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A VERY BRIEF INTRODUCTION TO VIRTUAL HAKEN CONJECTURE

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This note is a brief summary of my talk that I gave at RIMS Seminar "Representation spaces, twisted topological invariants and geometric structures of 3-manifolds" on May 30, 2012. The aim of this survey is to give an overview of works used to prove so-called *virtually Haken conjecture.* When I was preparing for my talk, the paper by Aschenbrenner–Friedl–Wilton [6] was very useful and I learned a lot from this.

1. THURSTON'S QUETIONS

In 1982, Thurston asked 24 questions, saying "Here are a few questions and projects concerning 3-manifolds and Kleinian groups which I find fascinating." The first question was the famous geometrization conjecture and the questions (15)–(18) were the following [19]:

(15) Are finitely-generated Kleinian groups LERF?

A group G is LERF if for every finitely generated subgroup $L < G$, and for all $g \in G \setminus L$, there exists a finite group K and a homomorphism $\phi : G \to K$ such that $\phi(g) \notin \phi(L)$. See section 2 for more details.

(16) Does every hyperbolic 3-manifold have a finite-sheeted cover which is Haken?

A compact, orientable, irreducible 3-manifold M is called Haken if M contains an orientable, incompressible surface. M is called virtually Haken if M has a finite-sheeted cover that is Haken. Waldhausen asked whether every compact, orientable, irreducible 3-manifold with infinite fundamental group is virtually Haken [20]. After the proof of the geometrization conjecture, the conjecture was only open for hyperbolic 3-manifolds.

(17) Does every aspherical 3-manifold have a finite-sheeted cover with positive first Betti number?

A 3-manifold M is called aspherical if all its higher homotopy groups ($\pi_i(M)$ for $i \geq 2$) vanish. A group G is said to have positive first Betti number if $\beta_1(G) = \operatorname{rank} \mathrm{H}_1(G; \mathbb{Q}) > 0$. A group G is said to have virtually positive first Betti number if G has a finite index subgroup $G' < G$ with $\beta_1(G') > 0$. A group G is said to have virtually infinite first Betti number if, for any $k > 0$, there exists finite index subgroup $G' < G$ with $\beta_1(G') > k$. A group G is called large if it has a finite index subgroup $G' < G$ and a epimorphism $\phi : G \twoheadrightarrow \mathbb{Z} * \mathbb{Z}$. A 3-manifold M is said to have corresponding properties if $\pi_1(M)$ has.

(18) Does every hyperbolic 3-manifold have a finite-sheeted cover which fibers over the circle?[1]

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[1]After this question, Thurston wrote, "This dubious-sounding question seems to have a definite chance for a positive answer"

Let Σ be a surface and $\phi : \Sigma \to \Sigma$ a homeomorphism. The mapping torus T_ϕ of ϕ is the manifold

$$T_\phi = \Sigma \times [0,1]/(x,0) \sim (\phi(x),1).$$

A 3-manifold M is said to fiber over the circle if M can be obtained as a mapping torus. M is called virtually fibered if M has a finite-sheeted cover which fibers over the circle. There are examples of graph manifolds which are not virtually fibered [15].

Now, we have the following:

Theorem 1.1 (Agol [3]). *All these conjectures are valid.*

2. Locally extended residually finite

We want to know when one can lift π_1-injective immersions to embeddings in finite-sheeted covers, and LERF allows this (Scott [18]).

2.1. **Residually finite.** A group G is residually finite (RF) if for every nontrivial $g \in G$, there exists a finite group K and a homomorphism $\phi : G \to K$ such that $\phi(g) \neq 1$.

Facts 2.1. *Suppose that G is residually finite and finitely generated. Then following hold:*
(1) G is Hopfian[2] *(Mal'cev).*
(2) Aut(G) *is residually finite. (Baumslag)*
(3) G has a solvable word problem.

