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A GRADIENT FLOW APPROACH TO THE KELLER-SEGEL SYSTEMS

ADRIEN BLANCHET[1]

ABSTRACT. These notes are dedicated to recent global existence and regularity results on the parabolic-elliptic Keller-Segel model in dimension 2, and its generalisation with nonlinear diffusion in higher dimensions, obtained throught a gradient flow approach in the Wassertein metric. These models have a critical mass M_c such that the solutions exist globally in time if the mass is less than M_c and above which there are solutions which blowup in finite time. The main tools, in particular the free energy, and the idea of the methods are set out.

1. INTRODUCTION

The Keller-Segel system can be seen as a first step toward the understanding of how, during the evolution of species, the passage from uni-cellular organisms to more complex structure was achieved. It is also a paradigm model for pattern formation of cells for meiose (e.g. [14]), embryo-genesis or angio-genesis, Balo disease (e.g. [25]), bio-convection (e.g. [18]) etc. In physics, this system models the motion of the mean field of many self-gravitating Brownian particles, see [17, 16].

Chemo-taxis is the phenomenon whereby organisms direct their movements according to certain chemicals in their environment. If the movement is toward a higher concentration of the chemical we speak about positive chemo-taxis and the attractant is called the *chemo-attractant.*

Some cells can produce this chemo-attractant themselves, creating thus a long-range non-local interaction between them. We are interested in a very simplified model of aggregation at the scale of cells by chemo-taxis: some myxamoebaes experience a random walk to spread in the space and find food. But in starvation conditions, they emit a chemical signal: the cyclic adenosine monophosphate (cAMP). They move towards a higher concentration of cAMP. Their behaviour is thus the result of a competition between a random walk-based diffusion process and a chemo-taxis-based attraction.

In nature the *dictyostelium discoideum* spread on the soil and then come together by chemo-taxis to form a motile pseudoplasmodium. This slug creeps to a few centimetres below the soil surface where it forms a fruiting body with spores and a stalk. The spores are then blown away by the wind to colonise a new place. See Figure 1.

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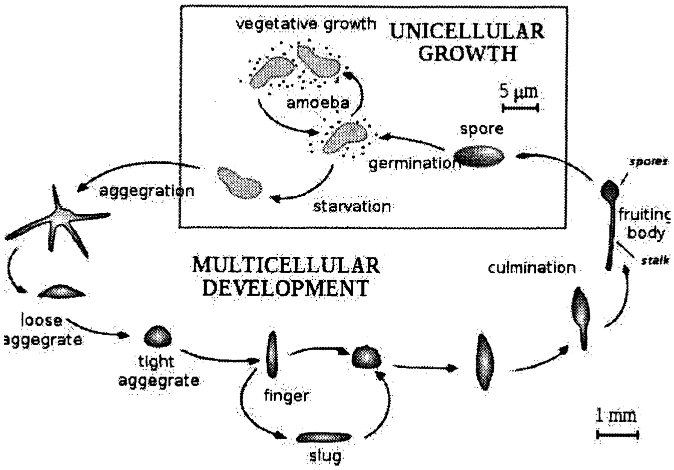

FIGURE 1. *Dictyostelium discoideum cycle (source: Wikipedia).*

The general form of the model is a competition between diffusion and aggregation:

$$\frac{\partial \rho}{\partial t} = \underbrace{\Delta(\rho^m)}_{\text{diffusion}} \underbrace{-\operatorname{div}(\rho \nabla \mathcal{K} * \rho)}_{\text{aggregation}} \quad \text{in } (0, +\infty) \times \mathbb{R}^d, \tag{1}$$

where $\mathcal{K}$ is a given attractive interaction potential.

The model of this aggregation phenomenon is due E. F. Keller and L. A. Segel in [24] and C. S. Patlak in [29]. The parabolic-parabolic Keller-Segel (thereafter KS) system is a drift-diffusion equation given by

$$\begin{cases} \dfrac{\partial \rho}{\partial t} = \Delta(\rho^m) - \operatorname{div}[\rho \nabla \phi], \\ \tau \partial_t \phi = \Delta \phi - \alpha \phi + \rho, & (t, x) \in (0, \infty) \times \mathbb{R}^d, \\ \rho_0 \geq 0 \qquad \phi_0 \geq 0 \end{cases} \tag{2}$$

where $m \in [1, 2)$, τ and α are given non-negative parameters and $d \geq 1$. Here ρ represents the cell density and c the concentration of chemo-attractant. This system corresponds to (1) with $\mathcal{K}$ being the kernel of the operator $\tau \partial_t - \Delta + \alpha$. For more references see [30, 22, 17].

It is immediate to notice that solution to such kind of problem have formally a mass which is preserved along time:

$$\int_{\mathbb{R}^d} \rho(x, t) \, dx = \int_{\mathbb{R}^d} \rho_0(x) \, dx =: M$$

so that birth and death of the organisms are ignored.

It was noticed experimentally that if there are enough bacteria they aggregate whereas if not they go on spreading, *e.g.* [12]. We thus expect the mass to play a crucial role. Let us then consider the following mass-preserving scaling: $\rho_\lambda(x) := \lambda^d \rho(t, \lambda x)$ with $\lambda > 0$. The diffusion term becomes $\lambda^{dm+2} \Delta(\rho^m)(t, \lambda x)$ while the interaction term gives $\lambda^{2d} \operatorname{div}(\rho \nabla(\mathcal{K} * \rho))(t, \lambda x)$. As a consequence if $dm + 2 > 2d$ then, whatever is the value of the mass M, we can always choose λ large enough, without changing the mass, so that the diffusion part dominates the aggregation part.

And reciprocally, if $d\,m + 2 < 2\,d$ then for any mass M we can always choose λ large enough such that the solution blowup in finite time. Results in this direction were proved rigorously by Sugiyama:

Theorem 1 (First criticality, [32, 33]). *let m_d be such that $d\,m_d + 2 = 2\,d$ i.e.*

$$m_d =: 2\left(1 - \frac{1}{d}\right) \in (1,2)\,.$$

- *if $m > m_d$ then the solutions to* (2) *exist globally in time,*
- *if $m < m_d$ then solutions to* (2) *with sufficiently large initial data blowup in finite time.*

In these notes, we will consider only the case $m = m_d$ (corresponding to 1 in dimension 2) and the indice d will be omitted. We are interested in the proof of the existence of global-in-time solutions using the gradient flow interpretation in the Wassertein metric. We will construct solutions using the minimising (or Jordan-Kinderlehrer-Otto) scheme. We will give formal arguments and try to make, as often as possible, the analogy with the usual gradient flow theory in the Euclidean setting. Sections 2 and 3 are dedicated to the parabolic-elliptic 2-dimensional KS system. Section 2 presents the minimising scheme and describe the discrete Euler-Lagrange equation satisfied by the minimisers. In this first application, passing to the limit in the Euler-Lagrange equation is straightforward. We however obtain very weak solutions. In Section 3, still consecrated to the parabolic-elliptic 2-dimensional KS system, we need to improve on this regularity to use the entropy/entropy production method in order to study the large-time asymptotics. Such a gain of regularity can be proved using the Matthes-McCann-Savaré technique [27] which we will describe in this section. Section 4 is dedicated to the non-linear parabolic-parabolic KS system in $\mathbb{R}^d$, $d \geq 3$. In this case also we need to prove more regularity at the discrete level but cannot rely on a non-increasing displacement convex functional as required by the Matthes-McCann-Savaré method. We thus have to generalise this technique.

2. The sub-critical mass parabolic-elliptic 2-dimensional KS system

2.1. The model. We consider the following classical simplified version of the KS system given by [23]:

$$\begin{cases} \dfrac{\partial \rho}{\partial t} = \Delta\rho - \nabla\cdot(\rho\nabla\phi) & x \in \mathbb{R}^2,\ t > 0\,, \\ -\Delta\phi = \rho & x \in \mathbb{R}^2,\ t > 0\,, \\ \rho(\cdot, t = 0) = \rho_0 \geq 0 & x \in \mathbb{R}^2\,. \end{cases} \tag{3}$$

Such a model can be seen as a limit case when the chemo-attractant diffuses much faster than the cells which emit it.

As the solution to the Poisson equation $-\Delta\phi = \rho$ is given up to a harmonic function, we choose the one given by $\phi = G * \rho$ where G is the Poisson kernel defined by

$$G(|x|) := -\frac{1}{2\pi}\log|x|\,.$$

The KS system (3) can thus be written as a non-local parabolic equation:

$$\frac{\partial \rho}{\partial t} = \Delta \rho - \operatorname{div}(\rho \nabla G * \rho) \quad \text{in } (0,+\infty) \times \mathbb{R}^2 .$$

Such a model has attracted a lot of attention these past years. The behaviour of the solutions is now better understood at least in the sub-critical regime. There actually exists a critical mass 8π such that all the solutions are global-in-time if the mass is below this critical mass, and all the solutions blowup in finite time if they start from an initial data of mass above 8π. The convergence toward a self-similar profile was initiated in [9, 2] and it was proved recently that such a convergence holds with rate for any mass below the critical mass [15]. The blowup profile was recently rigourously described in [31]. Above the critical mass the situation is less clear, for a more detailed display see [21].

