A stability estimate and numerical curiosities related to the flow of a curve by its binormal curvature

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1 Classical formulation of the binormal curvature flow

The equations for the evolution of a family $(\gamma_t)_{t\in I}$ of smooth curves in $\mathbb{R}^3$ according to their binormal curvature are written in terms of an arc-length parametrization $\gamma : I \times \mathbb{R} \to \mathbb{R}^3$ by

$$\partial_t\gamma = \partial_s\gamma \times \partial_{ss}\gamma \tag{1}$$

where $t \in I$ is the time variable, $s \in \mathbb{R}$ is the arc-length parameter, and $\times$ denotes the vector product in $\mathbb{R}^3$. In geometric terms, equation (1) takes its name from its equivalent form

$$\partial_t\gamma = \kappa b$$

where κ and b are the curvature function and the binormal vector field along γ_t respectively. Note that the arc-length parametrization condition $|\partial_s\gamma(t,s)|^2 = 1$ is always compatible with equation (1) since $\partial_t(|\partial_s\gamma|^2) = 2\partial_s\gamma \cdot \partial_{st}\gamma = 2\partial_s\gamma \cdot (\partial_s\gamma \times \partial_{sss}\gamma) = 0$ whenever (1) is satisfied. In particular, closed curves evolved by the binormal curvature flow equation (1) all have the same length.

Equation (1) first appeared in the 1906 Ph.D. thesis of L.S. Da Rios [5], whose work was promoted in a series of lectures in 1931 in Paris by its advisor T. Levi-Civita [17]. The problem considered by Da Rios and Levi-Civita goes back to the work of H. Helmholtz [7] in 1858 on the motion of a three dimensional incompressible fluid in rotation. In part of [7], Helmholtz considered configurations called "vortex-filaments of indefinitely small cross-section" (at least in its translation [8] by P.G. Tait) : in that case, the vorticity field $\omega := \mathrm{curl}(v)$ associated to the velocity field v of the fluid at a given time t is concentrated along a closed oriented curve γ_t, parallel to it and vanishing rapidly away from it. Helmholtz could not[1] rigorously answer the question of the persistence in time of such vortex-filaments under the Euler flow $\partial_t \omega + v \cdot \nabla \omega = \omega \cdot \nabla \omega$. Nevertheless, he obtained important contributions in that direction. A few years later in 1867, Lord Kelvin announced in [12] and published in [13] (thirteen years later!) the first result on linear stability of circular vortex-filaments. The latter, also called vortex rings, correspond in the asymptotic of infinitely small cross-section to the travelling wave solutions of equation (1) given by $\gamma(t,s) = \gamma_{r,\vec{e}}(s) + \frac{t}{r}\vec{e}$, where $\gamma_{r,\vec{e}}$ is an arc-length parametrization of a circle of radius r in a plane perpendicular to the unitary vector $\vec{e} \in \mathbb{R}^3$. It is only in 1906, with the help of progress made in potential theory, that Da Rios formally obtained the speculated general motion law (1). One hardly finds any reference to equation (1) in the literature in the interval of time between Da Rios thesis in 1906 and the two papers [1] by R.J. Arms and F.R. Hama and [3] by R. Betchov both in 1965. Actually, in a large part of the fluid dynamics community, equation (1) was long (mistakenly) attributed to Betchov and/or Arms and Hama. It is the author's pleasure to mention that not only Da Rios contribution was forgotten on those occasions, since it appears that a formal derivation of (1) was also conducted in 1937 (We were told it was presented as a (3 pages) homework for students!) in the paper [19] by the Japanese mathematicians Y. Murakami, H. Takahasi, Y. Ukita and S. Fujihara[2]. As will becomes clear in the remaining, Japan as played a prominent role in every perspective in the study of (1) so far.

2 A new form of (in)stability estimate

In [10], one of our goals with R.L. Jerrard was to extend the classical formulation (1) to a larger class the regular parametrized curves. The motivation to do so was two-fold : first, since it deals with parametrized curves it is necessarily insensitive to self-intersections, by which we mean lack of injectivity of the map $\gamma(t, \cdot)$. This is surely unsatisfactory if one believes that such flows

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[1]As anybody so far!

