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Tunneling for spatially cut-off $P(\phi)_2$-Hamiltonians

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Let $-L$ be the second quantization operator of $\sqrt{m^2-\Delta}$, where m is a positive number. Let $\lambda=1/\hbar$ be a large positive parameter. Let us consider an interaction potential function V_λ which is given by a Wick polynomial

$$V_\lambda(w)=\lambda\int_{\mathbb{R}}:P\left(\frac{w(x)}{\sqrt{\lambda}}\right):g(x)dx, \tag{1}$$

where g is a non-negative smooth function with compact support and $P(x)=\sum_{k=1}^{2M}a_kx^k$ is a polynomial bounded from below. The operator $-L+V_\lambda$ is called a spatially cut-off $P(\phi)_2$-Hamiltonian. Formally, $-L+V_\lambda$ is unitarily equivalent to the infinite dimensional Schrödinger operator:

$$-\Delta_{L^2(\mathbb{R})}+\lambda U(w/\sqrt{\lambda})-\frac{1}{2}\mathrm{tr}(m^2-\Delta)^{1/2} \quad \text{on } L^2(L^2(\mathbb{R}),dw) \tag{2}$$

where dw is an infinite dimensional Lebesgue measure. The function U is a potential function such that

$$U(w)=\frac{1}{4}\int_{\mathbb{R}}w'(x)^2dx+\int_{\mathbb{R}}\left(\frac{m^2}{4}w(x)^2+:P(w(x)):g(x)\right)dx$$

and $\Delta_{L^2(\mathbb{R})}$ denotes the "Laplacian" on $L^2(\mathbb{R},dx)$. Hence, by the analogy of Schrödinger operators in finite dimensions, it is natural to expect that asymptotic behavior of lowlying eigenvalues of $-L+V_\lambda$ in the semiclassical limit $\lambda\to\infty$ is related with the properties of global minimum points of U. In view of this, we consider the following assumptions.

Assumption 1. Let P be the polynomial in (1) and U be the function on H^1 which is given by

$$U(h)=\frac{1}{4}\int_{\mathbb{R}}h'(x)^2dx+\int_{\mathbb{R}}\left(\frac{m^2}{4}h(x)^2+P(h(x))g(x)\right)dx \qquad \text{for } h\in H^1. \tag{3}$$

(A1) The function U is non-negative and the zero point set

$$\mathcal{Z}:=\{h\in H^1 \mid U(h)=0\}=\{h_1,\ldots,h_n\} \tag{4}$$

is a finite set.

(A2) For all $1 \leq i \leq n$, the Hessian $\nabla^2 U(h_i)$ is non-degenerate. That is, there exists $\delta_i > 0$ for each i such that

$$\begin{aligned}\nabla^2 U(h_i)(h,h) &:= \frac{1}{2}\int_{\mathbb{R}} h'(x)^2 dx + \int_{\mathbb{R}} \left(\frac{m^2}{2}h(x)^2 + P''(h_i(x))g(x)h(x)^2\right) dx \\ &\geq \delta_i \|h\|^2_{L^2(\mathbb{R})} \qquad \text{for all } h \in H^1(\mathbb{R}).\end{aligned} \tag{5}$$

(A3) For all x, $P(x) = P(-x)$ and $\mathcal{Z} = \{h_0, -h_0\}$, where $h_0 \neq 0$.

Let $E_1(\lambda)$ be the lowest eigenvalue of $-L + V_\lambda$. The first main result is as follows.

Theorem 2. Assume that (A1) and (A2) hold. Let $E_1(\lambda) = \inf \sigma(-L + V_\lambda)$. Then

$$\lim_{\lambda\to\infty} E_1(\lambda) = \min_{1\leq i\leq n} E_i, \tag{6}$$

where

$$E_i = \inf \sigma(-L + Q_i) \tag{7}$$

and Q_i is given by

$$Q_i(w) = \frac{1}{2}\int_{\mathbb{R}} : w(x)^2 : P''(h_i(x))g(x)dx. \tag{8}$$

Remark 3. (1) In the case of finite dimensional Schrödinger operators, there exist eigenvalues near the approximate eigenvalues E_i when λ is large. In Theorem 2, if $E_i < m + \min_{1\leq i\leq n} E_i$, then the same results hold by the result of Hoegh-Krohn and Simon [11]. However, if it is not the case, it is not clear and they may be embedded eigenvalues in the essential spectrum. Under the assumptions in Theorem 5, $E_2(\lambda)$ is an eigenvalue for large λ. Simon [9] gave an example of $P(\phi)_2$-Hamiltonian for which an embedded eigenvalue exists.
(2) We refer the readers to [3] for the proofs of theorems in this note.

Let

$$E_2(\lambda) = \inf\{\sigma(-L + V_\lambda) \setminus \{E_1(\lambda)\}\}.$$

We can prove that $E_2(\lambda) - E_1(\lambda)$ is exponentially small when U is a symmetric double well potential function. The exponential decay rate is given by the Agmon distance which is defined below.

Definition 4. Let $0 < T < \infty$ and $h, k \in H^1(\mathbb{R})$. Let $AC_{T,h,k}(H^1(\mathbb{R}))$ be the set of all absolutely continuous paths $c : [0,T] \to H^1(\mathbb{R})$ satisfying $c(0) = h, c(T) = k$. Let U be the potential function in (3). Assume U is non-negative. We define the Agmon distance between h, k by

$$d_U^{Ag}(h,k) = \inf\left\{\ell_U(c) \mid c \in AC_{T,h,k}(H^1(\mathbb{R}))\right\}, \tag{9}$$

where

$$\ell_U(c) = \int_0^T \sqrt{U(c(t))}\|c'(t)\|_{L^2} dt. \tag{10}$$

The following estimate is the second main result.

