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Partial Differential Equations Arising from the Chern-Simons Gauged $O(3)$ Sigma Model

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Abstract

In this paper, we survey recent progress for the study of partial differential equations arising from the self-dual Chern-Simons gauged $O(3)$ sigma model. We review the classical $O(3)$ sigma model and its extension to gauge field models. Especially, the Maxwell gauged model and the Chern-Simons gauged models are described for broken and unbroken symmetries. We derive the self-dual equations and their reduction to the elliptic equations. We discuss recent progress for the existence of solutions for the reduced equations.

1 Introduction

The classical $O(3)$ sigma model originates from the description of the planar ferromagnet. This model allows an energy lower bound of Bogomol'nyi type which is saturated by the solutions of the self-dual equations. All minimal energy solutions can be obtained by solving these self-dual equations and they turn out to be meromorphic functions [3]. The corresponding minimal energy is given by the degree of solutions.

One of the important property of these soliton solutions is the scale invariance. Due to this conformal invariance, the size of these solitons can change arbitrarily during the time evolution without costing any energy. As a result, this model becomes unsuitable as a model for particles.

There have been several results to break scale invariance of the model. The inclusion of a Skyrme term and a potential term leads to cs-called baby Skyrme model which have soliton solutions of definite size [26, 27]. However, this model is nether integrable nor of Bogomol'nyi type. Another trial is to add a potential term and prevent the solitons from collapsing. A suitable choice of the potential make the model have self-dual structure and the corresponding soliton solutions are called Q-lump [23, 24].

The third possibility of breaking the scale invariance of the sigma model is to introduce a $U(1)$ gauge field. This was initiated by Schroers in [29], where he proposed a $U(1)$ gauged model whose dynamics is governed by the Maxwell term. He introduce a new gauge covariant derivative and derive the self-dual equations by adding a suitable potential. As a consequence, this model possesses topological solitons which gives the lower bound of

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energy. On the other hand, a different type of gauged $O(3)$ sigma model was studied in [1, 14, 21], where the Chern-Simons action is responsible for the dynamics. The addition of Chern-Simons action can give soliton solutions which have finite charge and angular momentum besides broken scale invariance. The vacuum of the potential consists of two symmetric phase which gives topological and nontopological solutions.

For both the Maxwell gauged sigma model and the Chern-Simons sigma gauged model, one can establish new Lagrangian by inserting a parameter which breaks the gauge symmetry. This approach was taken in [21, 30] and gives rich structures. As an vacuum manifold it allows asymmetric phase as well as symmetric phase such that one may consider the planar topological type solutions.

In this article, we review these models in detail and exhibit some mathematical problems related to them. One may refer to [33] for more general overview of the $O(3)$ models in the physics literature.

2 Classical O(3) Sigma Model

The Lagrangian of the classical planar ferromagnet model is given by

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi \cdot \partial^\mu\phi,$$

where the spin vector $\phi : \mathbb{R}^{2,1} \to S^2$ describes a Heisenberg ferromagnet. The static energy is given by

$$\mathcal{E}(\phi) = \int_{\mathbb{R}^2} \{(\partial_1\phi)^2 + (\partial_2\phi)^2\}dx. \tag{2.1}$$

Finite energy condition implies that ϕ goes to a constant unit vector at infinity which makes ϕ a continuous map from S^2 to S^2. Hence, the Hopf degree $\deg\phi \in \mathbb{Z}$ is well defined such that ϕ represents a homotopy class in the homotopy group $\pi_2(S^2) = \mathbb{Z}$. One of the fundamental problem is to find a energy-minimizing configuration among each topological class

$$\mathcal{C}_N = \{\phi : \mathbb{R}^2 \to S^2 \ : \ \mathcal{E}(\phi) < \infty,\ \deg(\phi) = N\}, \qquad N \in \mathbb{Z}.$$

In the work of Belavin-Polyakov [3], it was shown that this static model has a self-dual structure as follows. We first recall the formula of degree for a function $\phi : S^2 \to S^2$. It is convenient to express the degree in terms of the current density

$$k_\alpha = \frac{1}{2}\epsilon_{\alpha\beta\gamma}\phi \cdot \partial^\beta\phi \times \partial^\gamma\phi.$$

Then, the degree is given by

$$\deg\phi = \frac{1}{4\pi}\int_{\mathbb{R}^2} k_0\, dx = \frac{1}{4\pi}\int_{\mathbb{R}^2} \phi \cdot (\partial_1\phi \times \partial_2\phi)\, dx,$$

By an elementary calculation,

$$\mathcal{E}(\phi) = \int_{\mathbb{R}^2} \left(|\partial_1\phi \pm \phi \times \partial_2\phi|^2 + |\partial_2\phi \mp \phi \times \partial_1\phi|^2\right)dx \pm \int_{\mathbb{R}^2} \phi \cdot (\partial_1\phi \times \partial_2\phi) \geq 4\pi|\mathrm{deg}\phi|.$$

The lower bound saturated if and only if ϕ satisfying the following self-dual equations

$$\partial_j\phi = \mp\epsilon_{jk}(\phi \times \partial_k\phi). \tag{2.2}$$

If ϕ is a solution of (2.2) with upper signs, then $-\phi$ is a solution for lower signs. Hence, we only consider upper signs in the following.

