the boundary of $D_4 \times [0,1]$. In this situation, we will project the attaching curves to the punctured disk $D_4 \times \{0\}$ as a code (see Figure 2). Note that in the projection we do not have to care about the sign of a crossing, since we will look at the relations in the fundamental group.

Now, let us observe the mechanism which generates the equations giving $S_0(K_4)$. First, the attaching curves (codes) themselves give the following equations:



$$\begin{cases} x_{13}x_{23} - x_{12} = 2, & x_{13} = x_{23}, \\ x_{12}x_{24} - x_{14} = 2, & x_{12} = x_{24}, \\ x_{13}x_{14} - x_{34} = 2, & x_{13} = x_{14}, \\ x_{24}x_{34} - x_{23} = 2, & x_{23} = x_{24}. \end{cases}$$

To observe this, for example, we focus on the attaching curve corresponding to the code connecting ① and ②. Since the attaching curve is trivial in E_{4_1}, we have



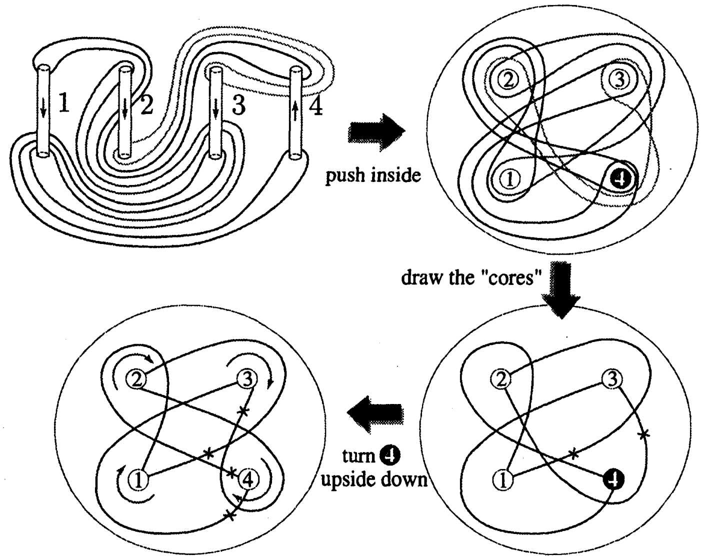
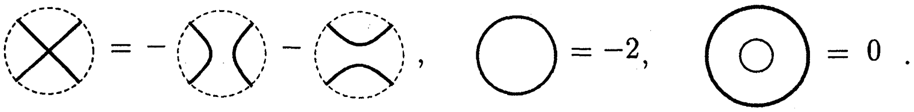


FIGURE 2. Depictions of attaching curves as codes. A cross "×" on a code presents a half-twist of the attaching curve.

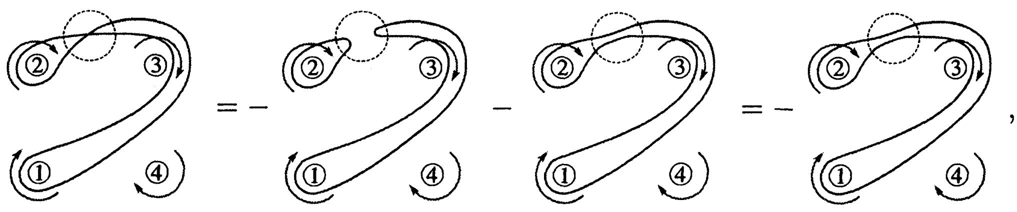
This means $1 = m_3 m_1 m_3^{-1} m_2^{-1}$ in the language of the fundamental group. For an arbitrary trace-free character χ_ρ, this gives us $-2 = -\mathrm{tr}\left(\rho(m_3 m_1 m_3^{-1} m_2^{-1})\right)$. By the $\mathrm{SL}_2(\mathbb{C})$-trace identity with trace-free condition $\mathrm{tr}(\rho(m_i)) = 0$, we obtain the following equation

$$(2)\quad -2 = -\mathrm{tr}\left(\rho(m_3 m_1 m_3^{-1})\right)\mathrm{tr}\left(\rho(m_2^{-1})\right) + \mathrm{tr}\left(\rho(m_3 m_1 m_3^{-1} m_2)\right) = \mathrm{tr}\left(\rho(m_3 m_1 m_3^{-1} m_2)\right)$$