Theorem 5. Assume that U satisfies (A1),(A2),(A3). Then it holds that

$$\limsup_{\lambda\to\infty} \frac{\log(E_2(\lambda)-E_1(\lambda))}{\lambda} \le -d_U^{Ag}(h_0,-h_0). \tag{11}$$

Agmon distance defined aboe can be extended to a continuous distance function on $H^{1/2}(\mathbb{R})$.

Definition 6. (1) Let $h,k\in H^{1/2}$. Let $\mathcal{P}^{loc}_{T,h,k,U}$ be all continuous paths $c=c(t)$ $(0\le t\le T)$ on $H^{1/2}$ such that $c(0)=h, c(T)=k$ and

(i) there exist finitely many times $0=t_0<\cdots<t_n=T$ such that for any closed interval $I\subset(t_i,t_{i+1})$ $(0\le i\le n-1)$, the restricted path $c|_I$ is an absolutely continuous path on $L^2(\mathbb{R})$.

(ii) $c(t)\in H^1(\mathbb{R})$ for $\|c'(t)\|dt$ -a.e. $t\in[0,T]$ and

$$\int_0^T \sqrt{U(c(t))}\|c'(t)\|_{L^2}dt<\infty. \tag{12}$$

We define the length $\ell_U(c)$ of $c\in\mathcal{P}^{loc}_{T,h,k,U}$ by the integral value of (12).
(2) Let $0<T<\infty$. We define the Agmon distance between $h,k\in H^{1/2}(\mathbb{R})$ by

$$d_U^{Ag}(h,k) \ = \ \inf\left\{\ell_U(c)\mid c\in\mathcal{P}^{loc}_{T,h,k,U}\right\}. \tag{13}$$

The above definition of d_U^{Ag} coincides with that in H^1. Moreover the topology defined by the Agmon distance coincides with the one defined by the Sobolev norm of $H^{1/2}(\mathbb{R})$. We can prove the existence of minimal geodesic between h_0 and $-h_0$ with respect to the Agmon metric. The uniqueness of the geodesics is not clear at the moment.

Theorem 7. Assume (A1), (A2) and $\mathcal{Z}$ consists of two points $\{h,k\}$. There exists a curve $c_\star\in\mathcal{P}^{loc}_{1,h,k,U}$ such that $\ell_U(c_\star)=d_U^{Ag}(h,k)$. This $c_\star$ has the following properties.
(1) $c_\star(t)\notin\mathcal{Z}$ for $0<t<1$.
(2) $c_\star=c_\star(t,x)$ is a C^∞ function of $(t,x)\in(0,1)\times\mathbb{R}$ and $c_\star\in H^1(\varepsilon,1-\varepsilon)\times\mathbb{R})$ for all $0<\varepsilon<1$.
(3) $\int_0^\varepsilon\|c_\star'(t)\|^2_{L^2}dt=\int_{1-\varepsilon}^1\|c_\star'(t)\|^2_{L^2}dt=+\infty$ for any $\varepsilon>0$.

The Agmon distance $d_U^{Ag}(h_0,-h_0)$ is equal to an Euclidean action integral of an instanton solution. This is an infinite dimensional example corresponding to the result of instanton in the case of Schrödinger operator which is due to Carmona and Simon [5]. The instanton equation in our model reads

$$\frac{\partial^2 u}{\partial t^2}(t,x)+\frac{\partial^2 u}{\partial x^2}(t,x)=m^2u(t,x)+2P'(u(t,x))g(x). \tag{14}$$

For $u = u(t,x)$, we define the Euclidean action integral:

$$I_{\infty,P}(u) = \frac{1}{4}\int_{-\infty}^{\infty} \|\partial_t u(t)\|_{L^2(\mathbb{R})}^2 dt + \int_{-\infty}^{\infty} U(u(t))dt. \quad (15)$$

We have the following theorem for the existence of instanton.

Theorem 8. There exists a solution $u_\star = u_\star(t,x)$ $((t,x) \in \mathbb{R}^2)$ to the equation (14) which satisfies the following properties.
(1) It holds that $u_\star|_{(-T,T)\times\mathbb{R}} \in H^1((-T,T)\times\mathbb{R}) \cap C^\infty((-T,T)\times\mathbb{R})$ for any $T > 0$. Also we have $\lim_{t\to-\infty} \|u_\star(t) - h\|_{H^{1/2}} = 0$ and $\lim_{t\to\infty} \|u_\star(t) - k\|_{H^{1/2}} = 0$.
(2) We have $I_{\infty,P}(u_\star) = d_U^{Ag}(h,k)$ and $u_\star$ is a minimizer of the functional $I_{\infty,P}$ in the set of functions u satisfying the following conditions:

(i) $u|_{(-T,T)\times\mathbb{R}} \in H^1((-T,T),\mathbb{R})$ for all $T > 0$,

(ii) $\lim_{t\to-\infty} \|u(t) - h\|_{H^{1/2}} = 0$ and $\lim_{t\to\infty} \|u(t) - k\|_{H^{1/2}} = 0$.

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