[2]The author would actually be grateful to receive some copy of it.

arise as limits from three dimensional fluid dynamics. Our formulation in [10] in principle would be able to detect such self-intersections, as well as possible collisions between elements of disconnected vortex filaments and changes of topology. Second, there are presumably important configurations of curves which are too singular to be considered under formulation (1). Invoking distributional derivatives on can give a meaning to equation (1) in a variety of spaces, but those spaces just fail to include the case of curves which are at most Lipschitz. On the other hand, in numerical simulations of the Euler equation or the Gross-Pitaevskii equation for quantum fluids, it is observed (see e.g. S. Kida and M. Takaoka [15] (still Japan!) or J. Koplik and H. Levine [16]) that vortex-filaments often tend to recombine by exchanging strands in cases of collisions or self-intersections. Those recombinations, when the intersections are transverse, inevitably create curves which are at most Lipschitz.

It is not our purpose here to present the details of the new formulation proposed in [10], these involve a number constructions based on notions related to Geometric Measure Theory which are not appropriate for a short survey. Instead, we will focus on the main ingredient in the corresponding (weak-strong) uniqueness theory which happens to have a counterpart with its own interest also in the framework of parametrized solutions of (1). As far as we know, it is the first rigorous stability (actually: control of instabilities) estimate for filament flows.

In order to measure distances between curves, without attaching too much importance in their parametrizations but still, we define the following notion of distance.

Definition 1. A curve in $\mathbb{R}^3$ parametrized by arc-length is a Lipschitz map $\gamma : \mathbb{R} \to \mathbb{R}^3$ such that

$$|\gamma'(s)| = 1 \quad \text{for a.e. } s \in \mathbb{R}.$$

A closed curve in $\mathbb{R}^3$ of period $\ell > 0$ is a curve in $\mathbb{R}^3$ parametrized by arc-length and such that

$$\gamma(s + \ell) = \gamma(s), \quad \forall s \in \mathbb{R}.$$

Given two closed curves γ and Γ in $\mathbb{R}^3$, we define the parametric distance

$$d_{\mathcal{P}}(\Gamma, \gamma) := \inf_{p \in \mathcal{P}(L, \ell)} \sup_{s \in \mathbb{R}} \left| \Gamma(s) - \gamma(p(s)) \right| = \inf_{p \in \mathcal{P}(\ell, L)} \sup_{s \in \mathbb{R}} \left| \gamma(s) - \Gamma(p(s)) \right|,$$

where ℓ and L are the periods of γ and Γ respectively, and for $0 < s_0, s_1$,

$$\mathcal{P}(s_0, s_1) = \{p \in \mathcal{C}(\mathbb{R}, \mathbb{R}) \text{ s.t. } p(s + s_0) = p(s) + s_1, \forall s \in \mathbb{R}\}.$$

$\square$

Clearly,

$$d_{\mathcal{H}}(\Gamma, \gamma) \leq d_{\mathcal{P}}(\Gamma, \gamma),$$

where $d_{\mathcal{H}}$ is the Hausdorff distance between sets.

In the sequel, $\gamma \in \mathcal{C}^2(\mathbb{R}, \mathbb{R}^3)$ is a periodic curve of period $\ell > 0$ and Γ is a periodic curve of period $L > 0$. We denote by r_γ the minimal radius of curvature of γ :

$$r_\gamma := \left(\max_{s \in [0, \ell]} |\gamma''(s)| \right)^{-1} \in (0, +\infty],$$

and by $\mathcal{C}_\gamma$ the tubular neighborhood :

$$\mathcal{C}_\gamma := \left\{ x \in \mathbb{R}^3, \quad \text{s.t.} \quad d(x, \gamma) < r_\gamma/8 \right\}.$$

For $s_0 \in \mathbb{R}$ and $x_0 \in \mathbb{R}^3$ such that $|x_0 - \gamma(s_0)| < r_\gamma/8$, it can be proved that there exists a unique $\xi_0 =: \xi(s_0, x_0) \in \mathbb{R}$ (the "best" orthogonal projection) such that

1. $\xi_0 \in (s_0 - r_\gamma/4, s + r_\gamma/4)$,

2. $\left(x_0 - \gamma(\xi_0) \right) \cdot \gamma'(\xi_0) = 0$.