In fact, this operation can be done through the Kauffman bracket skein relation at $t = -1$ with the trace-free condition:

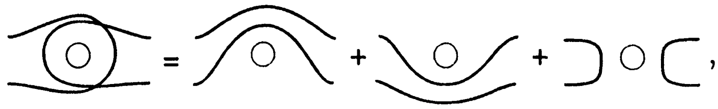


The first and the second equations are the Kauffman bracket skein relations at $t = -1$. The third equation corresponds to the trace-free condition. Again, we do not have to care about the sign of a crossing in the Kauffman bracket skein relations. To see (2),

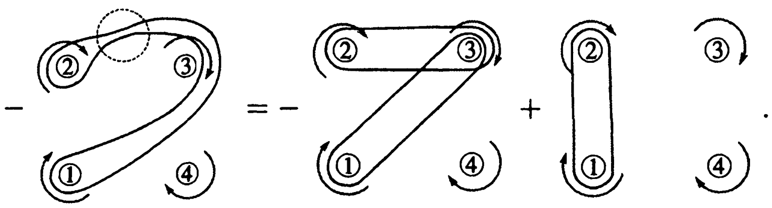


where this resulting loop presents $m_3 m_1 m_3^{-1} m_2$. In general, a loop γ in E_K up to the Kauffman bracket skein relations at $t = -1$ corresponds to $-\mathrm{tr}(\gamma)$ ([1], see also Theorem 3.1). Basically, we will apply this skein theoretic method throughout this paper instead of the $\mathrm{SL}_2(\mathbb{C})$-trace identity.

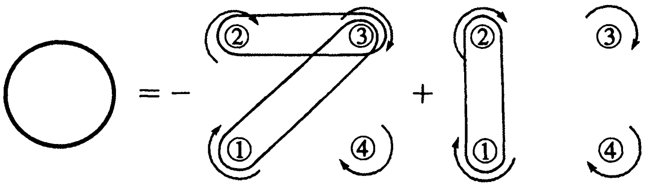
Now, by the skein relations, we obtain

(3) 

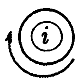
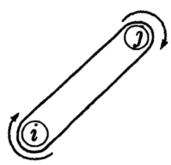
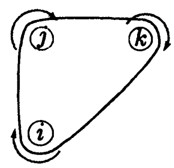
so we have



This means that $-2 = -\mathrm{tr}(\rho(m_1m_3))\mathrm{tr}(\rho(m_2m_3)) + \mathrm{tr}(\rho(m_1m_2))$ holds.



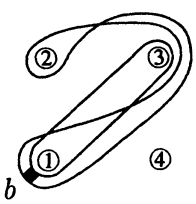
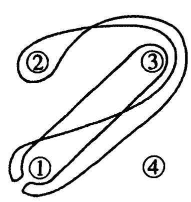
Setting the followings

$x_i :=$  $= 0,\ x_{ij} :=$  $,\ x_{ijk} :=$  ,

$\boxed{-\mathrm{tr}(\rho(m_i))} = 0 \qquad \boxed{-\mathrm{tr}(\rho(m_im_j))} \qquad \boxed{-\mathrm{tr}(\rho(m_1m_jm_k))}$

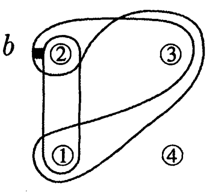
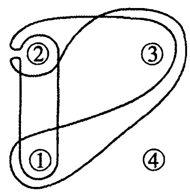
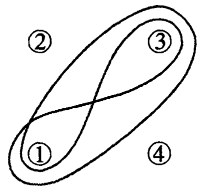
we obtain one of the desired equations $-2 = -x_{13}x_{23} + x_{12}$.

Also we can get $x_{13} = x_{23}$, $x_{12} = x_{24}$, $x_{13} = x_{14}$, $x_{23} = x_{24}$. In general, ***handle-slides*** along the attaching curves generate all equations giving the trace-free slice $S_0(K_2)$. Note that a handle-slide of a loop in H_4 along a attaching curve can be considered as ***a band-sum***[4] between them. For example, a handle-slide $sl_b(x_{13})$ of x_{13} along a band b connecting x_{13} to an attaching curve gives

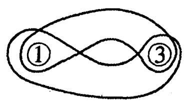
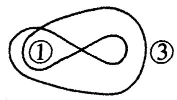
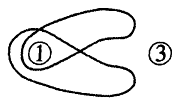
$x_{13} = sl_b(x_{13}) :=$  $=$  $= x_{23}.$

[4]Since a twisted band-sum can be reduced to a sum of band sums by resolving the twists, we only consider non-twisted band-sums.