2.2. **The free energy.** The main tool to study this system is the following natural free energy:

$$\mathcal{F}_{\mathrm{PKS}}[\rho] := \int_{\mathbb{R}^2} \rho \log \rho \, \mathrm{d}x - \frac{1}{2} \int_{\mathbb{R}^2} \rho \phi \, \mathrm{d}x .$$

A simple formal calculation shows that for all $u \in \mathcal{C}_c^\infty(\mathbb{R}^2)$ with zero mean,

$$\lim_{\epsilon \to 0} \frac{\mathcal{F}_{\mathrm{PKS}}[\rho + \epsilon u] - \mathcal{F}_{\mathrm{PKS}}[\rho]}{\epsilon} = \int_{\mathbb{R}^2} \frac{\delta \mathcal{F}_{\mathrm{PKS}}[\rho]}{\delta \rho}(x) \, u(x) \, \mathrm{d}x$$

where

$$\frac{\delta \mathcal{F}_{\mathrm{PKS}}[\rho]}{\delta \rho}(x) := \log \rho(x) - G * \rho(x) .$$

It is then easy to see that the KS system (3) can be rewritten as

$$\frac{\partial \rho}{\partial t}(t,x) = \operatorname{div}\left(\rho(t,x) \nabla \left[\frac{\delta \mathcal{F}_{\mathrm{PKS}}[\rho(t)]}{\delta \rho}(x) \right] \right) . \tag{4}$$

It follows that at least along well-behaved solutions to the KS system (3),

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{F}_{\mathrm{PKS}}[\rho(t)] = - \int_{\mathbb{R}^2} \rho(t,x) \left| \nabla \left[\frac{\delta \mathcal{F}_{\mathrm{PKS}}[\rho(t)]}{\delta \rho}(x) \right] \right|^2 \mathrm{d}x .$$

Or equivalently

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{F}_{\mathrm{PKS}}[\rho(t)] = - \int_{\mathbb{R}^2} \rho(t,x) \left| \nabla \left(\log \rho(t,x) - c(t,x) \right) \right|^2 \mathrm{d}x .$$

In particular, along such solutions, $t \mapsto \mathcal{F}_{\mathrm{PKS}}[\rho(t)]$ is monotone non-increasing. The main issue here is to study its boundedness.

The connection with the *logarithmic Hardy-Littlewood-Sobolev inequality* (LogHLS thereafter) was first made by [20]: Let f be a non-negative function in $\mathcal{L}^1(\mathbb{R}^2)$ such that $f \log f$ and $f \log(1+|x|^2)$ belong to $\mathcal{L}^1(\mathbb{R}^2)$. If $\int_{\mathbb{R}^2} f \, \mathrm{d}x = M$, then

$$\int_{\mathbb{R}^2} f \log f \, \mathrm{d}x + \frac{2}{M} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} f(x) f(y) \log|x-y| \, \mathrm{d}x \, \mathrm{d}y \geq -C(M) , \tag{5}$$

with $C(M) := M(1 + \log \pi - \log M)$. Moreover the minimisers of the LogHLS inequality (5) are the translations of

$$\bar{\varrho}_\lambda(x) := \frac{M}{\pi} \frac{\lambda}{(\lambda + |x|^2)^2} .$$

Using the monotony of $\mathcal{F}_{\rm PKS}[\rho]$ and the LogHLS inequality (5) it is easy to see that, for smooth solutions to the KS system (3):

$$\begin{aligned} \mathcal{F}_{\rm PKS}[\rho] &= \frac{M}{8\pi}\left(\int_{\mathbb{R}^2} \rho(x) \log \rho(x) \, \mathrm{d}x + \frac{2}{M} \iint_{\mathbb{R}^2\times\mathbb{R}^2} \rho(x) \log|x-y| \rho(y) \, \mathrm{d}x \, \mathrm{d}y\right) \\ &\quad + \left(1 - \frac{M}{8\pi}\right) \int_{\mathbb{R}^2} \rho(x) \log \rho(x) \, \mathrm{d}x \\ &\geq -\frac{M}{8\pi} C(M) + \left(1 - \frac{M}{8\pi}\right) \int_{\mathbb{R}^2} \rho(x) \log \rho(x) \, \mathrm{d}x . \end{aligned} \tag{6}$$

It follows that

$$\int_{\mathbb{R}^2} \rho(t,x) \log \rho(t,x) \, \mathrm{d}x \leq \frac{8\pi \mathcal{F}_{\rm PKS}[\rho_0] - M\, C(M)}{8\pi - M} . \tag{7}$$

Therefore, for $M < 8\pi$, the entropy stays bounded uniformly in time. This formally precludes the collapse of mass into a point mass for such initial data and will be the crucial argument in the proof.

It is worth noticing that for a given ρ, if we set $\rho_\lambda(x) = \lambda^{-2}\rho(\lambda^{-1}x)$ then

$$\mathcal{F}_{\rm PKS}[\rho_\lambda] = \mathcal{F}_{\rm PKS}[\rho] - 2M \left(1 - \frac{M}{8\pi}\right) \log \lambda . \tag{8}$$

So that as a function of λ, $\mathcal{F}_{\rm PKS}[\rho_\lambda]$ is bounded from below if $M < 8\pi$, and not bounded from below if $M > 8\pi$ in the set

$$\mathcal{K} := \{\rho : \int_{\mathbb{R}^2} \rho = M,\ \int_{\mathbb{R}^2} \rho(x) \log \rho(x) \, \mathrm{d}x < \infty \text{ and } \int_{\mathbb{R}^2} |x|^2 \rho(x) \, \mathrm{d}x < \infty\} . \tag{9}$$

2.3. **A gradient flow approach.** The above arguments can be made rigorous by a regularisation/passing to the limit procedure. We are interested in the the gradient flow interpretation of the KS system in the Wasserstein metric, formally described as:

$$\frac{\partial \rho}{\partial t} = -\text{"}\nabla_W\text{"} \mathcal{F}_{\rm PKS}[\rho(t)] . \tag{10}$$

A rigorous meaning to "∇_W" can be done using the approach developped by [28]. There is actually a riemannian structure on the probability space equipped with the Monge-Kantorovich (or 2-Wasserstein) distance. We do not aim to explain this structure in full details as we do not really need it but the interested reader could consult [34, 1].

We will indeed construct a solution using the minimising scheme, often known as the minimising or Jordan-Kinderlehrer-Otto (JKO) scheme: given a time step τ, we define the solution by

$$\rho_\tau^{k+1} \in \operatorname{argmin}_{\rho\in\mathcal{K}} \left[\frac{\mathcal{W}_2^2(\rho,\rho_\tau^k)}{2\tau} + \mathcal{F}_{\rm PKS}[\rho]\right] , \tag{11}$$

where $\mathcal{K}$ is defined in (9).

Let us develop here the analogy with the gradient flow structure in the Euclidean setting. In this situation the Euler-Lagrange equation associated to

$$X_\tau^{k+1} \in \operatorname{argmin} \left[\frac{|X - X_\tau^k|^2}{2\tau} + \mathcal{F}[X]\right] , \tag{12}$$

would be

$$\frac{X_\tau^{k+1} - X_\tau^k}{\tau} + \nabla\mathcal{F}[X_\tau^{k+1}] = 0 ,$$

which is nothing but the implicit Euler scheme associated to

$$\dot{X} = -\nabla\mathcal{F}[X(t)] .$$

We aim to contruct here a sequence $\{\rho_\tau^k\}_k$ using the scheme (11) and will obtain at the limit an gradient flow wich will can formally write as (10).

In the Euclidean setting, the next classical step is to built an interpolation between the constructed points. Here we interpolate between the terms of the sequence $\{\rho_\tau^k\}_{k\in\mathbb{N}}$ to produce a function from $[0,\infty)$ to $L^1(\mathbb{R}^2)$: For each positive integer k, let $\nabla\varphi^k$ be the optimal transportation plan with $\nabla\varphi^k\#\rho_\tau^{k+1} = \rho_\tau^k$, see the Appendix. Then for $k\tau \le t \le (k+1)\tau$ we define

$$\rho_\tau(t) = \left(\frac{t-k\tau}{\tau}\mathrm{id} + \frac{(k+1)\tau - t}{\tau}\nabla\varphi^k\right)\#\rho_\tau^{k+1} .$$

Note that $\rho_\tau(k\tau) = \rho_\tau^k$, $\rho_\tau((k+1)\tau) = \rho_\tau^{k+1}$ and $\mathcal{W}_2(\rho_\tau^k,\rho_\tau(t)) = (t-k\tau)\mathcal{W}_2(\rho_\tau^k,\rho_\tau^{k+1})$.

Theorem 2 (Convergence of the scheme as $\tau \to 0$, [5]). *If $M < 8\pi$ then the family $(\rho_\tau)_{\tau>0}$ admits a sub-sequence converging weakly in $\mathrm{L}^1(\mathbb{R}^2)$ to a weak solution to the KS system (3): for all $(t_1,t_2) \in [0,+\infty)$, for all smooth ζ*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2}\zeta(x)\rho(t,x)\,\mathrm{d}x = &\int_{\mathbb{R}^2}\Delta\zeta(x)\,\rho(s,x)\,\mathrm{d}x\,\mathrm{d}s\\ &-\frac{1}{4\pi}\iint_{\mathbb{R}^2\times\mathbb{R}^2}\rho(s,x)\,\rho(s,y)\,\frac{(x-y)\cdot(\nabla\zeta(x)-\nabla\zeta(y))}{|x-y|^2}\,\mathrm{d}y\,\mathrm{d}x .\end{aligned}$$

2.4. **Ideas of the proof.** The proof follows the main lines of the proof of the convergence of the scheme for euclidean gradient flow. It was done in full details in [5] and we present here a formal proof with the main ideas.