In order to simplify (2.2), we introduce the stereographic projection from the south pole $\mathbf{s} = (0, 0, -1)$ of S^2 to obtain a map $u : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$u = (u_1, u_2), \quad u_1 = \frac{\phi_1}{1+\phi_3}, \quad u_2 = \frac{\phi_2}{1+\phi_3}. \tag{2.3}$$

Conversely, we have

$$\phi_1 = \frac{2u_1}{1+|u|^2}, \quad \phi_2 = \frac{2u_2}{1+|u|^2}, \quad \phi_3 = \frac{1-|u|^2}{1+|u|^2}.$$

Let $\mathcal{P} = \phi^{-1}(\mathbf{s}) = \{p_1, \cdots, p_N\}$. We note that

$$\lim_{x\to p_j} |u(x)|^2 = \lim_{x\to p_j} \frac{1-\phi_3}{1+\phi_3} = \infty.$$

We note that away from p_j, (2.2) becomes

$$\partial_1 u_1 = \partial_2 u_2, \quad \partial_1 u_2 = -\partial_2 u_1, \tag{2.4}$$

which implies that $u(z) = u_1(z) + iu_2(z)$ with $z = x_1 + ix_2$ satisfies the Cauchy-Riemann equation. Let $\mathcal{Q} = \phi^{-1}(\mathbf{n}) = \{q_1, \cdots, q_N\}$. Then,

$$u(z) = c\,\frac{(z-q_1)\cdots(z-q_M)}{(z-p_1)\cdots(z-p_N)} \tag{2.5}$$

is a solution of (2.4).

We now calculate the degree of the solutions ϕ given by (2.5). It turns out that we have an obstruction $M \leq N-1$. The degree formula can be rewritten in terms of u as follows:

$$\mathrm{deg}\phi = \frac{i}{2\pi}\int_{\mathbb{R}^2} \frac{\partial_1 u\partial_2\bar{u} - \partial_1\bar{u}\partial_2 u}{(1+|u|^2)^2}\,dx = \frac{1}{4\pi}\int_{\mathbb{R}^2} J_{12}\,dx,$$

where

$$J_{12} = \partial_1 J_2 - \partial_2 J_1, \qquad J_k = \frac{i(u\partial_k\bar{u} - \bar{u}\partial_k u)}{(1+|u|^2)}.$$

Hence,

$$4\pi\mathrm{deg}\phi = \lim_{r\to\infty}\int_{|x|=r} J_k dx_k - \sum_{j=1}^{N}\lim_{r\to 0}\int_{|x-p_j|=r} J_k dx_k.$$

We note that $J_k = (1+|u|^2)^{-1} \cdot O(|x|^{2(M-N)-1})$. Hence, if $N > M$, the first term vanishes identically. It is easy to see that each integral in the second term equals to -4π. As a consequence, we have $\deg\phi = N$.

In conclusion, the classical $O(3)$ sigma model has an energy lower bound of Bobomol'nyi type which is saturated by the solutions of the self-dual equations. All minimal energy solutions can be obtained by solving this self-dual equations and they turn out to be meromorphic functions having N poles and M zeros. The corresponding minimal energy is given by the degree of solutions which equals N. Moreover, we have a constraint $M \leq N-1$.

In the next two sections, we will consider gauge field model which is extensions of the classical $O(3)$ sigma model in this section. We derive self-dual equations and find the minimal energy solutions as in this section.

3 Maxwell Gauged O(3) Sigma Model

3.1 Unbroken Model

We recall that the finite energy condition for (2.1) implies that $\phi \to \phi_\infty$ as $|x| \to \infty$. Since the functional (2.1) is invariant under the group of rotation $O(3)$, we may assume that $\phi_\infty = \mathbf{n}$. This choice breaks the $O(3)$ symmetry down to $SO(2) = U(1)$, from which one may consider a minimal potential like $(1 - \mathbf{n}\cdot\phi)^2$. In this case, we can consider the energy functional as

$$\mathcal{E}(\phi) = \int_{\mathbb{R}^2} \{(\partial_1\phi)^2 + (\partial_2\phi)^2 + (1-\mathbf{n}\cdot\phi)^2\}dx.$$

However, in this case there is no nontrivial critical points other than $\phi \equiv \mathbf{n}$.