Other relations $x_{12} = x_{13}^2 - 2$, $x_{24} = x_{13}^2 - 2$, $x_{34} = x_{13}^2 - 2$ can be obtained like this.

$x_{12} = sl_b(x_{12}) =$  $=$  $=$  .

Then by relation (3) the resulting loop is equal to

 $-$  $-$  $= x_{13}^2 - 2.$

Continuing this work, we obtain all (F2):

$$\left\{ \begin{array}{c} x_{13}x_{23} - x_{12} = 2,\ x_{12}x_{24} - x_{14} = 2,\ x_{13}x_{14} - x_{34} = 2,\ x_{24}x_{34} - x_{23} = 2 \\ x_{13} = x_{23},\ \boxed{x_{12} = x_{24}},\ \boxed{x_{13} = x_{14}},\ x_{23} = x_{24},\ \boxed{x_{12} = x_{13}^2 - 2} \\ x_{24} = x_{13}^2 - 2,\ x_{34} = x_{13}^2 - 2,\ \boxed{x_{13} = x_{14}x_{24} - x_{12}} \\ x_{14} = x_{23}x_{34} - x_{24},\ x_{13} = x_{23}x_{24} - x_{34},\ x_{23} = x_{12}x_{14} - x_{24} \end{array} \right\}.$$

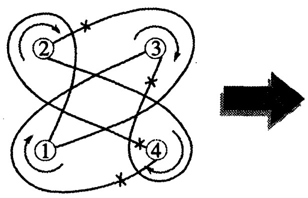
Here we define the algebraic set $F_2(4_1)$ which is the common zeros of the fundamental relations (F2):

$$F_2(4_1) := \left\{ (x_{12}, \cdots, x_{45}) \in \mathbb{C}^{10} \mid x_{ka} = x_{ik}x_{ia} - x_{ja}\ \text{(F2)} \right\}.$$

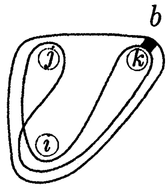
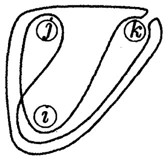
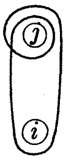
By reducing the variables in (F2), we see that $F_2(4_1)$ is parametrized by x_{13} and

$$\begin{aligned} x_{13} &= x_{14}x_{24} - x_{12}, \\ x_{13} &= x_{13}(x_{13}^2 - 2) - (x_{13}^2 - 2), \\ 0 &= (x_{13} - 2)(x_{13}^2 + x_{13} - 1). \end{aligned}$$

Hence we get $F_2(4_1) = \left\{2, (-1 \pm \sqrt{5})/2\right\}$. This shows that $F_2(4_1)$ coincides with $S_0(4_1)$. The reason is as follows. First, we see that (F3) become trivial:

 $\begin{cases} x_{123} = 0,\ x_{124} = 0, \\ x_{134} = 0,\ x_{234} = 0. \end{cases}$

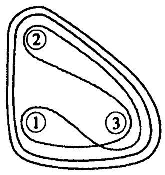
Indeed, for a Wirtinger triple (i, j, k)

$x_{ijk} = sl_b(x_{ijk}) =$  $=$  $=$   $= -x_i x_{ij} - x_i = 0.$

We can also check this by the hexagon relation (H):

$$x_{i_1 i_2 i_3} \cdot x_{j_1 j_2 j_3} = \frac{1}{2}\begin{vmatrix} x_{i_1 j_1} & x_{i_1 j_2} & x_{i_1 j_3} \\ x_{i_2 j_1} & x_{i_2 j_2} & x_{i_2 j_3} \\ x_{i_3 j_1} & x_{i_3 j_2} & x_{i_3 j_3} \end{vmatrix} \qquad \begin{matrix} (1 \le i_1 < i_2 < i_3 \le 4) \\ (1 \le j_1 < j_2 < j_3 \le 4) \end{matrix} .$$