(i) Existence of minimisers: Let us emphasise that the functional $\mathcal{F}_{\rm PKS}$ is not convex, so even the existence of a minimiser is not clear. When the functional is convex, or even displacement convex, general results from [34, 1] can be applied. However, we can construct a sequence of minimisers when $M < 8\pi$ by using Estimate (7).

(ii) The discrete Euler-Lagrange equation: The perturbation of the minimiser has to be done in the optimal transport way: Let ζ be a smooth vector field with compact

support, we introduce $\psi_\varepsilon := |x|^2/2 + \varepsilon\zeta$. We define $\overline{\rho_\varepsilon}$ the push-forward perturbation of ρ_τ^{n+1} by $\nabla\psi_\varepsilon$:

$$\overline{\rho_\varepsilon} = \nabla\psi_\varepsilon \# \rho_\tau^{n+1} .$$

Standard computations, see Appendix A.3 and A.4, give

$$\begin{aligned}\int_{\mathbb{R}^2} \nabla\zeta(x)\frac{x-\nabla\varphi^n(x)}{\tau}\rho_\tau^{n+1}(x)\,\mathrm{d}x \\ = \int_{\mathbb{R}^2}\left[\Delta\zeta(x) - \frac{1}{4\pi}\int_{\mathbb{R}^2}\frac{[\nabla\zeta(x)-\nabla\zeta(y)]\cdot(x-y)}{|x-y|^2}\rho_\tau^{n+1}(y)\,\mathrm{d}y\right]\rho_\tau^{n+1}(x)\,\mathrm{d}x ,\end{aligned}$$

which is the weak form of the Euler-Lagrange equation:

$$\boxed{\frac{\mathrm{id}-\nabla\varphi^n}{\tau}\rho_\tau^{n+1} = -\nabla\rho_\tau^{n+1} + \rho_\tau^{n+1}\nabla c_\tau^{n+1}} . \tag{13}$$

Using the Taylor's expansion

$$\zeta(x) - \zeta[\nabla\varphi^n(x)] = [x-\nabla\varphi^n(x)]\cdot\nabla\zeta(x) + O\left[|x-\nabla\varphi^n(x)|^2\right]$$

we obtain, for all $t_2 > t_1 \geq 0$,

$$\begin{aligned}\int_{\mathbb{R}^2}\zeta(x)\,[\rho_\tau(t_2,x)-\rho_\tau(t_1,x)]\,\mathrm{d}x = \int_{t_1}^{t_2}\int_{\mathbb{R}^2}\Delta\zeta(x)\,\rho_\tau(s,x)\,\mathrm{d}x\,\mathrm{d}s + O(\tau^{1/2}) \\ -\frac{1}{4\pi}\int_{t_1}^{t_2}\iint_{\mathbb{R}^2\times\mathbb{R}^2}\rho_\tau(s,x)\,\rho_\tau(s,y)\,\frac{(x-y)\cdot(\nabla\zeta(x)-\nabla\zeta(y))}{|x-y|^2}\,\mathrm{d}y\,\mathrm{d}x .\end{aligned} \tag{14}$$

(iii) A priori *estimates:* To pass to the limit, the scheme provides some *a priori* bounds: Taking ρ_τ^n as a test function in (11) we have:

$$\mathcal{F}_{\mathrm{PKS}}[\rho_\tau^{n+1}] + \frac{1}{2\tau}\mathcal{W}_2^2(\rho_\tau^n,\rho_\tau^{n+1}) \leq \mathcal{F}_{\mathrm{PKS}}[\rho_\tau^n] . \tag{15}$$

As a consequence we obtain an *energy estimate*

$$\sup_{n\in\mathbb{N}}\mathcal{F}_{\mathrm{PKS}}[\rho_\tau^n] \leq \mathcal{F}_{\mathrm{PKS}}[\rho_\tau^0] \tag{16}$$

and a *total square estimate*

$$\frac{1}{2\tau}\sum_{n\in\mathbb{N}}\mathcal{W}_2^2(\rho_\tau^n,\rho_\tau^{n+1}) \leq \mathcal{F}_{\mathrm{PKS}}[\rho_\tau^0] - \inf_{n\in\mathbb{N}}\mathcal{F}_{\mathrm{PKS}}[\rho_\tau^n] . \tag{17}$$

(iv) Passing to the limit: The energy estimate (16) together with (6) gives a bound on $\int\rho\log\rho$ at least as long as $M < 8\pi$. The bound on $\rho_\tau\log\rho_\tau$ prevents the solution from blowing up: indeed, using

$$\int_{\rho>K}\rho \leq \frac{1}{\log K}\int_{\rho>K}\rho|\log\rho| \leq \frac{C}{|\log(K)|} ,$$

we obtain that $(\rho_\tau)_\tau$ converges to a certain ρ in w-$\mathrm{L}^1(\mathbb{R}^2)$. It time, we can rely on the 1/2-Hölder continuity (17) and Ascoli's theorem to obtain a convergence in $\mathcal{C}^0([0,T];\mathcal{P}(\mathbb{R}^2))$.

We can thus pass to the limit in $\tau \to 0$ in (14) and prove that ρ is a weak solution. Note that the last term of (14) converges because the convergence of $(\rho_\tau)_\tau$ in w-$L^1(\mathbb{R}^2)$ ensures the convergence of $(\rho_\tau \otimes \rho_\tau)_\tau$ in w-$L^1(\mathbb{R}^2)$. The notion of constructed solutions is however weak.

3. The critical mass parabolic-elliptic 2-dimensional KS system

3.1. Preliminary remarks. We still consider the parabolic-elliptic 2-dimensional KS system (3). We focus is this section to the the case $M = 8\pi$. In this case, the remainder entropy which was controlled in (6) is thus entirely "eaten" by the logarithmic Hardy-Littlewood-Sobolev inequality (5). We however prove

Theorem 3 (Infinite Time Aggregation, [8]). *If the 2-moment is bounded, there is a global in time non-negative free-energy solution of the KS system* (3) *with initial data* ρ_0.

Moreover if $\{t_p\}_{p\in\mathbb{N}} \to \infty$ *as* $p \to \infty$, *then* $t_p \mapsto \rho(t_p, x)$ *converges to a Dirac peak of mass* 8π *concentrated at the centre of mass of the initial data weakly-* in the sense of measure as* $p \to \infty$.

We will not describe the proof of this result here but we are interested in the analysis of the existence of solutions in the critical case $M = 8\pi$ when the 2-moment is not assumed to be bounded. In this situation, nothing prevents the solutions from converging to the other minimisers of the LogHLS inequality (5) which are of the form:

$$\bar{\varrho}_\lambda(x) := \frac{1}{\pi} \frac{8\lambda}{(\lambda + |x|^2)^2} .$$

We can indeed prove the following theorem:

Theorem 4 (Existence of global solutions, [6]). *Let* ρ_0 *be any density in* $\mathbb{R}^2$ *with mass* 8π, *such that* $\mathcal{F}_{\mathrm{PKS}}[\rho_0] < \infty$. *If there is a minimiser* $\bar{\varrho}_\lambda$ *of the LogHLS inequality* (5) *such that* $\mathcal{W}_2(\rho_0, \bar{\varrho}_\lambda) < \infty$, *then there exists a global free energy solution of the Keller-Segel equation* (3) *with initial data* ρ_0. *Moreover,*

$$\lim_{t\to\infty} \mathcal{F}_{\mathrm{PKS}}[\rho(t)] = \mathcal{F}_{\mathrm{PKS}}[\bar{\varrho}_\lambda] \qquad \textit{and} \qquad \lim_{t\to\infty} \|\rho(t) - \bar{\varrho}_\lambda\|_1 = 0 .$$

Remember that the minimisers $\bar{\varrho}_\lambda$ of the logarithmic Hardy-Littlewood-Sobolev inequality (5) are of infinite 2-moment so that the condition $\mathcal{W}_2(\rho_0, \bar{\varrho}_\lambda) < \infty$ implies that ρ_0 is of infinite 2-moment. If we keep in mind that the 2-moment can be seen as the Monge-Kantorovich distance between the solution and the Dirac mass, we see that Theorem 4 completes the picture which emerged from Theorem 3.

As soon as we start at a finite distance from one of the minimisers $\bar{\varrho}_\lambda$ we can construct a solution which converges towards it. Note that this result is true for the solutions that we construct as we do not have uniqueness of the solution to the KS system, even if we strongly believe that this is the case. Also observe that the equilibrium solutions $\bar{\varrho}_\lambda$ are infinitely far apart: Indeed, let $\varphi(x) = \sqrt{\lambda/\mu}|x|^2/2$, one has $\nabla\varphi \# \varrho_\mu = \bar{\varrho}_\lambda$. Since the equilibrium densities $\bar{\varrho}_\lambda$ all have infinite second moments,

$$\mathcal{W}_2^2(\varrho_\mu, \bar{\varrho}_\lambda) = \frac{1}{2} \int_{\mathbb{R}^2} \left| \sqrt{\frac{\lambda}{\mu}} x - x \right|^2 \varrho_\mu(x) \, \mathrm{d}x = +\infty .$$

We will now give the ain ingredients of this proof.