Example 2.2. (1) Finitely-generated subgroup of GL(n,k), where k is a field. (Mal'cev)
(2) The fundamental group of any compact 3-manifold. (Hempel [13] and geometrization)
(3) Mapping class group of surfaces

It is known that the group $\langle a,b \,|\, b^{-1}a^2b = b^3\rangle$ is not Hopfian, in particular, not residually finite.

Question 2.3. *Is every hyperbolic group residually finite?*

The expected answer seems to be NO, but....

Theorem 2.4 (Agol–Groves–Manning [5]). *If every hyperbolic group is residually finite, then every quasi-convex subgroup of a hyperbolic group is separable.*

2.2. **LERF.** A group G is LERF (locally extended residually finite) if for every finitely generated subgroup $L < G$, and for all $g \in G \setminus L$, there exists a finite group K and a homomorphism $\phi : G \to K$ such that $\phi(g) \notin \phi(L)$.

Examples 2.5. (1) free group (Hall)
(2) surface group (Scott [18]),
(3) Bianchi groups (Agol–Long–Reid [1])
(4) Quasiconvex subgroups of word-hyperbolic Coxeter group (Haglund–Wise [12])

Not all 3-manifold groups are LERF (Burns–Karrass–Solitar [8]).

3. Cube complex

Surprisingly, cube complexes play an essential rule to solve Thurston's questions (15)–(18). Let us begin with the basic definitions.

[2] A group G is Hopfian if every homomorphic mapping of G onto itself is an automorphism.

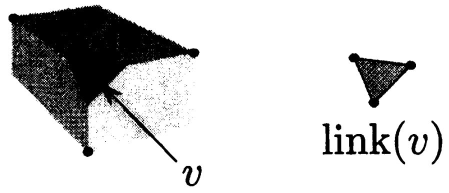

FIGURE 1. The link of a 0-cube

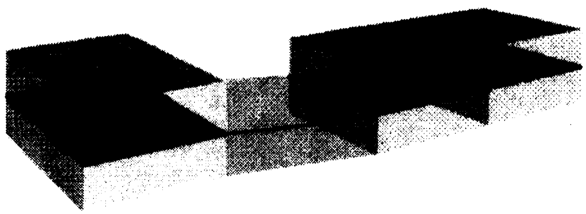

FIGURE 2. Hyperplane

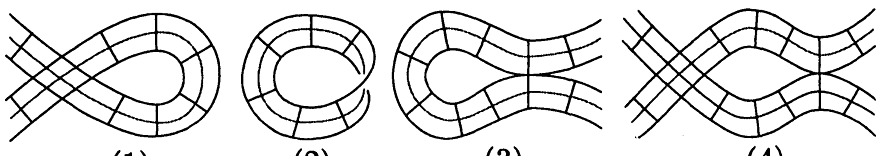

FIGURE 3. Immersed Hyperplane Pathologies

3.1. **Basic definitions.** An n-cube is a copy of $[-1,1]^n$ and a 0-cube is a single point. A cube complex is a cell complex formed from cubes by identifying subcubes. The link of a 0-cube v is a complex of simplices whose n-simplices correspond to corners of $n+1$-cubes meeting at v. See Figure 1. A flag complex is a simplicial complex such that $n+1$ vertices span an n-simplex if and only if they are pairwise adjacent. A cube complex C is nonpositively curved if link(v) is a flag complex for each 0-cube $v \in C^0$. A cube complex X is CAT(0) if it is simply connected and nonpositively curved.

A midcube in $[-1,1]^n$ is a subspace obtained by restricting one coordinate to 0. We then glue together midcubes in adjacent cubes whenever they meet, to get the hyperplanes of X. See Figure 2.

Definition 3.1 ([11]). A cube complex is spacial if all the following hold: See Figure 3.

(1) No immersed hyperplane crosses itself.
(2) Each immersed hyperplane is 2-sided.
(3) No immersed hyperplane self-osculates.
(4) No two immersed hyperplanes inter-osculate.