We note that the vacuum state $\phi_{\mathrm{v}} = \mathbf{n}$ is invariant under the $SO(2) = U(1)$ transformation

$$(\phi_1,\phi_2,\phi_3) \quad \to \quad (\phi_1\cos\theta - \phi_2\sin\theta, \phi_1\sin\theta + \phi_2\cos\theta, \phi_3), \qquad \theta \in \mathbb{R}.$$

Hence, if we set $\psi = \phi_1 + i\phi_2$, then the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\psi\cdot\partial^\mu\bar{\psi} + \frac{1}{2}\partial_\mu\phi_3\cdot\partial^\mu\phi_3 - \frac{1}{2}(1-\mathbf{n}\cdot\phi)^2 \tag{3.1}$$

is invariant under the global gauge transformation

$$\psi \to e^{i\theta}\psi, \quad \phi_3 \to \phi_3.$$

To enlarge this global symmetry into a local one, we need a gauge-covariant derivative $D_\mu\psi = \partial_\mu\psi + iA_\mu\psi$. The componentwise expression is

$$D_\mu\phi_1 = \partial_\mu\phi_1 - A_\mu\phi_2, \quad D_\mu\phi_2 = \partial_\mu\phi_2 + A_\mu\phi_1, \quad D_\mu\phi_3 = \partial_\mu\phi_3,$$

which is equal to

$$D_\mu\phi = \partial_\mu\phi + A_\mu(\mathbf{n}\times\phi).$$

We also introduce the Maxwell field $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Then, we can extend (3.1) to the following Maxwell gauged model

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}D_\mu\phi \cdot D^\mu\phi - \frac{1}{2}(1-\mathbf{n}\cdot\phi)^2. \tag{3.2}$$

We define a new current density

$$j_\alpha = \frac{1}{2}\epsilon_{\alpha\beta\gamma}\left[\phi\cdot D^\beta\phi \times D^\gamma\phi + F^{\beta\gamma}(1-\mathbf{n}\cdot\phi)\right] = k_\alpha + \epsilon_{\alpha\beta\gamma}\partial^\beta\left[(1-\mathbf{n}\cdot\phi)A^\gamma\right].$$

Then, the degree can be expressed as

$$\deg\phi = \frac{1}{4\pi}\int_{\mathbb{R}^2} j_0\,dx = \frac{1}{4\pi}\int_{\mathbb{R}^2}\left[\phi\cdot D_1\phi\times D_2\phi + F_{12}(1-\mathbf{n}\cdot\phi)\right]dx. \tag{3.3}$$

The static energy is given by

$$\begin{aligned}\mathcal{E}(\phi,A) &= \int_{\mathbb{R}^2}\left\{(D_1\phi)^2 + (D_2\phi)^2 + F_{12}^2 + (1-\mathbf{n}\cdot\phi)^2\right\}dx\\ &= \pm 4\pi\deg\phi + \frac{1}{2}\int_{\mathbb{R}^2}\left\{(D_1\phi \pm \phi\times D_2\phi)^2 + (F_{12}\mp[1-\mathbf{n}\cdot\phi])^2\right\}dx\\ &\geq 4\pi|\deg\phi|.\end{aligned}$$

Hence, we get a energy lower bound $4\pi|\deg\phi|$ which is saturated by the following self-dual equations: taking upper signs

$$D_1\phi + \phi\times D_2\phi = 0, \tag{3.4}$$
$$F_{12} - (1-\mathbf{n}\cdot\phi) = 0. \tag{3.5}$$

In terms of the new function u defined by (2.3), we can rewrite (3.4) and (3.5) as

$$D_1 u + iD_2 u = 0, \tag{3.6}$$
$$F_{12} - \frac{2|u|^2}{1+|u|^2} = 0, \tag{3.7}$$

where $D_j u = \partial_j u + iA_j u$.

It is obvious that $u(x) = 0$ if and only if $x \in \phi^{-1}(\mathbf{n})$. By virtue of the $\bar{\partial}$-Poincaré Lemma (see [18]), the equation of (3.6) implies that u is locally represented by a holomorphic factor up to a smooth function. Thus, the zeros of u have integer multiplicities and are realized by the points in $\phi^{-1}(\mathbf{n})$. Similar arguments show that ψ has poles at the points in $\phi^{-1}(\mathbf{s})$. Indeed, letting $\tilde{D}'_j = \partial_j - iA_j$ and $\tilde{u} = 1/u$, we see that

$$\tilde{D}'_1\tilde{u} = -\tilde{u}^2\tilde{D}_1 u = i\tilde{u}^2\tilde{D}_2 u = -i\tilde{D}'_2\tilde{u},$$

which leads us by the $\bar{\partial}$-Poincaré Lemma that $\tilde{u}$ has zeros of integer multiplicities. Now let

$$\left\{\begin{aligned} v &= |u|^2,\\ \mathcal{Q} &= \phi^{-1}(\mathbf{n}) = \{q_1, q_2, \cdots, q_M\},\\ \mathcal{P} &= \phi^{-1}(\mathbf{s}) = \{p_1, p_2, \cdots, p_N\}.\end{aligned}\right. \tag{3.8}$$

Each point p_j or q_j is counted as its multiplicity. Then, the equation (3.7) reduces to

$$\Delta v = \frac{4e^v}{1+e^v} - 4\pi \sum_{j=1}^{N} \delta_{p_j} + 4\pi \sum_{j=1}^{M} \delta_{q_j}. \tag{3.9}$$

The finite energy condition gives the boundary condition $\phi \to \mathbf{n}$ as $|x| \to \infty$, which is equivalent to $u(x) \to 0$ and hence $v(x) \to -\infty$ as $|x| \to \infty$.