3.2. **Another Lyapunov functional.** Consider first the fast diffusion Fokker-Planck equation:

$$(18)\qquad \begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \Delta\sqrt{u(t,x)} + 2\sqrt{\dfrac{\pi}{\lambda M}}\,\mathrm{div}(x\,u(t,x)) & t>0\,,\ x\in\mathbb{R}^2\,,\\ u(0,x) = u_0(x)\geq 0 & x\in\mathbb{R}^2\,. \end{cases}$$

This equation can also be written in a form analogous to (4): for $\lambda > 0$, define the relative entropy of the fast diffusion equation with respect to the stationary solution $\bar\varrho_\lambda$ by

$$\mathcal{H}_\lambda[u] := \int_{\mathbb{R}^2} \frac{\left|\sqrt{u(x)} - \sqrt{\bar\varrho_\lambda(x)}\right|^2}{\sqrt{\bar\varrho_\lambda(x)}}\,\mathrm{d}x\,.$$

Equation (18) can be rewritten as

$$\frac{\partial u}{\partial t}(t,x) = \mathrm{div}\left(u(t,x)\nabla\frac{\delta\mathcal{H}_\lambda[u(t)]}{\delta u}(x)\right)\,,$$

with

$$\frac{\delta\mathcal{H}_\lambda[u]}{\delta u} = \frac{1}{\sqrt{\bar\varrho_\lambda}} - \frac{1}{\sqrt{u}}\,.$$

The connection with the KS system (3) can be seen through the minimisers of $\mathcal{H}_\lambda$ which are the same as those of the LogHLS inequality (5). The functional $\mathcal{H}_\lambda$ is actually a weighted distance between the solution and its unique minimiser $\bar\varrho_\lambda$. It is thus tempting to compute the dissipation of $\mathcal{H}_\lambda$ along the flow of solutions to the KS system (3): Let ρ be a sufficiently smooth solution of the KS system (3). Then we compute

$$(19)\qquad \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_\lambda[\rho(t)] = -\frac{1}{2}\int_{\mathbb{R}^2}\frac{|\nabla\rho(t)|^2}{\rho(t)^{3/2}}\,\mathrm{d}x + \int_{\mathbb{R}^2}\rho(t)^{3/2}\,\mathrm{d}x + 4\sqrt{\frac{M\,\pi}{\lambda}}\left(1-\frac{M}{8\pi}\right)\,.$$

In the critical case $M = 8\pi$ the dissipation of the $\mathcal{H}_\lambda$ free energy along the flow of the KS system (3) is

$$\mathcal{D}[\rho] := \frac{1}{2}\int_{\mathbb{R}^2}\frac{|\nabla\rho|^2}{\rho^{3/2}}\,\mathrm{d}x - \int_{\mathbb{R}^2}\rho^{3/2}\,\mathrm{d}x\,.$$

We use the following Gagliardo-Nirenberg-Sobolev inequality in the form of [19]: For all functions f in $\mathbb{R}^2$ with a square integrable distributional gradient ∇f,

$$\pi\int_{\mathbb{R}^2}|f|^6\,\mathrm{d}x \leq \int_{\mathbb{R}^2}|\nabla f|^2\,\mathrm{d}x\int_{\mathbb{R}^2}|f|^4\,\mathrm{d}x\,,$$

and there is equality if and only if f is a multiple of a translate of $\bar\varrho_\lambda^{1/4}$ for some $\lambda > 0$.

As a consequence, taking $f = \rho^{1/4}$ so that $\int_{\mathbb{R}^2} f^4(x)\,\mathrm{d}x = 8\pi$, we obtain $\mathcal{D}[\rho] \geq 0$, and moreover, $\mathcal{D}[\rho] = 0$ if and only ρ is a translate of $\bar\varrho_\lambda$ for some $\lambda > 0$.

Remark 5. *This free energy $\mathcal{H}_\lambda[\rho]$ gives another proof of non existence of global-in-time solutions in the super-critical case $M > 8\pi$. Indeed, by (19) and as $\mathcal{D}[\rho]$ is non-negative,*

$$0 \le \mathcal{H}_\lambda[\rho(t)] \le 4\sqrt{\frac{M\pi}{\lambda}}\left(1 - \frac{M}{8\pi}\right) t\,.$$

So that in the case $M > 8\pi$, there cannot be global-in-time solutions even with infinite 2-moment as long as there is λ such that $\mathcal{H}_\lambda[\rho_0]$ is bounded.

We expect the propagation of the bounds on $\mathcal{F}_{\rm PKS}[\rho]$ and $\mathcal{D}[\rho]$ to give compactness. Unfortunately, $\mathcal{D}[\rho]$ is a difference of two functionals of ρ that can each be arbitrarily large even when $\mathcal{D}[\rho]$ is very close to zero. Indeed, for $M = 8\pi$ and each $\lambda > 0$, $\mathcal{D}[\bar\varrho_\lambda] = 0$ while

$$\lim_{\lambda\to 0} \|\bar\varrho_\lambda\|_{3/2} = \infty\,, \quad \lim_{\lambda\to 0} \|\nabla \bar\varrho_\lambda^{1/4}\|_2 = \infty \quad \text{and} \quad \lim_{\lambda\to 0} \bar\varrho_\lambda = 8\pi\delta_0\,.$$

Likewise, an upper bound on $\mathcal{F}_{\rm PKS}[\rho]$ provides no upper bound on the entropy $\int_{\mathbb{R}^2} \rho\log\rho$. Indeed, $\mathcal{F}_{\rm PKS}[\rho]$ takes its minimum value for $\rho = \bar\varrho_\lambda$ for each $\lambda > 0$, while

$$\lim_{\lambda\to 0} \int \bar\varrho_\lambda \log \bar\varrho_\lambda = \infty\,.$$

Fortunately, an upper bound on both $\mathcal{H}_\lambda[\rho]$ and $\mathcal{F}_{\rm PKS}[\rho]$ *does* provide an upper bound on $\int \rho\log\rho$:

Theorem 6 (Concentration control for $\mathcal{F}_{\rm PKS}$, [6])**.** *Let ρ be any density with mass $M = 8\pi$ such that $\mathcal{H}_\lambda[\rho] < \infty$ for some $\lambda > 0$. Then there exist $\gamma_1 > 0$ and an explicit $C > 0$ depending only on λ and $\mathcal{H}_\lambda[\rho]$ such that*

$$\gamma_1 \int_{\mathbb{R}^2} \rho\log\rho \,\mathrm{d}x \le \mathcal{F}_{\rm PKS}[\rho] + C\,.$$

Here we also prove that since $\mathcal{H}_\lambda$ controls concentration, a uniform bound on both $\mathcal{H}_\lambda$ and $\mathcal{D}$ does indeed provide compactness:

Theorem 7 (Concentration control for $\mathcal{D}$, [6])**.** *Let ρ be any density in $\mathcal{L}^{3/2}(\mathbb{R}^2)$ with mass 8π such that $\mathcal{F}_{\rm PKS}[\rho]$ is finite, and $\mathcal{H}_\lambda[\rho]$ is finite for some $\lambda > 0$. Then there exist constants $\gamma_1 > 0$ and an explicit $C > 0$ depending only on λ, $\mathcal{H}_\lambda[\rho]$ and $\mathcal{F}_{\rm PKS}[\rho]$ such that*

$$\gamma_2 \int_{\mathbb{R}^2} |\nabla\left(\rho^{1/4}\right)|^2 \,\mathrm{d}x \le \pi\mathcal{D}[\rho] + C\,.$$

Idea of the proof of Theorems 6 and 7: The trivial inequality

$$\int_{\mathbb{R}^2} \sqrt{\lambda + |x|^2}\,\rho(x)\,\mathrm{d}x \le 2\sqrt{\lambda}\,M + 2M^{3/4}(\lambda/\pi)^{1/4}\sqrt{\mathcal{H}_\lambda[\rho]}\,. \tag{20}$$

gives a vertical cut to prove Theorem 6. Indeed, we split the function ρ in two parts: given $\beta > 0$, define $\rho_\beta(x) = \min\{\rho(x)\,,\,\beta\}$. By (20), for β large enough, $\rho - \rho_\beta$ is such that:

$$\int_{\mathbb{R}^2} (\rho - \rho_\beta) \le \frac{C_1}{\beta} + C_2\sqrt{\mathcal{H}_\lambda[\rho]} \le \frac{C_1}{\beta} + \frac{8\pi - \varepsilon_0}{2} < 8\pi - \varepsilon_0\,.$$

We then apply the logarithmic Hardy-Littlewood-Sobolev inequality method as in (7) to the function $\rho - \rho_\beta$ whose mass is less than 8π.

The same idea works for the Gagliardo-Nirenberg-Sobolev inequality to prove Theorem 7: Let $f := \rho^{1/4}$, we split f in two parts by defining $f_\beta := \min\{f, \beta^{1/4}\}$ and $h_\beta := f - f_\beta$. We use (20) and apply the Gagliardo-Nirenberg-Sobolev inequality to control h_β.

3.3. **Ideas of the proof of Theorem** (4). The proof of Theorem 4 follows the line of the convergence of the JKO minimising scheme (11) exposed in the previous section to obtain the Euler-Lagrange equation (13). As in the previous section, we can rely on the same compactness to prove the existence of weak solutions. But as we want to study the large-time behaviour of the solution we need more regularity. We actually need to prove the existence of "free energy" solution satisfying the entropy/entropy production inequality:

$$\mathcal{F}_{\rm PKS}[\rho] + \int_0^T \int_{\mathbb{R}^2} \rho(t,x) \left|\nabla \left(\log \rho(t,x) - c(t,x)\right)\right|^2 \, \mathrm{d}x \le \mathcal{F}_{\rm PKS}[\rho_0] \, .$$

For this purpose more regularity has to be obtained on the solutions at the discrete level.