Theorem 3.2 (Haglund–Wise [11]). *If X is a compact special cube complex and its fundamental group $\pi_1(X)$ is word-hyperbolic, then every quasiconvex subgroup is separable.*

3.2. **Salvetti complex.** Let Σ be any graph. We build a cube complex S_Σ as follows:

(1) S_Σ has one 0-cell;
(2) S_Σ has one (oriented) 1-cell e_v for each vertex v of Σ;

(3) S_Σ has a square 2-cell with boundary reading $e_u e_v \overline{e_u} \overline{e_v}$ whenever u and v are joined by an edge in Σ;
(4) for $n > 2$, the n-skeleton is defined inductively — attach an n-cube to any subcomplex isomorphic to the boundary of n-cube which does not already bound an n-cube.

Let $V(\Sigma) = \{v_1, \ldots, v_k\}$ be the vertex set of the graph Σ. The right-angled Artin group (RAAG) associated to Σ is the group given as follows:

$$A_\Sigma = \langle v_1, \ldots, v_k \mid [v_i, v_j] = 1 \text{ if } v_i \text{ and } v_j \text{ are connected by an edge} \rangle$$

The fundamental group of the Salvetti complex S_Σ is right-angled Artin group.

The hyperplane graph of a cube complex X is the graph $\Sigma(X)$ with vertex-set equal to the hyperplanes of X, and with two vertices joined by an edge if and only if the corresponding hyperplanes intersect.

Typing map $\phi_X : X \to S_{\Sigma(X)}$ is defined as follows:

(0) Each 0-cell of X maps to the unique 0-cell x_0 of $S_{\Sigma(X)}$
(1) Each 1-cell e of X goes to the unique 1-cell in $S_{\Sigma(X)}$ which corresponds to the unique hyperplane that e crosses.
(2) ϕ_X is defined inductively on higher dimensional cubes.

Theorem 3.3 (Haglund–Wise [11]). *Let X be a non-positively curved cube complex. Then X is special if and only if there exists a graph Σ and there is an immersion $X \to S_\Sigma$ that is a local isometry at the level of the 2-skeleta.*

3.3. **Compact special group.** A group is called (compact) special if it is the fundamental group of a non-positively curved (compact) special cube complex.

Let X be a geodesic metric space. A subspace Y is said to be quasi-convex if there exists $\kappa \geq 0$ such that any geodesic in X with endpoints in Y is contained within the κ-neighborhood of Y.

Let π be a group with a fixed generating set S. A subgroup $H \subset \pi$ is said to be quasi-convex if it is a quasi-convex subspace of $\mathrm{Cay}_S(\pi)$, the Cayley graph of π with respect to the generating set S.

Corollary 3.4. *(See Corollary 5.8 and Corollary 5.9 in* [6]*.) A group is special if and only if it is a subgroup of a Right-Angled Artin Group. A group is compact special if and only if it is a quasi-convex subgroup of a Right-Angled Artin Group.*

3.4. **Virtually Compact Special Theorem.** Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f : X \to Y$ is called a quasi-isometric embedding[3] if there exist constants $K \geq 1$ and $C \geq 0$ such that

$$\frac{1}{K} d_X(x,y) - C \leq d_Y(f(x), f(y)) \leq K d_X(x,y) + C$$

for any $x, y \in X$.

Definition 3.5 (Quasiconvex hierarchy). The class QH is defined to be the smallest class of finitely generated groups that is closed under isomorphism and satisfies the following properties.

(1) The trivial group 1 is in QH.

[3]Here, it is not required to preserve any algebraic structure

(2) Amalgamated product $G \cong A *_C B$ is in $\mathcal{QH}$ if $A, B \in \mathcal{QH}$ and C is finitely generated and the inclusion map $C \hookrightarrow A *_C B$ is a quasi-isometric embedding.
(3) HNN extension $G \cong A*_C$ is in $\mathcal{QH}$ if $A \in \mathcal{QH}$ and C is finitely generated and the inclusion map $C \hookrightarrow A*_C$ is a quasi-isometric embedding.