For example, a Wirtinger triple $(1, 2, 3)$ gives us



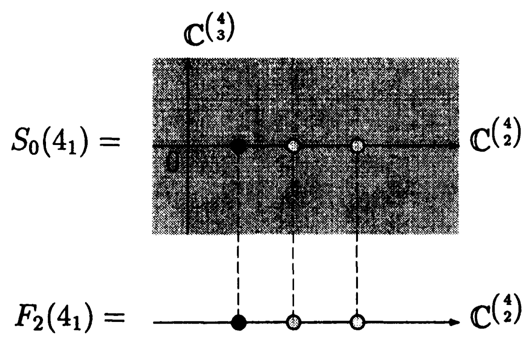
$$= x_{123}^2 = \frac{1}{2}\begin{vmatrix} 2 & x_{12} & x_{13} \\ x_{21} & 2 & x_{23} \\ x_{31} & x_{32} & 2 \end{vmatrix} = x_{12}x_{13}x_{23} - x_{12}^2 - x_{13}^2 - x_{23}^2 + 4$$

$$= (x_{13}^2 - 2)x_{13}^2 - (x_{13}^2 - 2)^2 - x_{13}^2 - x_{13}^2 + 4 = 0.$$

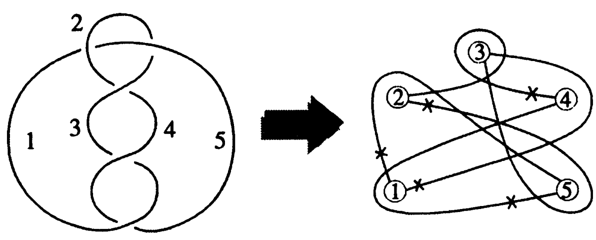
Then we can check that all point in $F_2(4_1)$ satisfy (H) and the rectangle relations (R):

$$\begin{vmatrix} 2 & x_{12} & x_{1a} & x_{1b} \\ x_{21} & 2 & x_{2a} & x_{2b} \\ x_{a1} & x_{a2} & 2 & x_{ab} \\ x_{b1} & x_{b2} & x_{ba} & 2 \end{vmatrix} = 0 \qquad (3 \le a < b \le 4).$$

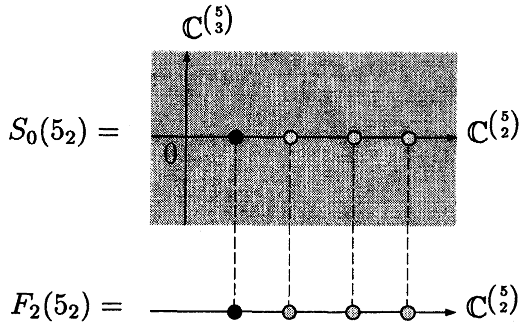
Hence every point in $F_2(4_1)$ lifts to a point in $S_0(4_1)$ and thus $F_2(4_1) = S_0(4_1)$ and the main theorem holds for $K_2 = 4_1$. We remark that to get $S_0(4_1)$ we calculate $F_2(4_1)$ first and then we check the liftability second.



We can also observe the case of $K = 5_2$.

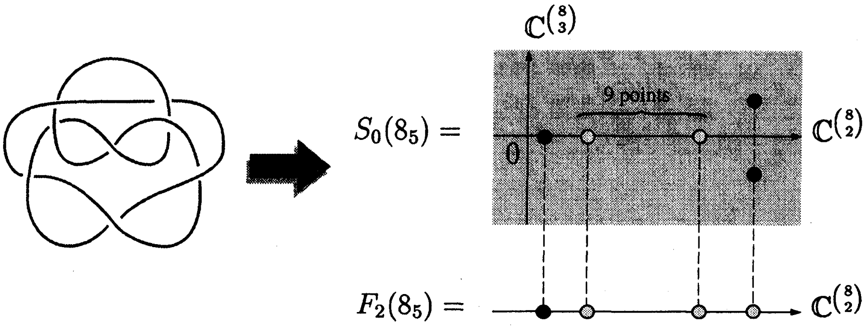


Then we obtain $S_0(5_2) = F_2(5_2) = \{x_{14} \in \mathbb{C} \mid (x_{14} - 2)\,(x_{14}^3 + x_{14}^2 - 2x_{14} - 1) = 0\}$.



Note that every point in $F_2(5_2)$ also lifts to a point in $S_0(5_2)$. Again we remark that to get $S_0(5_2)$ we calculate $F_2(5_2)$ first and then we check the liftability second.