Even if it was not clear at the time we wrote [6], we use a powerful method systematically described by Matthes-McCann-Savaré in [27]: Following their words, let us first consider the two ordinary differential equations describing gradient flow:

$$\dot{x}(t) = -\nabla\Phi[x(t)] \quad \text{and} \quad \dot{y}(t) = -\nabla\Psi[y(t)]$$

Then of course $\Phi[x(t)]$ and $\Psi[y(t)]$ are monotone decreasing. Differentiate each function along the other's flow gives:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\Phi[y(t)] &= -\langle \nabla\Phi[y(t)], \nabla\Psi[y(t)]\rangle \\ \frac{\mathrm{d}}{\mathrm{d}t}\Psi[x(t)] &= -\langle \nabla\Psi[x(t)], \nabla\Phi[x(t)]\rangle \end{aligned} \tag{21}$$

Thus, Φ is decreasing along the gradient flow of Ψ for any initial data if and only if Ψ is decreasing along the gradient flow of Φ for any initial data.

Let us now describe the consequences of this remark in the context of gradient flows in the Monge-Kantorovich metric. Consider the following variational problem:

$$\text{Find } u_{h,n} \text{ which minimises} \quad u \mapsto \frac{1}{2h}\mathcal{W}_2^2(u, u_{h,n-1}) + \mathcal{F}[u] \, . \tag{22}$$

Imagine now that we can find a displacement convex functional $\mathcal{H}$ such that the dissipation of $\mathcal{F}$ along the flow $S^{\mathcal{H}}$:

$$D^{\mathcal{H}}\mathcal{F}[\mu] := \limsup_{t\to 0} \frac{\mathcal{F}[\mu] - \mathcal{F}[S_t^{\mathcal{H}}\mu]}{t} \, .$$

is non-negative.

Definition (22) of the minimising scheme, means that for any u

$$\frac{1}{2\tau}\mathcal{W}_2^2(u_{\tau,n}, u_{\tau,n-1}) + \mathcal{F}[u_{\tau,n}] \le \frac{1}{2\tau}\mathcal{W}_2^2(u, u_{\tau,n-1}) + \mathcal{F}[u] \, .$$

Choosing $u = S_t^{\mathcal{H}}(u_{\tau,n})$, we obtain

$$\mathcal{F}[u_{\tau,n}] - \mathcal{F}[S_t^{\mathcal{H}} u_{\tau,n}] \leq \frac{1}{2\tau} \left[\mathcal{W}_2^2(S_t^{\mathcal{H}} u_{\tau,n}, u_{\tau,n-1}) - \mathcal{W}_2^2(u_{\tau,n}, u_{\tau,n-1}) \right] .$$

Dividing by t and letting $t \to 0$, we obtain

$$D^{\mathcal{H}} \mathcal{F}[u_{\tau,n}] \leq \frac{1}{2} \frac{\mathrm{d}^+}{\mathrm{d}t} \mathcal{W}_2^2(S_t^{\mathcal{H}} u, v) .$$

But as $\mathcal{H}$ is displacement convex and $S^{\mathcal{H}}$ is the associated semi-group we have

$$\frac{1}{2} \frac{\mathrm{d}^+}{\mathrm{d}t} \mathcal{W}_2^2(S_t^{\mathcal{H}} u, v) \leq \mathcal{H}[v] - \mathcal{H}[S_t^{\mathcal{H}} u] . \tag{23}$$

See the Appendix for more details. Taking $u = u_{\tau,n}$ and $v = u_{\tau,n-1}$ yields:

$$\boxed{D^{\mathcal{H}} \mathcal{F}[u_{\tau,n}] \leq \frac{\mathcal{H}[u_{\tau,n-1}] - \mathcal{H}[u_{\tau,n}]}{\tau}} . \tag{24}$$

So that the differential estimate of $\mathcal{F}$ is converted into a discrete estimate for the approximation scheme.

Here, as already discussed the functional $\mathcal{F}_{\mathrm{PKS}}$ is not displacement convex but the flow constructed from this functional is also non-increasing along the flow of $\mathcal{H}_\lambda$. Remark that the displacement convexity of $\mathcal{H}_\lambda$ is formally obvious from the fact that

$$\mathcal{H}_\lambda[u] = \int_{\mathbb{R}^2} \left(-2\sqrt{u(x)} + \sqrt{\frac{1}{2\lambda} \frac{|x|^2}{2} u(x)} \right) \mathrm{d}x + C .$$

where $-\sqrt{u(x)}$ and $|x|^2 u(x)$ are displacement convex. So that at each step, we can use the convexity estimate (24), which gives

$$\tau\, \mathcal{D}[\rho_\tau^n] \leq \mathcal{H}_\lambda[\rho_\tau^{n-1}] - \mathcal{H}_\lambda[\rho_\tau^n] . \tag{25}$$

This inequality together with Theorem 7 gives a bound on $\|\nabla(\rho_\tau^n)\|_2$. This is the crucial estimate which allows to apply the standard entropy/entropy dissipation method to study the asymptotics. There are main technical difficulties and the methods to turn around them are interesting by themselves but we do not present them in details here. For more details see [6].

4. The non-linear parabolic-parabolic KS system in $\mathbb{R}^d$, $d \geq 3$

4.1. **Main results.** We consider now the following *parabolic-parabolic generalisation of the Keller-Segel system:*

$$\begin{cases} \frac{\partial \rho}{\partial t} = \operatorname{div}\left[\nabla \rho^m - \rho \nabla \phi\right] , \\ \tau \frac{\partial \phi}{\partial t} = \Delta \phi - \alpha\, \phi + \rho , \end{cases} \qquad (t,x) \in (0,\infty) \times \mathbb{R}^d , \tag{26}$$

where $m \in [1,2)$ and $d \geq 2$. This system is known in theoretical physics as the generalised Smulochowski-Poisson system, see [17, 16].

For the case $d = 2$, global-in-time existence for a mass less that 8π was proved in [13]. But there are also global-in-time self-similar solutions for larger masses, see [4]. The question of the eventuality of blowing up solutions to this system remains opened.

For the parabolic-elliptic case, $\tau = 0$, the inequality which plays the role of the LogHLS inequality is a *variant to the Hardy-Littlewood-Sobolev (HLS) inequality*: for all $h \in \mathrm{L}^1(\mathbb{R}^d) \cap \mathrm{L}^m(\mathbb{R}^d)$, there exists an optimal constant C_* such that

$$\left|\frac{\Gamma(d/2)}{(d-2)2\,\pi^{d/2}}\iint_{\mathbb{R}^d\times\mathbb{R}^d}\frac{h(x)\,h(y)}{|x-y|^{d-2}}\,\mathrm{d}x\,\mathrm{d}y\right| \le C_*\,\|h\|_m^m\,\|h\|_1^{2/d}\,. \tag{27}$$

The critical mass can be expressed in terms of this inequality. Let us define

$$M_c := \left[\frac{2}{(m-1)\,C_*}\right]^{d/2}.$$

The available results of [7] can be summarised as follows:

- Sub-critical case: $0 < M < M_c$, solutions exist globally in time and there exists a radially symmetric compactly supported self-similar solution, although we are not able to show that it attracts all global solutions.
- Critical case: $M = M_c$, solutions exist globally in time. There are infinitely many compactly supported stationary solutions. We thus show a striking difference with respect to the classical KS system in two dimensions, namely, the existence of global in time solutions not blowing-up in infinite time. Recently [36] proved that radially symmetric solutions do not blowup in infinite time but this question remains opened for general solutions.
- Super-critical case: $M > M_c$, we prove that there exist solutions, corresponding to initial data with negative free energy, blowing up in finite time. However, we cannot exclude the possibility that solutions with positive free energy may be global in time. There are also solutions starting from positive free energy which blowup in finite time for any mass, see [3] but it is not clear if their free energy is still positive at the blowup time.

We will not describe the proof of these results but will focus on the extension of the global-in-time existence results to higher dimensions:

Theorem 8 (Global existence, [10]). *Let $\tau > 0$, $\alpha \ge 0$, ρ_0 be a non-negative function in $\mathrm{L}^1(\mathbb{R}^d,(1+|x|^2)\,\mathrm{d}x)\cap\mathrm{L}^m(\mathbb{R}^d)$ satisfying $\|u_0\|_1 = M$ and $\phi_0 \in \mathrm{H}^1(\mathbb{R}^d)$. If $M < M_c$ then there exists a weak solution (ρ,ϕ) to the parabolic-parabolic KS system* (26)*: almost-everywhere in $(0,t)\times\mathbb{R}^d$ and for all smooth function ξ*

$$\begin{cases} \displaystyle\int_{\mathbb{R}^d}\xi\,(\rho(t)-\rho_0)\,\mathrm{d}x + \int_0^t\int_{\mathbb{R}^d}(\nabla(\rho^m)-\rho\,\nabla\phi)\cdot\nabla\xi\,\mathrm{d}x\,\mathrm{d}s &= 0\,,\\ \tau\,\partial_t\phi - \Delta\phi + \alpha\,\phi &= \rho\,. \end{cases}$$

4.2. **Preliminary remarks.** The main difficulty stems from the fact that the system cannot easily be reduced to a single non-local parabolic equation. Actually the corresponding free energy has the two quantities ρ and ϕ:

$$\mathcal{E}_\alpha[\rho,\phi] := \int_{\mathbb{R}^d}\left\{\frac{|\rho(x)|^m}{(m-1)} - \rho(x)\,\phi(x) + \frac{1}{2}\,|\nabla\phi(x)|^2 + \frac{\alpha}{2}\,\phi(x)^2\right\}\mathrm{d}x\,. \tag{28}$$

The minimising scheme has thus to be replaced by a gradient flow of this energy in $\mathcal{K} := \mathcal{P}_2(\mathbb{R}^d)\times\mathrm{L}^2(\mathbb{R}^d)$ the probability measure with finite 2-moments endowed with the Monge-Kantorovich metric for the first component and the usual L^2-norm for the second component. Such a strategy has already been developed to prove existence of the thin film approximation of the Muskat problem in [26].