Theorem 3.6 (Virtually Compact Special Theorem for QH [21]). *If $G \in \mathcal{QH}$ is word-hyperbolic, then G is virtually compact special.*

This theorem has an application to one-relater groups.

Corollary 3.7 ([21]). *Every one-relater group with torsion is virtually compact special.*

Let N be a closed, hyperbolic 3-manifold which contains an incompressible geometrically finite surface. Thurston showed that N admits a hierarchy of geometrically finite surfaces. A subgroup of $\pi_1(N)$ is geometrically finite if and only if it is quasiconvex. Combining these results and virtually compact special theorem, we get the following:

Theorem 3.8 (Wise). *Let N be a closed hyperbolic 3-manifold which contains an incompressible geometrically finite surface, then $\pi_1(N)$ is virtually compact special.*

3.5. **Surface subgroups.** Let us recall some notions in Kleinian group theory. A Fuchsian group is a discrete subgroup of $\mathrm{PSL}(2, \mathbb{R})$. A Kleinian group is a discrete subgroup of $\mathrm{PSL}(2, \mathbb{C})$. A quasifuchsian group is a Kleinian group G that is conjugate to a Fuchsian group by a quasiconformal automorphism of $\hat{\mathbb{C}}$.

Fix an identification of $\pi_1(N)$ with a discrete subgroup of $\mathrm{PSL}(2, \mathbb{C})$. N is said to contain a dense set of quasifuchsian surface groups if for each great circle C of $\partial\mathbb{H}^3 = S^2$ there exists a sequence of π_1-injective immersions $\iota : \Sigma_i \to N$ of surfaces Σ_i such that the following hold:

(1) for each i, the group $\iota_*(\pi_1(\Sigma_i))$ is a quasifuchsian surface group.
(2) the sequence $\partial\Sigma_i \subset \partial\mathbb{H}^3$ converges to C in the Hausdorff metric.

Theorem 3.9 (Kahn–Markovic [14]). *Every closed hyperbolic 3-manifold contains a dense set of quasifuchsian surface groups.*

3.6. **Constructing cube complex.** Let G be a finitely generated group with Cayley graph $\mathrm{Cay}(G)$. A subgroup $H \subset G$ is codimension-1 if it has a finite neighborhood $N_r(H)$ such that $\mathrm{Cay}(G) \setminus N_r(H)$ contains at least two components that are deep in the sense that they do not lie in any $N_s(H)$.

Example 3.10. (1) $\mathbb{Z}^n$ in $\mathbb{Z}^{n+1}$.
(2) Any infinite cyclic subgroup of a closed surface subgroup.

Let $H_1, \ldots, H_k$ be a collection of codimension-1 subgroups. The wall associated to H_i is a fixed partition $\{\overleftarrow{N_i}, \overrightarrow{N_i}\}$ of $\mathrm{Cay}(G)$. The translated wall associated to gH_i is the partition $\{g\overleftarrow{N_i}, g\overrightarrow{N_i}\}$.

The (1-skeleton of) "dual cube complex" due to Sageev is defined as follows:

(1) A 0-cube is a choice of one halfspace from each wall such that each element of G lies in all but finitely many of these chosen halfspaces.
(2) Two 0-cubes are joined by a 1-cube precisely when their choices differ on exactly one wall.

Let G be a word-hyperbolic group, and $H_1, \ldots, H_k$ be a collection of quasiconvex codimension-1 subgroups. Then the action of G on the dual cube complex is cocompact. (See Sageev [16], [17].)