We are interested in the solutions v of (3.9) having finite magnetic flux which is given by

$$\Phi = \int_{\mathbb{R}^2} F_{12} dx = \int_{\mathbb{R}^2} \frac{2e^v}{1+e^v}\, dx < \infty. \tag{3.10}$$

The corresponding energy is also finite and the solution of (3.9) satisfying (3.10) is referred as a finite-energy solution. If we set

$$v_0(x) = -\sum_{j=1}^{N} \ln |x-p_j|^2 + \sum_{k=1}^{M} \ln |x-q_k|^2,$$

then $w = v - v_0$ satisfies

$$\Delta w = \frac{4e^{u_0+w}}{1+e^{u_0+w}} \equiv F \in L^\infty(\mathbb{R}^2) \cap L^1(\mathbb{R}^2).$$

From a standard argument it follows that

$$\lim_{|x|\to\infty} \frac{w(x)}{\ln |x|} = \frac{1}{2\pi} \int_{\mathbb{R}^2} F(x)\, dx \equiv \alpha > 0.$$

As a consequence, v enjoys the following decays:

$$v(x) = -\beta \ln |x| + o(\ln |x|) \quad \text{as} \quad |x| \to \infty, \tag{3.11}$$

where $\beta = 2N - 2M - \alpha < 2(N-M)$. By the condition (3.10), we also see that $\beta \geq 2$. In the sequel, if v is a finite-energy solution of (3.9), then v must satisfy (3.11) with

$$\beta \in [2, 2N-2M) \quad \text{and} \quad N - M \geq 2. \tag{3.12}$$

In this point of view, the main question about the existence of finite-energy solutions is to verify whether the condition (3.12) is also sufficient. Concerning this problem, it was proved in [31] that if $N - M \geq 2$, then for each $\beta \in (2,4)$ there exists a unique finite-energy solution v of (3.9) satisfying that

$$v(x) = -\beta \ln |x| + O(1) \quad \text{as} \quad |x| \to \infty. \tag{3.13}$$

More general result is the following.

Theorem 3.1 ([15]). *Suppose that $N - M \geq 2$.*

(a) *If u is a finite-energy solution of* (3.9) *satisfying* (3.11) *for some $\beta \in (2, 2N-2M)$, then u satisfies* (3.13).

(b) *Conversely, for any $\beta \in (2, 2N-2M)$, there exists a unique finite-energy solution v of* (3.9) *satisfying* (3.13). *Moreover, we have*

$$\int_{\mathbb{R}^2} \frac{4e^v}{1+e^v}\, dx = 2\pi(2N - 2M - \beta). \tag{3.14}$$

The critical case is that $\beta = 2$. The decay rate of the solution turns out to be different from (3.13). We have the following partial result for the existence of solutions.

Theorem 3.2 ([15]). *Let $N \geq 2$ be an integer and $M = 0$. Then, for $\beta = 2$,* (3.9) *possesses a unique finite-energy solution u satisfying* (3.14). *The solution v satisfies the following asymptotic behavior*

$$u(x) = -2\ln|x| - 2\ln\ln|x| + O(1) \quad \text{as} \quad |x| \to \infty. \tag{3.15}$$

It is still open to find solutions of (3.9) for the general case $M \neq 0$ when $\beta = 2$. Once we find a solution v of (3.9), we can find the corresponding solution of (3.4) and (3.5) by using the relations of v, u and (ϕ, A). So, the physical quantities of solutions satisfy the following properties.

Theorem 3.3. *Let (ϕ_β, A_β) of* (3.4) *and* (3.5) *corresponding to $\beta \in (2, 2N-2M)$. Then, we have*

$$\begin{aligned} &\textit{energy}: && \mathcal{E}(\phi_\beta, A_\beta) = \deg\phi_\beta = 4\pi N, \\ &\textit{magnetic flux}: && \Phi(\phi_\beta, A_\beta) = \int_{\mathbb{R}^2} \frac{2e^v}{1+e^v}\, dx = \pi(2N - 2M - \beta), \\ &\textit{decays}: && (\phi_\beta)_1^2, (\phi_\beta)_2^2, (F_{12})_\beta = 1 - (\phi_\beta)_3^2, |D\phi_\beta|^2 = O(|x|^{-\beta}), \end{aligned} \tag{3.16}$$

where $\beta = 2(N - M) - 2N\alpha$.

