

# Multiplicity of solutions in the Lotka-Volterra competition with cross-diffusion

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## 1 Introduction

This is a joint project with Yuan Lou (Ohio State University), Wei-Ming Ni (University of Minnesota and East China Normal University) concerning mathematical analysis, and Masaharu Nagayama (Hokkaido University), Tatsuki Mori (Ryukoku University) concerning numerical computation.

In an attempt to model segregation phenomena in population dynamics, Shigesada, Kawasaki and Teramoto [7] in 1979 incorporated the inter-competition system. In particular, the following system was proposed

$$\begin{cases} u_t = \Delta[(d_1 + \rho_{11}u + \rho_{12}v)u] + u(a_1 - b_1u - c_1v), & \text{in } \Omega \times (0, \infty), \\ v_t = \Delta[(d_2 + \rho_{21}u + \rho_{22}v)v] + v(a_2 - b_2u - c_2v), & \text{in } \Omega \times (0, \infty), \\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0, & \text{on } \partial\Omega \times (0, \infty), \\ u(x, 0) = u_0(x), v(x, 0) = v_0(x), & \text{in } \Omega, \end{cases} \quad (1.1)$$

where  $\Omega$  is a bounded domain  $R^N (N \geq 1)$  with smooth boundary  $\partial\Omega$ . Here  $u$  and  $v$  represent the densities of two competing species. The constants  $a_j, b_j, c_j$  and  $d_j$  ( $j = 1, 2$ ) are all positive, where  $a_1, a_2$  denote the intrinsic growth rates of these two species,  $b_1$  and  $c_2$  account for intra-specific competitions while  $b_2, c_1$  account for inter-specific competitions, and  $d_1, d_2$  are their diffusion rates. The constants  $\rho_{11}, \rho_{22}$  represent intra-specific population pressures, also known as self-diffusion rates, and  $\rho_{12}, \rho_{21}$  are the coefficients of inter-specific population pressures, also known as cross-diffusion rates.

For convenience, we set  $A := a_1/a_2, B := b_1/b_2, C := c_1/c_2$ . If  $B < C$ , we call it the strong competition case and  $B > C$  the weak competition case.

If  $\rho_{11} = \rho_{12} = \rho_{21} = \rho_{22} = 0$ , no nonconstant steady state can exist for any diffusion rates  $d_1, d_2$ . On the other hand, it seems not entirely reasonable to add just diffusions to models in population dynamics, since individuals do not move around completely randomly. In particular, while modeling segregation phenomena for two competing species one must take into account the cross-diffusion pressures.

Mimura and his collaborators started mathematical analysis around 1980 (see, e.g. Mimura [4]). Considerable work has been done concerning the global existence of solutions to systems (1.1) under various hypotheses. A priori estimates are crucial to obtain

the global existence. As for recent progress including stationary problems, see Ni [5], Ni [6], Yagi[9] and Yamada [10].

We are interested in the case  $\rho_{11} = \rho_{21} = \rho_{22} = 0$  to clarify the effect of the cross-diffusion. By putting  $r := \rho_{12}/d_1$ , we have

$$\begin{cases} u_t = d_1 \Delta[(1+r v)u] + u(a_1 - b_1 u - c_1 v) & \text{in } \Omega \times (0, \infty), \\ v_t = d_2 \Delta v + v(a_2 - b_2 u - c_2 v) & \text{in } \Omega \times (0, \infty), \\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u(x, 0) = u_0(x), v(x, 0) = v_0(x) & \text{in } \Omega. \end{cases} \quad (1.2)$$

The stationary problem for the above equation is

$$\begin{cases} d_1 \Delta[(1+r v)u] + u(a_1 - b_1 u - c_1 v) = 0 & \text{in } \Omega, \\ d_2 \Delta v + v(a_2 - b_2 u - c_2 v) = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 & \text{on } \partial\Omega, \\ u(x) > 0, \quad v(x) > 0 & \text{in } \Omega. \end{cases} \quad (1.3)$$

Even for these special cases, it is not easy to understand the structure of stationary solutions, and the stability of them.

## 2 Limiting equations

Let us consider about a time-dependent limiting equation for (1.2) as  $r \rightarrow \infty$ .

It seems from numerical computations for (1.2) and limiting equations for (1.3) derived in Lou-Ni [1], [2] that we suspect that limiting equations with  $r \rightarrow \infty$  for (1.2) is the following equation.

Unknown functions are  $\tau(t)$ ,  $v(x, t)$ , and

$$\begin{cases} \frac{d}{dt} \left( \int_{\Omega} \frac{\tau}{v} dx \right) = \int_{\Omega} \frac{\tau}{v} (a_1 - b_1 \frac{\tau}{v} - c_1 v) dx & \text{in } (0, \infty), \\ \frac{\partial v}{\partial t} = d_2 \Delta v + v(a_2 - c_2 v) - b_2 \tau & \text{in } \Omega \times (0, \infty), \\ \frac{\partial v}{\partial n} = 0 & \text{on } \partial\Omega \times (0, \infty), \\ \tau(0) = \tau_0 > 0, \quad v(x, 0) = v_0(x) > 0 & \text{in } \Omega. \end{cases} \quad (2.1)$$

This is formally derived by rewriting the first equation in (1.2) as

$$u_t = d_1 r \Delta \left[ \left( \frac{1}{r} + v \right) u \right] + u(a_1 - b_1 u - c_1 v),$$

and letting  $r \rightarrow \infty$  after dividing by  $r$ . In fact, this implies that  $u(x, t)v(x, t)$  is equal to a constant  $\tau(t)$  depending only on  $t$ . On the other hand, by integrating the above equation in  $\Omega$ , we get

$$\frac{d}{dt} \left( \int_{\Omega} u dx \right) = \int_{\Omega} u(a_1 - b_1 u - c_1 v) dx.$$

Thus, we obtain (3.1).