We also observe the case of $K = 8_5$. $F_2(8_5)$ consists of 11 points and $S_0(8_5)$ consists of 12 points.

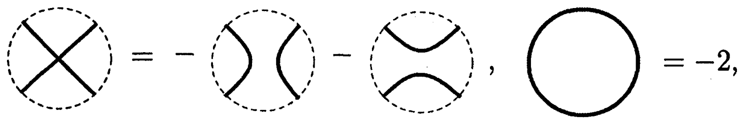


Note that there exists a point in $F_2(8_5)$ which lifts to two points in $S_0(8_5)$ and so $F_2(8_5) \neq S_0(8_5)$. Again we remark that to obtain $S_0(8_5)$ we calculate $F_2(8_5)$ first and then we check the liftability second.

So far, any point of $F_2(K)$ can lift to $S_0(K)$. It would be interesting to research whether or not any point of $F_2(K)$ can lift to $S_0(K)$ for any knot K. In the next section, we first show a sketch of the proof of Theorem 1.1 in Subsection 3.1 and then we will speculate this question in Subsection 3.2.

3. A SKETCH OF THE PROOF OF THEOREM 1.1

3.1. A sketch of the proof of Theorem 1.1. In general, the skein theory observed in Subsection 2.2 is realized as the Kauffman bracket skein algebra[5] (KBSA for short) of a 3-manifold. The KBSA of a 3-manifold M, denoted by $\mathcal{K}_{-1}(M)$, is the quotient of the module over $\mathbb{C}$ generated by all free homotopy classes of loops in M by the Kauffman bracket skein relations:



where in the first relation loops coincide each other outside dashed circles (refer to [1, 15, 16, 17]). Actually, a loop (a homotopy class of a loop) $s \in \mathcal{K}_{-1}(M)$ has the same

[5] This is the specialization of the Kauffman bracket skein module at the parameter $t = -1$.

properties as $-\mathrm{tr}(s)$. In this correspondence, the Kauffman bracket skein relations can be thought of as the $SL_2(\mathbb{C})$-trace identities. This gives a correspondence between $\mathcal{K}_{-1}(M)$ and *the coordinate ring* $\chi(M) := \chi(\pi_1(M))$ of the character variety $X(\pi_1(M))$.

Theorem 3.1 ([1, 17]). *There exists a surjective homomorphism $\varphi : \mathcal{K}_{-1}(M) \to \chi(M)$ defined by $\varphi(\gamma) := -t_\gamma$ for a loop $\gamma \in \mathcal{K}_{-1}(M)$. Moreover* $\mathrm{Ker}(\varphi)$ *is the nilradical $\sqrt{0}$.*

This gives $\mathcal{K}_{-1}(M)/\sqrt{0} = \chi(M)$ and thus a method to calculate the character varieties using the Kauffman bracket skein theory. The next theorem is basic to calculate the KBSA.

Theorem 3.2 (cf. [15]).

$$\mathcal{K}_{-1}(E_K) = \frac{\mathcal{K}_{-1}(H_n)}{\langle z - sl_b(z) \mid z\text{: any loop in } \mathcal{K}_{-1}(H_n)\rangle}$$

Theorem 3.2 immediately gives the trace-free version:

$$\mathcal{K}_{-1,TF}(E_K) := \frac{\mathcal{K}_{-1,TF}(H_n)}{\langle z - sl_b(z) \mid z\text{: any loop in } \mathcal{K}_{-1,TF}(H_n)\rangle},$$

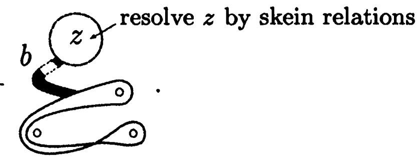
where $\mathcal{K}_{-1,TF}(H_n)$ denotes the KBSA $\mathcal{K}_{-1}(H_n)$ with the trace-free condition. Now we define two ideals in $\mathcal{K}_{-1,TF}(H_n)$, *the sliding ideal* S_K and *the fundamental ideal* F_K:

$$S_K := \langle z - sl_b(z) \mid z\text{: any loop in } \mathcal{K}_{-1,TF}(H_n)\rangle$$

$$F_K := \langle x_{ka} - x_{ij}x_{ia} + x_{ja} \text{ (F2)},\ x_{kab} - x_{ij}x_{iab} + x_{jab} \text{ (F3)}\rangle$$

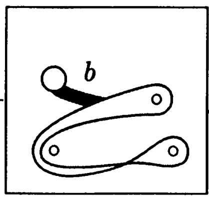
By definition, $S_K \supset F_K$ holds. To show Theorem 1.1, we will first show that they coincide, i.e., $S_K = F_K$.