3.2 Broken Model

We consider a more general situation:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}D_\mu\phi \cdot D^\mu\phi - \frac{1}{2}(\tau + \mathbf{n}\cdot\phi)^2, \tag{3.17}$$

where τ is a real number in $[-1, 1]$. We note that if $|\tau| = 1$,then the vacuum state is given by two symmetric phases $\phi_{\mathrm{v}} = \pm\mathbf{n}$ which is invariant under $U(1)$ transformation. On the other hand, if $|\tau| < 1$, then we have broken phases $\mathbf{n}\cdot\phi_{\mathrm{v}} = 1$.

We define a new current density

$$j_\alpha = \frac{1}{2}\epsilon_{\alpha\beta\gamma}\left[\phi \cdot D^\beta\phi \times D^\gamma\phi - F^{\beta\gamma}(\tau + \mathbf{n}\cdot\phi)\right] = k_\alpha - \epsilon_{\alpha\beta\gamma}\partial^\beta\left[(\tau + \mathbf{n}\cdot\phi)A^\gamma\right].$$

The corresponding conserved topological charge is defined by

$$T = \frac{1}{4\pi}\int_{\mathbb{R}^2} j_0\, dx = \frac{1}{4\pi}\int_{\mathbb{R}^2} \left[\phi\cdot D_1\phi\times D_2\phi - F_{12}(\tau+\mathbf{n}\cdot\phi)\right] dx. \tag{3.18}$$

Comparing this formula with (3.3), we can see that if $\tau=-1$, then T is the degree of ϕ. However, T may not be an integer in general. The static energy is given by

$$\begin{aligned}
\mathcal{E}(\phi,A) &= \int_{\mathbb{R}^2}\left\{(D_1\phi)^2+(D_2\phi)^2+F_{12}^2-(\tau+\mathbf{n}\cdot\phi)^2\right\}dx \\
&= \pm 4\pi T + \frac{1}{2}\int_{\mathbb{R}^2}\left\{(D_1\phi\pm\phi\times D_2\phi)^2-(F_{12}\pm[\tau+\mathbf{n}\cdot\phi])^2\right\}dx \\
&\geq 4\pi|T|.
\end{aligned}$$

Hence, we get a energy lower bound $4\pi|T|$ which is saturated by the following self-dual equations: taking upper signs

$$D_1\phi+\phi\times D_2\phi = 0, \tag{3.19}$$
$$F_{12}+(\tau+\mathbf{n}\cdot\phi) = 0. \tag{3.20}$$

In terms of the new function u defined by (2.3), we can rewrite (3.4) and (3.5) as

$$D_1u+iD_2u = 0, \tag{3.21}$$
$$F_{12}-\frac{(1-\tau)|u|^2-(1+\tau)}{1+|u|^2} = 0, \tag{3.22}$$

where $D_ju=\partial_ju+iA_ju$. Using the notations of (3.8) and proceeding as before, we change (3.7) into

$$\Delta v = \frac{2(1-\tau)e^v-2(1+\tau)}{1+e^v} - 4\pi\sum_{j=1}^N\delta_{p_j}+4\pi\sum_{j=1}^M\delta_{q_j}. \tag{3.23}$$

It is not difficult to see that if v is a solution of (3.23) for $\tau\in(-1,0]$, then $-v$ is also a solution for $-\tau$ with the change of the roles of p_j's and q_j's. Therefore, from now on we may assume that $-1\leq\tau\leq 0$. When $\tau=-1$, (3.23) corresponds to the unbroken case (3.9). So, we consider only the broken case $|\tau|<1$ in the following. The finite energy condition gives the boundary condition $\phi_3\to-\tau$ as $|x|\to\infty$, which is equivalent to $|u(x)|^2\to(1+\tau)/(1-\tau)$. Letting

$$w = v+\ln a, \qquad a=(1-\tau)/(1+\tau)\geq 1, \tag{3.24}$$

we have

$$\begin{aligned}
&\Delta w = \frac{4a(e^w-1)}{(1+a)(e^w+a)} - 4\pi\sum_{j=1}^N\delta_{p_j}+4\pi\sum_{j=1}^M\delta_{q_j}, \\
&w \to 0 \text{ as } |x|\to\infty.
\end{aligned} \tag{3.25}$$

Concerning this equation, we have the following existence result.

Theorem 3.4 ([32]). *Suppose that* $|\tau| < 1$.