Moreover, the function $\xi : \Xi_\gamma \to \mathbb{R}$, $(s, x) \mapsto \xi(s, x)$ defined on the open set

$$\Xi_\gamma := \left\{ (s, x) \in \mathbb{R}^4 \text{ s.t. } |x - \gamma(s)| < r_\gamma/8 \right\}$$

is of class $\mathcal{C}^1$ on Ξ_γ and for every $(s, x) \in \Xi_\gamma$ we have

3. $(s + \ell, x) \in \Xi_\gamma$ and $\xi(s + \ell, x) = \xi(s, x) + \ell$,

4. $\partial_s \xi(s, x) = 0$,

5. $D_x \xi(s, x) = \left(1 + \frac{(x - \gamma(\xi(s,x))) \cdot \gamma''(\xi(s,x))}{1 - (x - \gamma(\xi(s,x))) \cdot \gamma''(\xi(s,x))} \right) \gamma'(\xi(s, x))$.

We have

Corollary 2. Assume that $d_{\mathcal{P}}(\Gamma, \gamma) < r_\gamma/8$ and let $p \in \mathcal{P}(L, \ell)$ be such that

$$\sup_{s \in \mathbb{R}} |\Gamma(s) - \gamma(p(s))| < r_\gamma/8.$$

The function $\sigma : \mathbb{R} \to \mathbb{R}$, $s \mapsto \sigma(s) := \xi(p(s), \Gamma(s))$ is the unique continuous function which satisfies, for any s in $\mathbb{R}$,

1. $\sigma(s + L) = \sigma(s) + \ell$,

2. $\left| \Gamma(s) - \gamma(\sigma(s)) \right| < r_\gamma/8$,

3. $\Big(\Gamma(s) - \gamma(\sigma(s))\Big) \cdot \gamma'(\sigma(s)) = 0,$

4. $\big|\sigma(s) - p(s)\big| < r_\gamma/4.$

$\square$

A continuous function σ that satisfies 1, 2 and 3 in Corollary 2 is called a reparametrization of γ for Γ. Notice that when each point x in $\mathcal{C}_\gamma$ has a unique orthogonal projection $P(x)$ on γ (this can happen only if γ has no self-intersection), then there exist a unique reparametrization σ of γ for Γ (modulo a constant multiple of ℓ), and it is determined by $\gamma(\sigma(s)) = P(\Gamma(s))$.

The following discrepancy measure between Γ and γ is better suited to our needs than d_P.

Definition 3. For $r > 0$, we set

$$F_{\gamma,r}(\Gamma) := \inf_\sigma F_{\gamma,\sigma,r}(\Gamma) \equiv \inf_\sigma \int_0^L \mathcal{F}(\Gamma,\gamma,\sigma,r,s)\,ds, \qquad (2)$$

where

$$\mathcal{F}(\Gamma,\gamma,\sigma,r,s) := 1 - f\big(|\Gamma(s) - \gamma(\sigma(s))|^2\big)\gamma'(\sigma(s)) \cdot \Gamma'(s),$$

the function $f \equiv f_r : [0,+\infty) \to [0,+\infty)$ is given by

$$f(d^2) := \left\{ \begin{array}{ll} 1 - \left(\frac{d}{r}\right)^2, & \text{for } 0 \le d^2 \le r^2, \\ 0, & \text{for } d^2 \ge r^2, \end{array} \right.$$

and the infimum in (2) is taken over all possible reparametrizations of γ for Γ.

The functional $F_{\gamma,r}$, which plays a central role in the following (in)stability estimate, can be related to more standard discrepancy measures, like the distances $d_{\mathcal{H}}$ or d_P between the curves, or the L^2 distance between the parametrizations of their tangents (see [9] Section 3 for details).

We come back now to equation (1) and assume that $\gamma \in L^\infty(I, H^4_{\text{loc}}(\mathbb{R},\mathbb{R}^3))$ and $\Gamma \in L^\infty(I, H^{\frac{3}{2}}_{\text{loc}}(\mathbb{R},\mathbb{R}^3))$ be two solutions[3] of the binormal curvature flow equation (1) on $I \times \mathbb{R}$, where $I = (-T,T)$ for some $T > 0$. Assume moreover that γ and Γ are both periodic with respective periods $\ell > 0$ and $L > 0$. For $t \in I$, we set

$$\gamma_t := \gamma(t,\cdot), \qquad \Gamma_t := \Gamma(t,\cdot),$$

and define[4]

$$r_\gamma := \max_{t \in \bar{I}} \left(\max_{s \in \mathbb{R}} |\partial_{ss}\gamma(t,s)| \right)^{-1} \in (0,+\infty].$$

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[3] Then necessarily $\gamma \in \mathcal{C}(\bar{I}, \mathcal{C}^3(\mathbb{R},\mathbb{R}^3))$ and $\Gamma \in \mathcal{C}(I, \mathcal{C}(\mathbb{R},\mathbb{R}^3))$.