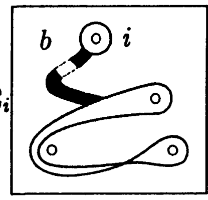
Step1 For a loop z, take a band b for a handle-slide $sl_b(z)$

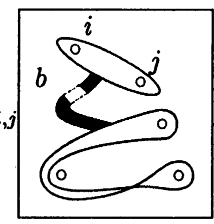
$z - sl_b(z) = z -$ 

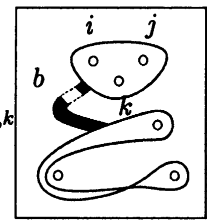
.

Note that the dashed band in the band b express omitting the way of b. We first resolve the loop z with b by the skein relations. Similar to the property on $\mathrm{tr}(\rho(g))$ as seen in Section 1, any loop can be presented by a sum $\boxed{\bigcirc} f - \Sigma_i \boxed{x_i} f_i - \Sigma_{i,j} \boxed{x_{ij}} f_{ij} - \Sigma_{i,j,k} \boxed{x_{ijk}} f_{ijk}$, where f, f_i, f_{ij}, f_{ijk} are polynomials in $\mathbb{C}[x_{ij}; x_{ijk}]$. Here the rectangles means the loops which connect to the band b. Then we see that $sl_b(z)$ is equal to

$-$ 

$f - \Sigma_i$ 

$f_i - \Sigma_{i,j}$ 

$f_{ij} - \Sigma_{i,j,k}$ 

f_{ijk}.

So $z - sl_b(z)$ turns out to be

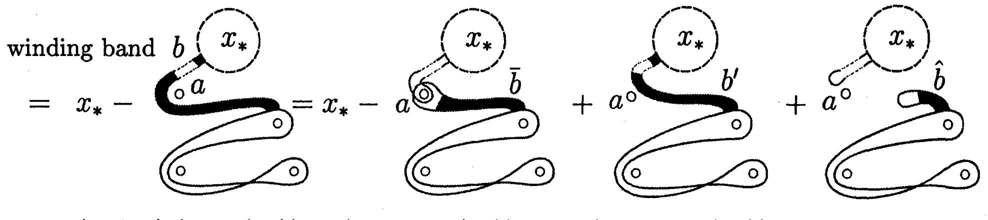
$$(\bigcirc - sl_b(\bigcirc))f + \Sigma_i(x_i - sl_b(x_i))f_i + \Sigma_{i,j}(x_{ij} - sl_b(x_{ij}))f_{ij} + \Sigma_{i,j,k}(x_{ijk} - sl_b(x_{ijk}))f_{ijk}.$$

Hence any handle-slide can be generated by $\bigcirc - sl_b(\bigcirc)$, $sl_b(x_i)$, $x_{ij} - sl_b(x_{ij})$, $x_{ijk} - sl_b(x_{ijk})$ and thus we obtain

$$S_K = \langle \bigcirc - sl_b(\bigcirc),\ sl_b(x_i),\ x_{ij} - sl_b(x_{ij}),\ x_{ijk} - sl_b(x_{ijk}) \mid b\text{: any band}\rangle\,.$$

$\boxed{\textbf{Step2}}$ Consider $sl_b(x_*)$ for $x_* \in \{\bigcirc,\ x_i = 0,\ x_{ij},\ x_{ijk}\}$. If the band b is "winding", i.e., b goes around at least a puncture, then we can actually "straighten" b by the skein relations:

$$x_* - sl_b(x_*) \;=$$



$$= \;(x_* \sharp x_a)\,(-sl_{\bar{b}}(x_a)) - (x_* - sl_{b'}(x_*)) - x_*\,(-2 - sl_{\hat{b}}(\bigcirc))$$

where $x_* \sharp x_a$ denotes the band sum between x_* and x_a in the above equation. Continuing this work until the winding bands disappear, we obtain

$$x_* - sl_b(x_*) = \Sigma_i\,(-sl_*(x_i))\,f + \Sigma\,(x_* - sl_*(x_*))\,g + \Sigma\,(-2 - sl_*(\bigcirc))\,h,$$

where sl_* denotes the band-sum along an unspecified non-winding band $*$, and f, g and h are polynomials in $\mathbb{C}[x_{ij}; x_{ijk}]$. Therefore we see that

$$S_K = \langle \bigcirc - sl_*(\bigcirc),\ sl_*(x_i),\ x_{ij} - sl_*(x_{ij}),\ x_{ijk} - sl_*(x_{ijk}) \mid *\text{: any non-winding band}\rangle$$