(a) *For any* $N, M > 0$, *there exists a unique solution* w *of* (3.25) *decaying exponentially at the infinity .*

(b) *The corresponding solution* (ϕ, A) *of* (3.19) *and* (3.20) *enjoys the following:*

$$\begin{aligned} &\textit{energy :} && \mathcal{E}(\phi, A) = \frac{1}{2}(1-\tau)(N-M), \\ &\textit{magnetic flux :} && \Phi(\phi, A) = 2\pi(N-M). \end{aligned} \tag{3.26}$$

Proof. Since the proof of the part (b) was not explicitly given in [32], we provide it here. It is easy to see that $\Phi(\phi, A) = 2\pi(N-M)$. It follows from (3.3) and (3.18) that

$$T = \deg \phi - \frac{1}{4\pi}\int_{\mathbb{R}^2} H_{12},$$

where $H_{12} = \partial_1 H_2 - \partial_2 H_1$ and

$$H_k \;=\; A_k(\tau + \mathbf{n}\cdot\phi) \;=\; A_k\,\frac{(1+\tau) - (1-\tau)|u|^2}{1+|u|^2}, \qquad k = 1, 2.$$

If we set $J_{12} = \partial_1 J_2 - \partial_2 J_1$ and $G_{12} = \partial_1 G_2 - \partial_2 G_1$, where with $k = 1, 2$,

$$\begin{aligned} J_k \;&=\; \frac{iu\overline{\tilde{D}_k u} - i\bar{u}\tilde{D}_k u}{1+|u|^2} \;=\; \frac{\varepsilon_{kl}\partial_l v}{1+e^v}, \\ G_k \;&=\; -\frac{2|u|^2 A_k}{1+|u|^2} \;=\; \frac{(1+\tau)e^v - (1-\tau)}{2(1+e^v)}\,\varepsilon_{kl}\partial_l\Big(v - \sum_{j=1}^{N}\ln|x-p_j|^2 + \sum_{j}^{M}|x-q_j|^2\Big), \end{aligned}$$

then we obtain from [33] that

$$\left\{ \begin{aligned} &T = \frac{1}{4\pi}\int_{\mathbb{R}^2} J_{12} - \frac{1+\tau}{4\pi}\,\Phi(\phi, A), \\ &\deg\phi = T + \frac{1}{4\pi}\int_{\mathbb{R}^2} H_{12}. \end{aligned} \right. \tag{3.27}$$

Here, we used the identity $H_k - G_k = (1+\tau)A_k$. Since v and its derivative decay exponentially to zero at infinity, we obtain that

$$\int_{\mathbb{R}^2} J_{12}dx = 4\pi(N-M), \qquad \int_{\mathbb{R}^2} H_{12}dx = 0. \tag{3.28}$$

As a consequence, we have

$$T = \frac{1}{2}(1-\tau)(N-M).$$

The proof is finished. □

It is worth while to mention the difference between Theorem 3.3 and Theorem 3.4. In the broken model (3.17), the magnetic flux and the energy are quantized by (3.26). This is one of the features of classical $U(1)$ gauged planar vortices in $(2+1)$ (for example the Abelian Higgs model [18]): they either have quantized magnetic flux in which case they are topologically stable, or have arbitrary flux in which case they are not topologically stable. So the quantization of the magnetic flux is responsible for the topological stability of soliton solutions. However, in the unbroken model (3.2), the topological stability of soliton solutions is independent of the magnetic flux. The solitons can carry arbitrary magnetic flux in a certain range, but yet be topologically stable such that the energy is quantized by the degree of the scalar field. See (3.16).

4 Chern-Simons Gauged O(3) Sigma Model

In this section, we consider the model in the previous section when the action is governed by the Chern-Simons force instead of the Maxwell force. We consider both the broken and the unbroken models. The Lagrangian is given by

$$\mathcal{L} = -\frac{1}{4}\varepsilon^{\mu\nu\rho}F_{\mu\nu}A_\rho + \frac{1}{2}D_\mu\phi\cdot D^\mu\phi + \frac{1}{2\kappa^2}(\tau + \mathbf{n}\cdot\phi)^2(\mathbf{n}\times\phi)^2.$$

Here, the constant $\kappa > 0$ is the Chern-Simons coupling constant representing the strength of the Chern-Simons action. While $\tau \in [-1,1]$ determines the vacuum manifold of the potential. If $|\tau| = 1$, we have only the symmetric vacua $\phi_{\mathrm{v}} = \pm\mathbf{n}$ which is fixed under the $U(1)$ gauge group. So, the case $|\tau| = 1$ gives the unbroken model. If $|\tau| < 1$, we have not only the symmetric vacuum but also a asymmetric vacuum $\mathbf{n}\cdot\phi_{\mathrm{v}} = \tau$ which is invariant but not fixed under the $U(1)$ gauge group. Hence, we have the broken model for $|\tau| < 1$.