The minimising scheme is as follows: given an initial condition $(\rho_0, \phi_0) \in \mathcal{K}$ and a time step $h > 0$, we define a sequence $(\rho_{h,n}, \phi_{h,n})_{n\geq 0}$ in $\mathcal{K}$ by

$$(29) \qquad \begin{cases} (\rho_{h,0}, \phi_{h,0}) = (\rho_0, \phi_0)\,, \\ (\rho_{h,n+1}, \phi_{h,n+1}) \in \mathrm{Argmin}_{(\rho,\phi)\in\mathcal{K}} \mathcal{F}_{h,n}[\rho, \phi]\,, \qquad n \geq 0\,, \end{cases}$$

where

$$\mathcal{F}_{h,n}[\rho, \phi] := \frac{1}{2h}\ \left[\mathcal{W}_2^2(\rho, \rho_{h,n}) + \tau\ \|\phi - \phi_{h,n}\|_2^2\right] + \mathcal{E}_\alpha[\rho, \phi]\,.$$

The kernel which appears in the parabolic-parabolic KS system is the Bessel kernel, $\mathcal{Y}_\alpha$, defined for $\alpha \geq 0$ by:

$$\mathcal{Y}_\alpha(x) := \int_0^\infty \frac{1}{(4\pi s)^{d/2}} \exp\left(-\frac{|x|^2}{4s} - \alpha s\right)\,\mathrm{d}s\,, \quad x \in \mathbb{R}^d\,,$$

the case $\alpha = 0$ corresponding to the already defined Poisson kernel. For $u \in \mathrm{L}^1(\mathbb{R}^d)$, $S_\alpha(u) := \mathcal{Y}_\alpha * u$ solves

$$(30) \qquad -\Delta S_\alpha(u) + \alpha S_\alpha(u) = u \quad \text{in } \mathbb{R}^d$$

in the sense of distributions. The Bessel kernel is also referred to as the screened Poisson or Yukawa potential in the literature. The crucial inequality is thus a modified Hardy-Littlewood-Sobolev inequality valid for the Bessel kernel $\mathcal{Y}_\alpha$ for $\alpha > 0$: For $\alpha > 0$,

$$(31) \qquad \sup\left\{\frac{\displaystyle\int_{\mathbb{R}^d} h(x)\,(\mathcal{Y}_\alpha * h)(x)\,\mathrm{d}x}{\|h\|_m^m\,\|h\|_1^{2/d}}\ :\ h \in (\mathrm{L}^1 \cap \mathrm{L}^m)(\mathbb{R}^d), h \neq 0\right\} = C_{\mathrm{HLS}}\,,$$

where C_{HLS} is defined in (27). Note that the constant is the exact same as for the case $\alpha = 0$ so that the critical mass below which all the solutions exist globally-in-time is the same as for the parabolic-elliptic version.

Several difficulties arise in the proof of the well-posedness and convergence of the previous minimising scheme. First, as the energy $\mathcal{E}_\alpha$ is not displacement convex, standard results from [34, 1] do not apply and even the existence of a minimiser is not clear. Nevertheless, the modified Hardy-Littlewood-Sobolev inequality (27) trivially implies:

$$(32) \qquad \mathcal{E}_\alpha[\rho, \phi] \geq \frac{C_{\mathrm{HLS}}}{2}\ \left(M_c^{2/d} - M^{2/d}\right)\ \|\rho\|_m^m\,.$$

which permits in particular to pass to the limit in the term in $\mathcal{E}_\alpha[\rho, \phi]$ involving the product $\rho\phi$, and proves the existence of a minimiser.

To obtain the Euler-Lagrange equation satisfied by a minimiser $(\bar\rho, \bar\phi)$ of $\mathcal{F}_{h,n}$ in $\mathcal{K}$, the parameters h and n being fixed, we consider, as before, an "optimal transport" perturbation for $\bar\rho$ and a L^2-perturbation for $\bar\phi$ defined for $\delta \in (0,1)$ by

$$\rho_\delta = (\mathrm{id} + \delta\,\zeta)\#\bar\rho\,, \quad \phi_\delta := \bar\phi + \delta\,w\,,$$

where $\zeta \in C_0^\infty(\mathbb{R}^d;\mathbb{R}^d)$ and $w \in C_0^\infty(\mathbb{R}^d)$. Identifying the Euler-Lagrange equation requires to pass to the limit as $\delta \to 0$ in

$$\frac{\mathcal{W}_2^2(\rho_\delta,\rho_{h,n}) - \mathcal{W}_2^2(\bar\rho,\rho_{h,n})}{2\delta} \quad \text{and} \quad \frac{\|\rho_\delta\|_m^m - \|\bar\rho\|_m^m}{\delta},$$

which can be performed by standard arguments, see the Appendix, but also in

$$\frac{1}{\delta}\int_{\mathbb{R}^d} (\bar\rho\bar\phi - \rho_\delta\phi_\delta)(x)\,\mathrm{d}x = \int_{\mathbb{R}^d} \bar\rho(x)\left[\frac{\bar\phi(x) - \bar\phi(x+\delta\zeta(x))}{\delta} - w(x+\delta\zeta(x))\right]\mathrm{d}x\,.$$

This is where the main difficulty lies: indeed, since $\bar\phi \in \mathcal{H}^1(\mathbb{R}^d)$, we only have

$$\frac{\bar\phi\circ(\mathrm{id}+\delta\zeta) - \bar\phi}{\delta} \rightharpoonup \zeta\cdot\nabla\bar\phi \quad \text{in } \mathrm{L}^2(\mathbb{R}^d),$$

while $\bar\rho$ is only in $(\mathrm{L}^1\cap\mathrm{L}^m)(\mathbb{R}^d)$ with $m<2$. So even the product $\bar\rho\zeta\cdot\nabla\bar\phi$ which is the candidate for the limit is not well defined and the regularity of $(\bar\rho,\bar\phi)$ has to be improved. We develop in the next section a generalisation to the Matthes-McCann-Savaré technique.

4.3. **A generalisation of Matthes-McCann-Savaré's approach.** Actually, the cornerstone of Matthes-McCann-Savaré's method is the availability of another functional $\mathcal{G}$ and the simplest situation is the case where the flow has a displacement convex Lyapunov functional which is different from the energy, which was the case in the previous section. Unfortunately, there does not seem to be a natural choice of such a functional $\mathcal{G}$ here. A first try is to choose $\mathcal{G}$ as the displacement convex part of $\mathcal{E}_\alpha$, that is,

$$\mathcal{G}[u,v] := \int_{\mathbb{R}^d}\left(\frac{|u(x)|^m}{(m-1)} + \frac{1}{2}\,|\nabla v(x)|^2 + \frac{\alpha}{2}\,|v(x)|^2\right)\mathrm{d}x\,.$$

The associated gradient flow is the solution (u,v) to

$$\partial_s u - \Delta u^m = 0 \quad \text{in } (0,\infty)\times\mathbb{R}^d, \qquad u(0) = \bar\rho\,,$$

and

$$\partial_s v - \Delta v + \alpha v = 0 \quad \text{in } (0,\infty)\times\mathbb{R}^d, \qquad v(0) = \bar\phi\,.$$

Computing $\mathrm{d}\mathcal{E}_\alpha[u(s),v(s)]/\,\mathrm{d}s$ leads to the sum of a negative term and a remainder but the remainder terms cannot be controlled. Despite this failed attempt, it turns out that, somehow unexpectedly, the following functional

$$\mathcal{G}[u,v] := \int_{\mathbb{R}^d}\left(u(x)\log(u(x)) + \frac{1}{2}\,|\nabla v(x)|^2 + \frac{\alpha}{2}\,|v(x)|^2\right)\mathrm{d}x$$

provide the right information. Indeed, its associated gradient flow is the solutions U and V to the initial value problems

$$\partial_s u - \Delta u = 0 \quad \text{in } (0,\infty)\times\mathbb{R}^d, \qquad u(0) = \bar\rho\,,$$

and

$$\partial_s v - \Delta v + \alpha v = 0 \quad \text{in } (0,\infty)\times\mathbb{R}^d, \qquad v(0) = \bar\phi\,,$$

and, as we shall see below, $\mathrm{d}\mathcal{E}_\alpha[u(s),v(s)]/\,\mathrm{d}s$ is in that case the sum of a negative term and a remainder which we are able to control. For sake on simplicity in the presentation let us take $\alpha=0$. We compute