[4] We have defined above a quantity called r_γ for a curve with no time dependence. One is the natural extension of the other and there shouldn't be any possible confusion.

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Assume that at time zero we have

$$d_{\mathcal{P}}(\Gamma_0, \gamma_0) < r_\gamma/8,$$

and let therefore σ_0 be a reparametrization of γ_0 for Γ_0 (the existence of which being ensured by Corollary 2).

Theorem 4. Fix $0 < r \le r_\gamma/8$ and assume that

$$F(0) := F_{\gamma_0,\sigma_0,r}(\Gamma_0) < F_r := r\left(\sqrt{2} + \frac{r}{L}\right)^{-1}. \tag{3}$$

Define then

$$T_r := \tfrac{1}{K}\log\left(\tfrac{F_r}{F(0)}\right),$$

where

$$K := \frac{8}{r^2} + \frac{32}{r_\gamma^2} + \left(2 + 32\frac{r}{r_\gamma}\right)\|\partial_{sss}\gamma\|_{L^\infty(I\times\mathbb{R})},$$

and set $J_r = [-T_r, T_r] \cap (-T, T)$. There exists a unique $\sigma \in \mathcal{C}(J_r \times \mathbb{R}, \mathbb{R})$ such that $\sigma(0, \cdot) = \sigma_0$ and $\sigma_t := \sigma(t, \cdot)$ is a reparametrization of γ_t for Γ_t, for every $t \in J_r$. The function F defined on J_r by $F(t) := F_{\gamma_t,\sigma_t,r}(\Gamma_t)$ is Lipschitz continuous on J_r and satisfies for almost every $t \in J_r$ the inequality

$$|F'(t)| \le KF(t).$$

Therefore by Gronwall Lemma

$$F(t) \le \exp(K|t|)F(0), \qquad \forall\, t \in J_r,$$

and in particular

$$d_{\mathcal{P}}(\Gamma_t, \gamma_t) < r, \qquad \forall\, t \in J_r.$$

Since, as mentionned above, F can be related to more usual forms of discrepancy measures (like Haussdorff disntance e.g.), the final estimate in the previous theorem allows to control the "distance" between a reference smooth solution and a rough solution by constants which *only* depend on the smooth solution.

3 A short excursion into two related worlds (after H. Hasimoto)

If $\gamma \in L^\infty(I, H_{loc}^{3/2}(\mathbb{R}, \mathbb{R}^3))$ is a solution to the binormal curvature flow equation (1), then the map $u := \partial_s\gamma \in L^\infty(I, H_{loc}^{1/2}(\mathbb{R}, S^2))$, parametrizing the evolution in time of the unit tangent vectors to the curves $\gamma(t, \cdot)$, satisfies

$$\partial_t u = \partial_t\partial_s\gamma = \partial_s\partial_t\gamma = \partial_s(u \times \partial_s u) \tag{4}$$

in the sense of distributions on $I \times \mathbb{R}$. In other words, u is a solution of the Schrödinger map equation (4) for maps from $\mathbb{R}$ to S^2, and the binormal curvature flow equation is therefore a primitive equation of the Schrödinger map equation.

Conversely, let $u \in L^{\infty}(I, H_{loc}^{1/2}(\mathbb{R}, S^2))$ be a solution to the Schrödinger map equation (4) and define the function $\Gamma_u \in L^{\infty}(I, H_{loc}^{3/2}(\mathbb{R}, \mathbb{R}^3))$ by

$$\Gamma_u(t, s) := \int_0^s u(t, z) \, dz. \tag{5}$$

In the sense of distributions on $I \times \mathbb{R}$, we have

$$\partial_s \left(\partial_t \Gamma_u - \partial_s \Gamma_u \times \partial_{ss} \Gamma_u \right) = 0. \tag{6}$$

By construction, the primitive curves $\Gamma_u(t, \cdot)$ all have their base point $\Gamma_u(t, 0)$ fixed at the origin. For smooth solutions, equation (6) would directly imply the existence of a function c_u depending only time only and such that

$$\gamma_u(t, s) := \Gamma_u(t, s) + c(t)$$

is a solution to the binormal curvature flow equation (1). The function c_u indeed represents the evolution in time of the actual base point of the curves. For solutions barely in $H^{3/2}$ the same conclusion holds with slightly more involved argument.