The Gauss law equation yields

$$\kappa F_{12} - 2A_0\big(\tau + (\mathbf{n}\cdot\phi)^2\big) = 0.$$

Then, by means of the Gauss law equation, the static energy is given by

$$\begin{aligned}\mathcal{E}(\phi) &= \int_{\mathbb{R}^2} \frac{\kappa^2 F_{12}^2}{(\mathbf{n}\times\phi)^2} + (D_1\phi)^2 + (D_2\phi)^2 - \frac{1}{\kappa^2}(\tau + \mathbf{n}\cdot\phi)^2(\mathbf{n}\times\phi)\\ &= \pm 4\pi T + \int_{\mathbb{R}^2}\left\{\left(\frac{\kappa F_{12}}{|\mathbf{n}\times\phi|} \pm \frac{1}{\kappa}(\tau+\mathbf{n}\cdot\phi)|\mathbf{n}\times\phi|\right)^2 + (D_1\phi \pm \phi\times D_2\phi)^2\right\}dx\\ &\geq 4\pi|T|,\end{aligned}$$

where the equality holds if and only if ϕ satisfies the self-dual equations

$$\begin{aligned} D_1\phi \pm \phi\times D_2\phi &= 0,\\ F_{12} \pm \frac{2}{\kappa^2}(\tau + \mathbf{n}\cdot\phi)(\mathbf{n}\times\phi)^2 &= 0.\end{aligned} \tag{4.1}$$

We take the upper signs of (4.1).

In terms of the new function u defined by (2.3), we can rewrite (4.1) as

$$D_1u + iD_2u = 0, \tag{4.2}$$

$$F_{12} - \frac{4|u|^2\{(1-\tau)|u|^2 - (1+\tau)\}}{(1+|u|^2)^3} = 0, \tag{4.3}$$

where $D_ju = \partial_j u + iA_ju$. Under the setting of (3.8) and proceeding as before, we change (4.2) and (4.3) into

$$\Delta v + \frac{1}{\varepsilon^2} f(u,\tau) = -4\pi \sum_{j=1}^{N} \delta_{p_j} + 4\pi \sum_{j=1}^{M} \delta_{q_j}, \tag{4.4}$$

where $\varepsilon^2 = 8/\kappa^2$ and

$$f(v,\tau) = \frac{e^v\{(1-\tau) - (1+\tau)e^v\}}{(1+e^v)^3}.$$

We have three types of boundary condition from the finite energy condition:

$$\begin{cases} \text{topological BC : } v(x) \to \ln \dfrac{1-\tau}{1+\tau} \quad \text{for} \quad |\tau| < 1, \\ \text{nontopological BC of type I : } v(x) \to -\infty, \\ \text{nontopological BC of type II : } v(x) \to +\infty. \end{cases} \tag{4.5}$$

For the topological solitons, we have the following result.

Theorem 4.1. (a) [32] *There exists a solution of* (4.4) *which decays exponentially to zero at infinity. The corresponding energy and the magnetic flux are quantized.*

(b) [13] *If ε is small enough, the solution is unique.*

The uniqueness of solutions for general $\varepsilon > 0$ is still an open problem. In the following, we focus on the nontopological solutions. In particular, let us consider the case when there is only one vortex or antivortex point. In this case, we want to obtain radial solutions. To see this, let $v(r;\tau,s)$ denote the solution of the following initial value problem:

$$\begin{cases} v'' + \dfrac{1}{r}v' + f(v,\tau) = 0 & \text{for} \quad r > 0\,, \\ v(r;\tau,s) = 2N\ln r + s + o(1) & \text{as} \quad r \to 0. \end{cases} \tag{4.6}$$

Here, v' always denotes $\frac{dv}{dr}(r;\tau,s)$. Since $f(-v,-\tau) = -f(v,\tau)$, $w = -v$ satisfies (4.4) with τ and $2N$ replaced by $-\tau$ and $-2N$. So without loss of generality, we might assume that $N \geq 0$. Define $\beta(\tau,s)$ by

$$\beta(\tau,s) = \frac{1}{2\pi}\int_{\mathbb{R}^2} f\big(v(r;\tau,s),\tau\big)dx = \int_0^\infty f\big(v(r;\tau,s),\tau\big)dr.$$

Then, we have

$$v(r;\tau,s) = \big[2N - \beta(\tau,s)\big]\ln r + I_{N,\beta,\tau} + O(r^{2+2N-\beta}) \quad \text{for large } r. \tag{4.7}$$

For the unbroken case $|\tau| = 1$, we have a result for the existence of solutions from [1] for certain range of β. This result is extended in [9] as follows.

Theorem 4.2 ([9]). (a) *Let $\tau = -1$. If $N \geq 0$ and $\beta > 4N+4$, then there exists a nontopological solution u_β of* (4.6) *which satisfies* (4.7). *If (ϕ_β, A_β) is the corresponding solution of* (4.1), *then*

$$\mathcal{E}(\phi_\beta, A_\beta) = \Phi(\phi_\beta, A_\beta) = 2\pi\beta. \tag{4.8}$$

(b) *If $N \geq 1$ and $\beta \in (0, 2N-1]$, then* (4.6) *possesses a unique nontopological solution u_β of type II such that u_β satisfies* (4.7) *for $\beta \in (0, 2N-1)$, and*

$$u_\beta(r) = \ln r + \ln\ln r + O(1) \quad \textit{near } \infty \quad \textit{for} \quad \beta = 2N-1. \tag{4.9}$$

(c) *Let $\tau = 1$. If $N \geq 0$ and $\beta < \min\{0, 4N-4\}$, then* (4.6) *admits a nontopological solution u_β of type II which satisfies* (4.7). *There is no nontopological solution of type I to the equation* (4.6).