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_0[u,v]=\underbrace{-\frac{4}{m}\|\nabla\left(u^{m/2}(t)\right)\|_2^2-\|(\Delta v+u)(t)\|_2^2}_{:=\mathcal{D}[u,v]}+\|u(t)\|_2^2\,,\qquad t>0\,.$$

So that the discrete estimate (24) gives:

$$\mathcal{D}[\rho_{h,n},\phi_{h,n}]-\|\rho_{h,n}\|_2^2\le\frac{\mathcal{G}[\rho_{h,n-1},\phi_{h,n-1}]-\mathcal{G}[\rho_{h,n},\phi_{h,n}]}{h}\,. \tag{33}$$

Remains to prove that we can control $\|\rho_{h,n}\|_2^2$ by $\mathcal{D}[\rho_{h,n},\phi_{h,n}]$. This can be done using the Hölder and Sobolev inequalities:

$$\|w\|_2^2\le\|w\|_m\,\|w\|_{m/(m-1)}\le C\,\|w\|_m\,\left\|\nabla(|w|^{m/2})\right\|_2^{2/m}\,. \tag{34}$$

Combining the above estimate with Young's inequality gives

$$\|\rho_{h,n}\|_2^2\le\frac{2}{m}\left\|\nabla(\rho_{h,n}^{m/2})\right\|_2^2+C\,\|\rho_{h,n}\|_m^{m/(m-1)}\,,$$

and thus

$$\|\rho_{h,n}\|_2^2\le\frac{1}{2}\mathcal{D}[\rho_{h,n},\phi_{h,n}]+C\,\|\rho_{h,n}\|_m^{m/(m-1)}\,. \tag{35}$$

By (32) we obtain, for any $M<M_c$

$$\|\rho_{h,n}\|_2^2\le\frac{1}{2}\mathcal{D}[\rho_{h,n},\phi_{h,n}]+C\,\mathcal{E}_0[\rho_{h,n},\phi_{h,n}]^{1/(m-1)}\,.$$

And finally (33) implies

$$\frac{1}{2}\mathcal{D}[\rho_{h,n},\phi_{h,n}]\le\frac{\mathcal{G}[\rho_{h,n-1},\phi_{h,n-1}]-\mathcal{G}[\rho_{h,n},\phi_{h,n}]}{h}+C\,\mathcal{E}_0[\rho_{h,n},\phi_{h,n}]^{1/(m-1)}\,.$$

Which gives a bound in $\mathrm{H}^1(\mathbb{R}^2)$ for $(\rho_{h,n})^{m/2}$. By the Gagliardo-Nirenberg-Sobolev inequality $\{\rho_{h,n}\}_n$ is thus bounded in $\mathrm{L}^p(\mathbb{R}^2)$, for any $p\in[1,\infty)$. Such a regularity is now enough to pass to the limit in the Euler-Lagrange equation and obtain the stated result.

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Appendix A. An optimal transport toolbox

We just give some basic results from optimal transport theory that we use in the proof, for a detailed exposition of this rich and rapidly developing subject, we refer the interested reader to the very accessible textbook [34] or [1, 35].

A.1. **Kantorovich and Monge's problems.** Let X and Y be two spaces equipped respectively with the Borel probability measures with finite 2-moment $\mu \in \mathcal{P}(X)$ and $\nu \in \mathcal{P}_2(Y)$. For $\mu \in \mathcal{P}_2(X)$ and T, Borel: $X \to Y$, $T_\# \mu$ denotes the *push forward* (or image measure) of μ through T which is defined by $T_\#\mu(B) = \mu(T^{-1}(B))$ for every Borel subset B of Y or equivalently by the change of variables formula

$$\int_Y \varphi \, \mathrm{d}T_\#\mu = \int_X \varphi(T(x)) \, \mathrm{d}\mu(x), \ \forall \varphi \in \mathcal{C}^0_{\mathrm{b}}(X). \tag{36}$$

A transport map between μ and ν is a Borel map such that $T_\#\mu = \nu$. Now, let $c \in \mathcal{C}(X \times Y)$ be some transport cost function, the *Monge optimal transport* problem for the cost c consists in finding a transport T between μ and ν that minimises the total transport cost $\int_X c(x, T(x)) \, \mathrm{d}\mu(x)$. A minimiser is then called an *optimal transport.* Monge problem is in general difficult to solve (it may even be the case that there is no transport map, for instance it is impossible to transport one Dirac mass to a sum of distinct Dirac masses), this is why Kantorovich relaxed Monge's formulation as

$$\mathcal{W}_c(\mu, \nu) := \inf_{\gamma \in \Pi(\mu,\nu)} \iint_{X \times Y} c(x, y) \, \mathrm{d}\gamma(x, y) \tag{37}$$

where $\Pi(\mu, \nu)$ is the set of transport plans between μ and ν *i.e.* Borel probability measures on $X \times Y$ having μ and ν as marginals. Since $\Pi(\mu, \nu)$ is weakly $*$ compact and c is continuous, it is easy to see that the infimum of the linear program defining $\mathcal{W}_c(\mu, \nu)$ is attained at some γ, such optimal γ's are called *optimal transport plans* (for the cost c) between μ and ν. If there is an optimal γ which is induced by a *transport map i.e.* is of the form $\gamma = (\mathrm{id}, T)_\#\mu$ for some transport map T then T is obviously an optimal solution to Monge's problem.

A.2. **The quadratic case and Monge-Ampère equation.** We now restrict ourselves to the quadratic case:

Theorem 9 (Brenier's theorem, [11]). *Let $\mu \in \mathcal{P}(\mathbb{R}^d)$ be absolutely continuous with respect to the Lebesgue measure and compactly supported and $\nu \in \mathcal{P}(\mathbb{R}^d)$ be compactly supported, then the quadratic optimal transport problem*

$$\mathcal{W}_2^2(\mu, \nu) := \inf_{\gamma \in \Pi(\mu,\nu)} \iint_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^2 \, \mathrm{d}\gamma(x, y)$$

possesses a unique solution γ which is in fact a Monge solution $\gamma = (\mathrm{id}, T)_\#\mu$. Moreover $T = \nabla u$ μ-a.e. for some convex function u and ∇u is the unique (up to μ-a.e. equivalence) gradient of a convex function transporting μ to ν; $T = \nabla u$ is called the Brenier map *between μ and ν.*

When we have additional regularity, *i.e.* when μ and ν have regular densities (still denoted f and g) and ∇u is a diffeomorphism which transports $f(x) \, \mathrm{d}x$ onto $g(y) \, \mathrm{d}y$ we have

$$\int_{\mathbb{R}^d} \zeta(y) g(y) \, \mathrm{d}y = \int_{\mathbb{R}^d} \zeta[\nabla u(x)] f(x) \, \mathrm{d}x \quad \forall \zeta : \mathcal{C}^0_b \to \mathcal{C}^0_b \, .$$

By performing the change of variable $y = \nabla u(x)$ on the left hand side we obtain

$$\int_{\mathbb{R}^d} \zeta(\nabla u(x)) g(\nabla u(x)) |\det D^2 u(x)| \, \mathrm{d}x = \int_{\mathbb{R}^d} \zeta[\nabla u(x)] f(x) \, \mathrm{d}x \quad \forall \zeta : \mathcal{C}^0_b \to \mathcal{C}^0_b \, .$$

By equalling the two integrands we obtain the Monge-Ampère equation:

$$f(x) = g(\nabla u(x)) \det(D^2 u(x)) \quad \text{or equivalently} \quad g(y) = \frac{f(\nabla u^{-1}(y))}{\det(D^2 u(\nabla u^{-1}(y))} \,. \tag{38}$$

A.3. **Differentiating the internal and the interaction energies.** Introduce $\nabla\psi_\varepsilon := \mathrm{id} + \varepsilon\zeta$ and define ρ_ε the push-forward perturbation of ρ_τ^{n+1} by $\nabla\psi_\varepsilon$:

$$\rho_\varepsilon = \nabla\psi_\varepsilon \# \rho_\tau^{n+1} \,.$$

By (38) and the change of variable $x = \nabla\psi_\varepsilon^{-1}(y)$, the differential of the $\int F(u)\,\mathrm{d}x$ where $F(x) = x\log x$ or $F(x) = x^m$ formally gives

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\varepsilon}_{|\varepsilon=0} \int_{\mathbb{R}^d} F(\rho_\varepsilon)\,\mathrm{d}y &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}_{|\varepsilon=0} \int_{\mathbb{R}^d} F\left(\frac{\rho(\nabla\psi_\varepsilon^{-1}(y))}{\det(D^2\psi_\varepsilon(\nabla\psi_\varepsilon^{-1}(y)))}\right)\mathrm{d}y \\
&= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}_{|\varepsilon=0} \int_{\mathbb{R}^d} F\left(\frac{\rho(y)}{\det(D^2\psi_\varepsilon(y))}\right)\det(D^2\psi_\varepsilon(y))\,\mathrm{d}y \\
&= -\int_{\mathbb{R}^d} \rho\,[\Delta\psi - d]\,F'(\rho)\,\mathrm{d}y + \int_{\mathbb{R}^d} F(\rho)\,[\Delta\psi - d]\,\mathrm{d}y \\
&= \int_{\mathbb{R}^d} [F(\rho) - \rho F'(\rho)]\,[\Delta\psi - d]\,\mathrm{d}y \,. \qquad (39)
\end{aligned}$$