Besides the above sort of equivalence between the binormal curvature flow equation and the Schrödinger map equation presented just above, H. Hasimoto [6] (Japan once more!) exhibited in 1972 an intimate relation between the binormal curvature flow equation (1) and the cubic focusing nonlinear Schrödinger equation. Let $\gamma : I \times \mathbb{R} \to \mathbb{R}^3$ be a smooth and biregular solution of the binormal curvature flow equation (1), and denote by κ and T respectively the curvature and torsion functions of γ. Then, the function Ψ defined on $I \times \mathbb{R}$ by the Hasimoto transform

$$\psi(t, s) := \kappa(t, s) \exp \left(i \int_0^s T(t, z) \, dz \right)$$

is a solution to

$$\partial_t \psi + \partial_{ss} \psi + \frac{1}{2}(|\psi|^2 - A(t))\psi = 0$$

where

$$A(t) := \left(2\frac{\partial_{ss}\kappa - \kappa T^2}{\kappa} + \kappa^2 \right) (t, 0).$$

If γ is 2π-periodic in s, that is if $\gamma : I \times T^1 \to \mathbb{R}^3$, then Ψ is only quasiperiodic unless $\int_0^{2\pi} T(t, z) \, dz \in 2\pi\mathbb{Z}$. Nevertheless, it is possible to recover a 2π-periodic

function Ψ by means of a Galilean transform. One can also get rid of the $A(t)$ factor by means of a phase shift. More precisely, the function

$$\Psi(t, s) := \psi(t, s - \frac{b}{2}t) \exp\left(i(bs - b^2 t - \int_0^t \frac{1}{2} A(z)\, dz)\right),$$

where

$$b := 1 - \frac{1}{2\pi} \int_0^{2\pi} T(0, z)\, dz = 1 - \frac{1}{2\pi} \int_0^{2\pi} T(t, z)\, dz$$

is well-defined on $I \times T^1$ and is a solution to the cubic focusing nonlinear Schrödinger equation

$$\partial_t \Psi + \partial_{ss}\Psi + \frac{1}{2}|\Psi|^2 \Psi = 0 \tag{7}$$

on $I \times T^1$.

Equation (7) is integrable and known to be solvable by the inverse scattering method since the works of Zakharov and Shabat [20] in 1971 for the vanishing case and Ma and Ablowitz [18] in 1981 for the periodic case. Therefore the binormal curvature flow equation (1) and the Schrödinger map equation (4) are also integrable, in the weak sense that they can be mapped to an integrable equation. In particular, solitons of NLS give rise to solitons on a filament (Hasimoto's original title). Notice however that the inverse of the Hasimoto transform, whenever it is well defined[5], involves a big deal of nonlinearity.

4 A beautiful class of special solutions (after S. Kida)

Besides the soliton solutions of Hasimoto, another special class of solutions of (1) was singled out in 1981 by S. Kida [14] (guess what ?!). He studied the set of initial curves in $\mathbb{R}^3$ for which the solution map of the binormal curvature flow equation reduces to a family of rigid motions. Using the symmetries of (1) and its conservations laws, such motions are necessarily the superposition of a constant speed rotation around a fixed axis (which we may always assume to be the x_3-axis after a fixed rotation) and a constant speed translation parallel to that same axis. Following [14], we denote by Ω and V the speeds of rotation around e_3 and of translation along e_3 respectively, and by C the speed of the slipping motion of the curve along itself[6], so that

$$\gamma(t, s) = \gamma(0, s - Ct) \cdot \begin{pmatrix} \cos(\Omega t) & \sin(\Omega t) & 0 \\ -\sin(\Omega t) & \cos(\Omega t) & 0 \\ 0 & 0 & 1 \end{pmatrix} + V(0, 0, t)$$

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[5] Vanishing of Ψ yields underdetermination.