Theorem 3.11 ([7]). *Let G be a word-hyperbolic group. Suppose that for each pair of distinct points $(u, v) \in (\partial G)^2$ there exists a quasiconvex codimension-1 subgroup H such that u and v lie in distinct components of $\partial G \setminus \partial H$. Then there is a finite collection $H_1, \ldots, H_k$ of quasiconvex codimension-1 subgroups such that G acts properly and cocompactly on the resulting dual CAT(0) cube complex.*

Combining theorem 3.9 and theorem 3.11, Bergeron and Wise showed that

Theorem 3.12 ([7]). *Let M be a closed hyperbolic 3-manifold. Then $\pi_1(M)$ acts properly and cocompactly on a CAT(0) cube complex.*

3.7. **RFRS and virtual fiber.** For a group G, set $G_r^{(1)} = \{x \in G \mid \exists k \neq 0, x^k \in [G, G]\}$.

Definition 3.13 (RFRS [4]). A group G is residually finite $\mathbb{Q}$-solvable (RFRS) if there is a sequence of subgroups $G = G_0 > G_1 > G_2 > \ldots$ such that $G \triangleright G_i$, $[G : G_i] < \infty$, $\cap_i G_i = \{1\}$ and $G_{i+1} \geq (G_i)_r^{(1)}$.

Theorem 3.14 (Agol [4]). *The following hold:*

- *If G is RFRS, then any subgroup $H < G$ is as well.*
- *Right angled Artin groups are virtually RFRS. (Agol [4])*

Examples 3.15. The following are other examples of (virtually) RFRS:

- surface groups,
- reflection groups,
- arithmetic hyperbolic groups defined by a quadratic.

Theorem 3.16 (Agol [4]). *If M is aspherical and $\pi_1(M)$ is RFRS, then M virtually fibers.*

4. The final step

This is what Agol showed for the final step of the conjectures.

Theorem 4.1 (Agol [3]). *Let G be a word-hyperbolic group acting properly and cocompactly on a CAT(0) cube complex X. Then G has a finite index subgroup F acting specially on X.*

Combining and theorem 3.8, 3.12, 4.1 and other cases, the situation can be described in a very nice way.

Theorem 4.2 (Virtually Compact Special Theorem). *If N is a hyperbolic 3-manifold, then $\pi_1(N)$ is virtually compact special.*

If $\pi_1(N)$ is virtually (compact) special, then it is a subgroup of a RAAG (theorem3.3). By theorem 3.14 and 3.16, N virtually fibers.

To show that $\pi_1(M)$ is LERF, we need the next theorem.

Theorem 4.3 (Tameness [2],[9]). *Let N be a hyperbolic 3-manifold, not necessarily of finite volume. If $\pi_1(N)$ is finitely generated, then N is topologically tame, i.e., N is homeomorphic to the interior of a compact 3-manifold.*

A 3-manifold N is fibered if there exists a fibration $N \to S^1$. A surface fiber in a 3-manifold N is the fiber of a fibration $N \to S^1$. $\Gamma \subset \pi_1(N)$ is a surface fiber subgroup if there exists a surface fiber Σ such that $\Gamma = \pi_1(\Sigma)$. $\Gamma \subset \pi_1(N)$ is a virtual surface fiber subgroup if N admits a finite cover $N' \to N$ such that $\Gamma \subset \pi_1(N')$ and such that Γ is a surface fiber subgroup of N'.

Theorem 4.4 (Subgroup Tameness Theorem). *Let N be a hyperbolic 3-manifold and let $\Gamma \subset \pi_1(N)$ be a finitely generated subgroup. Then either*

(1) Γ *is a virtual surface fiber group, or*
(2) Γ *is geometrically finite.*

For the proof of the theorem, theorem 4.3 and the covering theorem (due to Canary) is used.

Theorem 3.2 and 4.4 are used to show the next theorem.

Theorem 4.5 (Agol). *Let M be a closed hyperbolic 3-manifold. Then there is a finite-sheeted cover $\widetilde{M} \to M$ such that $\widetilde{M}$ fibers over the circle. Moreover, $\pi_1(M)$ is LERF and large.*

Then, a standard argument in 3-manifold theory shows the next theorem.

Theorem 4.6 (Agol). *Let M be a closed aspherical 3-manifold. Then there is a finite-sheeted cover $\widetilde{M} \to M$ such that $\widetilde{M}$ is Haken.*

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