Where, as $\det(I + H) = 1 + \mathrm{tr}(H) + o(\|H\|)$, we have used

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}_{|\varepsilon=0} \det(D^2\psi_\varepsilon(y)) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}_{|\varepsilon=0} \det(I + \varepsilon(D^2\psi - I)) = \Delta\psi - d \,.$$

By integrating by parts (39) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}_{|\varepsilon=0} \int_{\mathbb{R}^d} F(\rho_\varepsilon)\,\mathrm{d}y = -\int_{\mathbb{R}^d} \nabla\,[F(\rho) - \rho F'(\rho)]\,[\nabla\psi - \mathrm{id}]\,\mathrm{d}y \,.$$

By convexity of F, $x \mapsto F(x) - xF'(x)$ is non-increasing from $F(0) = 0$. So that the boundary term is non-positive and

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}_{|\varepsilon=0} \int_{\mathbb{R}^d} F(\rho_\varepsilon)\,\mathrm{d}y \leq -\int_{\mathbb{R}^d} \nabla\,[F(\rho) - \rho F'(\rho)]\,[\nabla\psi - \mathrm{id}]\,\mathrm{d}y \,.$$

As $\nabla\,[F(\rho) - \rho F'(\rho)] = -\rho\nabla\,[F'(\rho)] = \rho\nabla\,[f(\rho)]$, we have

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}_{|\varepsilon=0} \int_{\mathbb{R}^d} F(\rho_\varepsilon)\,\mathrm{d}y \leq -\int_{\mathbb{R}^d} \rho\nabla\,[f(\rho)]\,[\nabla\psi - \mathrm{id}]\,\mathrm{d}y \,.$$

• By symmetry of ϕ and definition of the push-forward, the interaction term formally gives

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\varepsilon}_{|\varepsilon=0} \iint_{\mathbb{R}^{2d}} \phi(y,z)\,\mathrm{d}\rho_\varepsilon(y)\,\mathrm{d}\rho_\varepsilon(z) &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}_{|\varepsilon=0} \iint_{\mathbb{R}^{2d}} \phi(\nabla\psi_\varepsilon(y), \nabla\psi_\varepsilon(z))\,\mathrm{d}\rho\otimes\rho \\
&= 2\iint_{\mathbb{R}^{2d}} \nabla\phi(y,z)(\nabla\psi(y) - y)\,\mathrm{d}\rho\otimes\rho
\end{aligned}$$

A.4. **Differentiability of the Wasserstein distances.** We need first to recall the following classical characteristics method, see [34, Theorem 5.34] [1, Theorem 8.3.1]:

Proposition 10 (Characteristics method for linear transport equation). *Let ρ be in $\mathcal{P}(Y)$ and $(T_t)_{t\in[0,T_*]}$ be a family of diffeomorphism locally Lipschitz with $T_0 = \mathrm{id}$ and let v be the associated velocity field* i.e. $\dot{T}_t(x) = v(t, T_t(x))$. *Then $\rho_t = T_{t\#}\rho$ is a solution to the following linear transport equation in $C(0, T_*; \mathcal{P}(Y))$:*

$$\begin{cases} \dfrac{\partial \rho_t}{\partial t} + \nabla \cdot (v\,\rho_t) = 0\,, \qquad \forall t \in [0, T_*] \\ \rho_0 = \rho\,. \end{cases}$$

The idea of the proof is formally as follows: Let ϕ be any test function. By the definition of the push-forward and using $\dot{T}_t(x) = v(t, T_t(x))$ we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} \phi(y)\,\mathrm{d}\rho_t(y) &= \frac{\mathrm{d}}{\mathrm{d}t}\int_Y \phi(T_t(x))\,\mathrm{d}\rho(y) \\ &= \int_{\mathbb{R}^d} \nabla\phi(T_t(x))\,\dot{T}_t(x)\,\mathrm{d}\rho(y) \\ &= \int_{\mathbb{R}^d} \nabla\phi(T_t(x))\,v(T_t(x))\,\mathrm{d}\rho(y) \\ &= \int_{\mathbb{R}^d} \nabla\phi(y)\,v(y)\,\mathrm{d}\rho_t(y)\,. \end{aligned}$$

Which gives the desire result. Actually it can be proven that ρ_t is the only solution to the linear transport equation.

Proposition 11 (Differentiability of the Monge-Kantorovich distance). *Let $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ and $\nu \in \mathcal{P}_2(\mathbb{R}^d)$ be given. Let $(T_t)_{t\in[0,T_*]}$ be a family of $C^1(Y)$ function with $T_0 = \mathrm{id}$ and let v be the associated velocity field* i.e. $\dot{T}_t(x) = v(t, T_t(x))$. *Consider $\nu \in \mathcal{P}(Y)$ and $\nu_t = T_t\#\nu$. Then we have*

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{W}_2^2(\mu, \nu_t) = \int \langle y - \nabla\varphi^*, v(y)\rangle\,\mathrm{d}\nu(y)\,.$$

where $\nabla\varphi^*$ is the Legendre transform of $\nabla\varphi$ the optimal map between μ and ν.

Once again we do not aim to give a rigorous proof of this proposition and will refer the interested reader to [34, Theorem 8.13] and [1, Corollary 10.2.7]. We however give a formal idea of the proof:

The map $T_t \circ \nabla\varphi$ pushes forward μ onto ν_t. We do not know if it the optimal map but by definition of the Monge-Kantorovich distance we have

$$\frac{1}{2}\mathcal{W}_2^2(\mu, \nu_t) \le \int_{\mathbb{R}^d} |x - T_t[\nabla\varphi(x)]|^2\,\mathrm{d}\mu(x)\,.$$

As a consequence, for any $t \ge 0$, using $A^2 - B^2 = (A+B)(A-B)$ we have

$$\begin{aligned} \frac{\mathcal{W}_2^2(\mu, \nu_t) - \mathcal{W}_2^2(\mu, \nu)}{t} &\le \int_{\mathbb{R}^d} |x - T_t[\nabla\varphi(x)]|^2\,\mathrm{d}\mu(x) - \int_{\mathbb{R}^d} |x - \nabla\varphi(x)|^2\,\mathrm{d}\mu(x) \\ &\le \int_{\mathbb{R}^d} (2x - T_t[\nabla\varphi] - \nabla\varphi)\,(\nabla\varphi - T_t[\nabla\varphi])\,\mathrm{d}\mu\,. \end{aligned}$$

As, by (10)

$$T_t[\nabla\varphi(x)] - \nabla\varphi(x) = T_t[\nabla\varphi(x)] - T_0[\nabla\varphi(x)] = t\dot{T}_t[\nabla\varphi(x)] + o(t)$$
$$= tv\,[T_t(\nabla\varphi(x))] + o(t)$$

taking the limit when $t \to 0$, we obtain

$$\lim_{t\to 0} \frac{\mathcal{W}_2^2(\mu,\nu_t) - \mathcal{W}_2^2(\mu,\nu)}{t} \leq \int_{\mathbb{R}^d} \langle 2x - 2\nabla\varphi(x), -v\,[\nabla\varphi(x)]\rangle \,\mathrm{d}\mu(x)\ .$$

As $\nabla\varphi$ pushes-forward μ onto ν and using Theorem 9, we obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{W}_2^2(\mu,\nu_t) &= \int_{\mathbb{R}^d} \langle \nabla\varphi(x) - x, v\,[\nabla\varphi(x)]\rangle \,\mathrm{d}\mu(x)\\ &= \int_{\mathbb{R}^d} \langle \nabla\varphi(x) - \nabla\varphi^*[\nabla\varphi(x)], v\,[\nabla\varphi(x)]\rangle \,\mathrm{d}\mu(x)\\ &= \int_{\mathbb{R}^d} \langle y - \nabla\varphi^*(y), v(y)\rangle \,\mathrm{d}\nu(y)\ .\end{aligned}$$

A.5. **Displacement convexity.** In concrete terms, a functional $\mathcal{G}$ is said to be *displacement convex* when the following is true: for any two densities ρ_0 and ρ_1 of the same mass M, let φ be such that $\nabla\varphi\#\rho_0 = \rho_1$. For $0 < t < 1$, define

$$\varphi_t(x) = (1-t)\frac{|x|^2}{2} + t\varphi(x) \qquad \text{and} \qquad \rho_t = \nabla\varphi_t\#\rho_0\ .$$

The *displacement interpolation* between ρ_0 and ρ_1 is the path of densities $t \mapsto \rho_t$, $0 \leq t \leq 1$. Let γ be any real number. To say that $\mathcal{G}$ is γ*-displacement convex* means that for all such mass densities ρ_0 and ρ_1, and all $0 \leq t \leq 1$,

$$(1-t)\mathcal{G}(\rho_0) + t\mathcal{G}(\rho_1) - \mathcal{G}(\rho_t) \geq \gamma t(1-t)\mathcal{W}_2^2(\rho_0,\rho_1)\ .$$

$\mathcal{G}$ is simply *displacement convex* if this is true for $\gamma = 0$, and $\mathcal{G}$ is *uniformly displacement convex* if this is true for some $\gamma > 0$.

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[1] TSE (GREMAQ, CNRS UMR 5604, INRA UMR 1291, Université de Toulouse), 21 Allée de Brienne, F–31000 Toulouse, France
E-mail address: Adrien.Blanchet@univ-tlse1.fr