[6] Even though the speed given by the binormal curvature flow equation is perpendicular to the tangent vector, in a non-orthogonal frame like $(\partial_s \gamma, \gamma \times e_3, e_3)$ the component C of $\partial_t \gamma$ along $\partial_s \gamma$ may not be zero. For the rigid motions we consider, C is a constant function of space and time.

and therefore γ satisfies the additional equation

$$\partial_t\gamma = -C\partial_s\gamma + \Omega e_3 \times \gamma + Ve_3. \tag{8}$$

Combining (1) and (8) we write

$$\partial_s\gamma \times \partial_{ss}\gamma = -C\partial_s\gamma + \Omega e_3 \times \gamma + Ve_3. \tag{9}$$

Taking the scalar product of (9) with $\partial_s\gamma$ yields

$$-C + \Omega r^2\theta' + Vz' = 0, \tag{10}$$

where we wrote $(\gamma^1(0,\cdot),\gamma^2(0,\cdot)) =: (r(\cdot)\cos(\theta(\cdot)), r(\cdot)\sin(\theta(\cdot)))$ and $\gamma^3(0,\cdot) =: z(\cdot)$. Taking the vector product of (9) with $\partial_s\gamma$ instead yields

$$z'' = -\Omega rr' = -\frac{\Omega}{2}R', \tag{11}$$

where $R = r^2$. After integration, (11) leads to

$$z' = \frac{\Omega}{2}[A - R] \tag{12}$$

for some integration constant $A \in \mathbb{R}$. Assume that $\Omega \neq 0$. Combining (10) and (12) we obtain

$$\theta' = \frac{1}{2}V + (C - \frac{1}{2}AV\Omega)/(\Omega R), \tag{13}$$

and then combining (1) with (12) and (13) we finally obtain

$$(R')^2 + f(R) = 0, \tag{14}$$

where

$$f(R) = \Omega^2 R^3 + (V^2 - 2A\Omega^2)R^2 + (4V(C - \frac{1}{2}AV\Omega)/\Omega + \Omega^2 A^2 - 4)R \\ + 4(C - \frac{1}{2}AV\Omega)^2/\Omega^2. \tag{15}$$

Playing on the many parameters A, C, V, Ω, it is then possible to identify a subclass leading to smooth closed curves. Nice examples and pictures can be found in the original paper [14]. In [9], we have used this family in order to construct counter-examples to some form of continuity properties for the Schrödinger map equation (4) mentionned in the previous section.

5 A puzzling class of special solutions

Since open problems are particularly welcomed in RIMS proceedings, we shall end this short survey by numerical curiosities which presumably have interesting theoretical roots (and may be related to integrability).
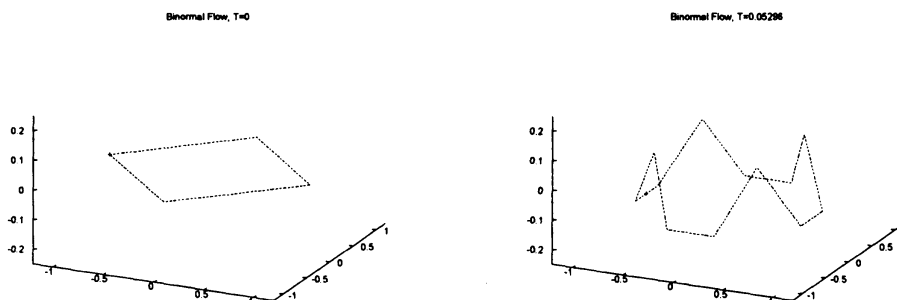
As mentionned in the second section, our existence theory in [10] allows to consider initial curves for (1) (or rather for its extended formulation) that are barely Lipschitz, and in particular polygonal lines. In order to get some insight on the corresponding solutions, we performed numerical simulations according to an algorithm due to Buttke [4] (after all even smooth curves when discretized become polygonal lines...). If γ is a solution to (1), the corresponding tangent vector $u := \partial_s \gamma$: $I \times (\mathbb{R}/\ell\mathbb{Z}) \to S^1$ satisfies the Schrödinger map equation (4). Buttke's algorithm simulates the binormal curvature flow equation (1) by the Crank-Nicolson type discretization