(d) *In both cases (b) and (d), we have*

$$\mathcal{E}(\phi_\beta, A_\beta) = 4\pi N, \quad \Phi(\phi_\beta, A_\beta) = \pi\beta. \tag{4.10}$$

For the unbroken case $|\tau| < 1$, we have the following result.

Theorem 4.3 ([12]). *Suppose N is a nonnegative integer and $\tau \in (-1, 1)$ is given.*

(a) *There exists a unique $s_* = s_*(\tau)$ such that $\beta(\tau, s_*) = 2N$, i.e.* (4.6) *has a unique topological solution $v(r; \tau, s)$.*

(b) *$\beta(\tau, s) > 0$ if and only if $s < s_*$. In this case, $v(r; \tau, s)$ is a nontopological solution of type I. Moreover, $\beta(\tau, \cdot) : (-\infty, s_*) \to (4N+4, \infty)$ is strictly increasing in s and satisfies*

$$\lim_{s \to -\infty} \beta(\tau, s) = 4N+4, \quad \lim_{s \to s_*} \beta(\tau, s) = +\infty.$$

(c) *For $\beta < \min\{0, 4N-4\}$, there exists an unique $s \in (s_*, \infty)$ such that $\beta(\tau, s) = \beta$, and $v(r; \tau, s)$ is a nontopological solution of type II.*

(d) *Let (ϕ, A) be the corresponding solution of* (4.1). *The energy $\mathcal{E}(\phi, A)$, which equals the topological charge $4\pi T(\phi, A)$, is given by*

$$\begin{aligned} \mathcal{E}(\phi_*, A_*) &= 2\pi N(1-\tau), \\ \mathcal{E}(\phi_{(\beta)}, A_{(\beta)}) &= \begin{cases} \pi(1-\tau)\beta, & \beta > 4N+4, \\ 4\pi N - \pi\beta(1+\tau), & \beta < \min\{0, 4N-4\}. \end{cases} \end{aligned} \tag{4.11}$$

(e) *The degree of ϕ is given by*

$$\begin{aligned} \deg \phi_* &= \frac{1}{2}(1-\tau)N, \\ \deg \phi_{(\beta)} &= \begin{cases} 0, & \beta > 4N+4, \\ N, & \beta < \min\{0, 4N-4\}. \end{cases} \end{aligned} \tag{4.12}$$

(f) *The total magnetic flux* Φ *and the total charge* $Q = -\kappa\Phi$ *are given by*

$$\begin{aligned} \Phi(\phi_*, A_*) &= 2\pi N, \\ \Phi(\phi_{(\beta)}, A_{(\beta)}) &= \pi\beta. \end{aligned} \tag{4.13}$$

Radially symmetric solutions may be used to the study of non-radial solutions. In particular, the linearized operator at each radial solution plays a crucial role in constructing blow-up solutions. Indeed, using Theorem 4.2 and Theorem 4.3, one can show that the linearized operator $\Delta + f'(u,\tau)$ is an isomorphism on suitable function spaces. Then, following the argument of [7, 22], we can construct bubbling solutions [10, 11].

Now let us investigate the properties of solutions for broken and unbroken models. In the broken case $|\tau| = 1$, the vacuum of the potential consists of two symmetric phase which gives nontopological solutions of type I and II. The type II solution is topologically stable and the degree is an integer. The energy is quantized but the magnetic flux is fractional. See Theorem 4.2 (b)-(d). For the nontopological solution of type I, the energy and the magnetic flux are fractional. So it looks like the classical planar nontopological solutions [4]. See Theorem 4.2 (a).

In the broken model $|\tau| < 1$, we have two symmetric phases and one asymmetric phase, depending on the parameter τ. Two symmetric phases gives nontopological solutions of type I and II. The asymmetric phase yields a topological solution. The nontopological solutions have fractional magnetic flux and energy. The type II solution has integer degree but the type II solution has fractional degree. In the Schroers's model, the energy is equal to the degree of the solution. However, in the Chern-Simons model they are different for type II solutions. In contrast, the topological solutions have quantized magnetic flux and energy and look like the classical planar Chern-Simons system [19, 20]. In this case, the degree is fractional. See Theorem 4.3 (d)-(f).

We close this section with two remarks. First, we can consider sigma models under the t'Hooft type periodic condition. This leads us to consider (4.4) on a flat torus. One can find some results in [6, 11, 22, 25] for this direction. Second, one may consider the partial differential equations arising from the self-dual Maxwell-Chern-Simons gauged $O(3)$ sigma model [21]. This model unifies the Maxwell $O(3)$ sigma model and the Chern-Simons $O(3)$ sigma model. Recent progress for this model can be found in [5, 8, 16, 17, 28].

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