Since there exist only finitely many non-winding bands for a loop up to homotopy, this shows that S_K is finitely generated. By the same argument[6], we can reduce the finitely many generators to (F). Therefore, we obtain $S_K = F_K$.

Now we can show Theorem 1.1. It follows from the above argument that

$$\mathcal{K}_{-1,TF}(E_K) = \frac{\mathcal{K}_{-1,TF}(H_n)}{\langle x_{ka} - x_{ik}x_{ia} + x_{ja}\ \text{(F2)},\ x_{kab} - x_{ik}x_{iab} + x_{jab}\ \text{(F3)}\rangle}.$$

By [7][7], we have

$$\mathcal{K}_{-1,TF}(H_n)/\sqrt{0} = \frac{\mathbb{C}[x_{ij}; x_{ijk}]}{\sqrt{\left\langle \text{(H)},\ \text{(R)},\ (\star) = \begin{vmatrix} 2 & x_{12} & x_{13} & x_{1a} \\ x_{21} & 2 & x_{23} & x_{2a} \\ x_{31} & x_{32} & 2 & x_{3a} \\ x_{b1} & x_{b2} & x_{b3} & x_{ab} \end{vmatrix}\ (4 \le a < b \le n) \right\rangle}}.$$

---

[6]If $sl_*(x_*)$ is a band sum of $x_* \in \{\bigcirc,\ x_i = 0,\ x_{ij},\ x_{ijk}\}$ and an attaching curve disjoint from x_*, then the resulting relation essentially comes from the fundamental relations (F). So we only need to focus on the band-sums between x_* and attaching curves intersecting with x_*. Then the remaining generators turns out to be essentially (F). We will omit the details.

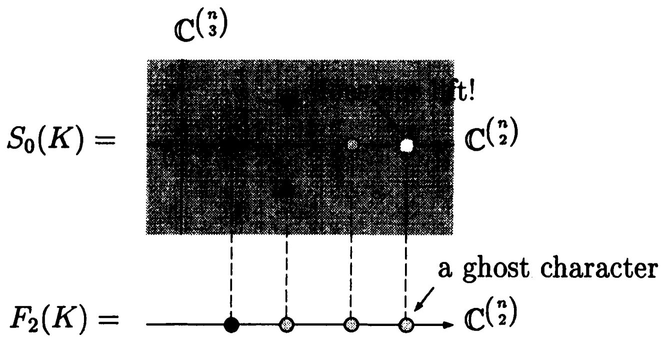
[7]The relations given in [7, p.639] can be realized by the skein relations. So it follows from Theorem 3.1 that this equality holds.

In fact, by taking $(1,2,3)$ as a Wirtinger triple, the relations $(\star)$ become trivial as follows:

$$\begin{aligned}(\star) &= x_{ab}\begin{vmatrix} 2 & x_{12} & x_{13} \\ x_{21} & 2 & x_{23} \\ x_{31} & x_{32} & 2 \end{vmatrix} - x_{b3}\begin{vmatrix} 2 & x_{12} & x_{1a} \\ x_{21} & 2 & x_{2a} \\ x_{31} & x_{32} & x_{3a} \end{vmatrix} \\ &\quad + x_{b2}\begin{vmatrix} 2 & x_{13} & x_{1a} \\ x_{21} & x_{2a} & x_{2a} \\ x_{31} & 2 & x_{3a} \end{vmatrix} - x_{b1}\begin{vmatrix} x_{12} & x_{13} & x_{1a} \\ 2 & x_{23} & x_{2a} \\ x_{32} & 2 & x_{3a} \end{vmatrix} \\ &= x_{ab}x_{123}^2 - x_{b3}x_{123}x_{12a} + x_{b2}x_{123}x_{13a} - x_{b1}x_{123}x_{23a} \\ &= x_{123}(x_{ab}x_{123} - x_{b3}x_{12a} + x_{b2}x_{13a} - x_{b1}x_{23a}) = 0.\end{aligned}$$

Therefore, we obtain Theorem 1.1 with the condition that $(1,2,3)$ is a Wirtinger triple.