$$\frac{u_n^{j+1} - u_n^j}{\Delta t} = \left(\frac{u_n^j + u_n^{j+1}}{2} \right) \times \left(\frac{u_{n-1}^j + u_{n+1}^j}{2(\Delta x)^2} + \frac{u_{n-1}^{j+1} + u_{n+1}^{j+1}}{2(\Delta x)^2} \right)$$
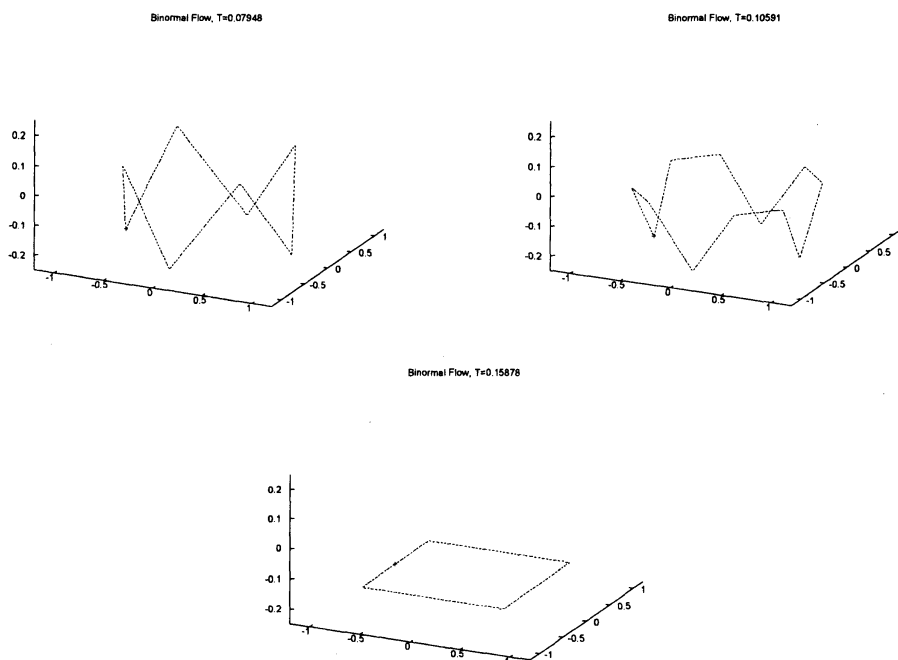
of (4), and numerical integration to recover γ from u. The implicit scheme for u can be resolved by a fixed point method if $\Delta t < \sigma (\Delta x)^2$ for some explicit $\sigma > 0$; it has the advantage that the constraint $|u_n^j| = 1$, the mean $\sum_n u_n^j$, and the discrete squared $\dot{H}^1$ norm $\sum_n |u_n^j - u_{n+1}^j|^2$ are conserved quantities of the scheme.

Doing so, we have observed some phenomena which we did not expect, and for which we have no clear explanation[7].

In the following pictures, we present the shape of the simulated solution at different (well chosen) times for a 5000 points discretization of a unit square parallel to the xy-plane as initial datum.

Binormal Flow, T=0

Binormal Flow, T=0.05296

[7]L. Vega (Univ. of Bilbao) recently told us that he believes to have some kind of explanation and should publish it shortly. See also [2] for related questions.

Binormal Flow, T=0.07948

Binormal Flow, T=0.10591

Binormal Flow, T=0.15878

As it may suggests, at some times close to 0.05296, 0.07948, 0.10591 and 0.15878, the (or "a") solution seems to turn again polygonal. Notice that the symmetries of the square are preserved (intermediate shapes have 8 or 12 sides), and that the square in the last picture is rotated by $\pi/4$ with respect to the initial one. At times intermediate between those special moments the simulated solution looks quite irregular and has not been represented. Also, running the simulation further in time suggests that this sequence is reproduced in a (quasi)periodic manner. It is of course tempting to believe that solitons could play a role here (notice in particular the ratios of those special times); on the other hand polygons are the worst possible examples for the Hasimoto transform (the solution is not smooth and the curvature vanishes almost everywhere!).

This kind of phenomena seems rather robust to some changes in the initial polygon. This history of (1) so far should convince japanese mathematicians to spend some time on that problem !

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