3.2. **Ghost characters and liftability problem of $F_2(K)$ to $S_0(K)$.** Can any point of $F_2(K)$ lift to $S_0(K)$ for any knot? If there exists a point in $F_2(K)$ which does not lift to $S_0(K)$, then we call it *a ghost character.*



More precisely, if a point (x_{ij}) in $F_2(K)$ does not satisfy one of (F3), (H) and (R), then (x_{ij}) does not lift to $S_0(K)$ and thus (x_{ij}) turns out to be a ghost character. Before we look into ghost characters, we focus on the meanings of (H) from the algebraic set $F_2(K)$ point of view.

First, for a point (x_{ij}) in $F_2(K)$ the hexagon relations (H)

$$x_{i_1i_2i_3}\cdot x_{j_1j_2j_3} = \frac{1}{2}\begin{vmatrix} x_{i_1j_1} & x_{i_1j_2} & x_{i_1j_3} \\ x_{i_2j_1} & x_{i_2j_2} & x_{i_2j_3} \\ x_{i_3j_1} & x_{i_3j_2} & x_{i_3j_3} \end{vmatrix}$$
$$(1 \le i_1 < i_2 < i_3 \le n,\ 1 \le j_1 < j_2 < j_3 \le n)$$

give a 2-fold branched covering structure to $S_0(K)$, i.e., (H) show that a point in $F_2(K)$ can lift at most two points. In particular, (H) give each x_{ijk} two possibility as follows:

$$x_{ijk} = \pm\sqrt{\frac{1}{2}\begin{vmatrix} 2 & x_{ij} & x_{ik} \\ x_{ji} & 2 & x_{jk} \\ x_{kj} & x_{kj} & 2 \end{vmatrix}}.$$

Next, the hexagon relations (H) give always a solution of (F3). Namely, if $(x_{ij}; x_{klm}) \in F_2(K) \times \mathbb{C}^{\binom{n}{3}}$ satisfies (H), then $(x_{ij}; x_{klm})$ satisfies (F3), because

(1) if all $x_{klm} = 0$, then (F3) are trivial.

(2) if there exists a coordinate $x_{stu} \neq 0$, then it follows from (F2) and (H) that for any Wirtinger triple (i,j,k) and $1 \leq a, b \leq n$

$$\begin{aligned}
x_{stu}x_{kab} &= \frac{1}{2}\begin{vmatrix} x_{sk} & x_{sa} & x_{sb} \\ x_{tk} & x_{ta} & x_{tb} \\ x_{uk} & x_{ua} & x_{ub} \end{vmatrix} = \frac{1}{2}\begin{vmatrix} x_{ij}x_{si} - x_{sj} & x_{sa} & x_{sb} \\ x_{ij}x_{ti} - x_{tj} & x_{ta} & x_{tb} \\ x_{ij}x_{ui} - x_{uj} & x_{ua} & x_{ub} \end{vmatrix} \\
&= x_{ij}\frac{1}{2}\begin{vmatrix} x_{si} & x_{sa} & x_{sb} \\ x_{ti} & x_{ta} & x_{tb} \\ x_{ui} & x_{ua} & x_{ub} \end{vmatrix} - \frac{1}{2}\begin{vmatrix} x_{sj} & x_{sa} & x_{sb} \\ x_{tj} & x_{ta} & x_{tb} \\ x_{uj} & x_{ua} & x_{ub} \end{vmatrix} \\
&= x_{ij}x_{stu}x_{iab} - x_{stu}x_{jab} \\
&= x_{stu}(x_{ij}x_{iab} - x_{jab}).
\end{aligned}$$

So $x_{kab} = x_{ij}x_{iab} - x_{jab}$ holds.

Therefore, the hexagon relations (H) and the rectangle relations (R) give an obstruction to lift a point in $F_2(K)$ to $S_0(K)$. Namely, we have the following.

Theorem 3.3. *A point in $F_2(K)$ is a ghost character if and only if the point does not satisfy* (H) *or* (R).

We are now researching relationships between (H) and (R), and trying to find knots with ghost characters. We will report this research in another paper.

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