The stationary limiting equation for (3.1) becomes as follows:

$$\begin{cases} \int_{\Omega} \frac{\tau}{v} (a_1 - b_1 \frac{\tau}{v} - c_1 v) dx = 0, \\ d_2 \Delta v + v(a_2 - c_2 v) - b_2 \tau = 0 \quad \text{in } \Omega, \\ \frac{\partial v}{\partial n} = 0 \quad \text{on } \partial\Omega, \\ \tau > 0, \quad v(x) > 0 \quad \text{in } \Omega. \end{cases} \quad (2.2)$$

### 3 1-dimensional case

Now, we consider the 1-dimensional case with  $\Omega = (0, 1)$ . The stationary limiting equation (2.2) becomes as follows:

$$\begin{cases} \int_0^1 \frac{1}{v} (a_1 - b_1 \frac{\tau}{v}) dx - c_1 = 0, \\ d_2 v_{xx} + v(a_2 - b_2 \frac{\tau}{v} - c_2 v) = 0, \quad \text{in } (0, 1), \\ v_x(0) = v_x(1) = 0, \\ v > 0, \quad \text{in } (0, 1). \end{cases}$$

Due to the scaling and reflection properties of solutions to autonomous ordinary differential equations, all solutions to the (3.1) are obtained by several reflections and a suitable re-scaling from solutions of the following system:

$$\begin{cases} \int_0^1 \frac{1}{v} (a_1 - b_1 \frac{\tau}{v}) dx - c_1 = 0, \\ d_2 v_{xx} + v(a_2 - b_2 \frac{\tau}{v} - c_2 v) = 0 \quad \text{in } (0, 1), \\ v_x(0) = v_x(1) = 0, \\ \tau > 0, \quad v > 0, \quad v_x > 0, \quad \text{in } (0, 1). \end{cases} \quad (3.1)$$

Now, we will discuss about the structure of stationary solutions and their stability.

This system (3.1) consists of a nonlinear elliptic equation and an integral constraint. As far as existence and non-existence in one dimensional domain are concerned, Lou-Ni-Yotsutani [3] obtained nearly complete knowledge. They also obtained the precise qualitative behavior of solutions to this limiting system as the diffusion rate varies.

Their basic approach is to convert the problem of solving the system to a problem of solving its “representation” in a different parameter space. This is first done *without* the integral constraint, and then they use the integral constraint to find the “solution curve” in the new parameter space. This turns out to be a powerful method as it gives fairly precise information about the solutions.

We have recently made clear the remained delicate parts due to the explicit representation by elliptic functions.

We summarized the structure of solutions of (3.1). We concentrate on the case

$$B < C \quad (\text{strong competition case}).$$

The following two theorem are due to [3].

**Theorem 3.1** (Existence) *Suppose that  $B < C$ . If*

$$\max \left\{ 0, \frac{B + C - 2A}{C - B} \right\} \frac{a_2}{\pi^2} < d_2 < \frac{a_2}{\pi^2},$$

*then there exists a solution  $(v(x), \tau)$  of (3.1).*

**Theorem 3.2** (Nonexistence) *Suppose that  $B < C$ .*

*(i) If  $d_2 \geq \frac{a_2}{\pi^2}$ , then there exists no solution of (3.1).*

*(ii) If  $A < B$ , there exists no solution of (3.1).*

*(iii) If  $B \leq A < \frac{B+C}{2}$ , then there exists a  $d_2^* = d_2^*(A, B, C, a_2) > 0$  such that there exists no solution of (3.1) for  $d_2 \in (0, d_2^*]$ .*

We see that the above theorem is sharp by the following theorems. The existence region depending on the the ratio  $C/B$ . The situation drastically changes at  $C/B = 7/3$ .

**Theorem 3.3** *Suppose that  $B < C \leq 7B/3$ . (3.1) has a solution  $(v(x), \tau)$  if and only if  $d_2$  satisfies*

$$\max \left\{ 0, \frac{B + C - 2A}{C - B} \right\} \frac{a_2}{\pi^2} < d_2 < \frac{a_2}{\pi^2}.$$

*Moreover, the solution is unique.*

**Theorem 3.4** *Suppose that  $7B/3 < C$ . (3.1) has the unique solution  $(v(x), \tau)$  if*

$$\max \left\{ 0, \frac{B + C - 2A}{C - B} \right\} \frac{a_2}{\pi^2} < d_2 < \frac{a_2}{\pi^2}$$

*Moreover, there exists the only one connected non-empty open set  $D$  with*

$$D \subset \left\{ (A, d_2) : B < A < \frac{B+C}{2}, 0 < d_2 < \left\{ \frac{B+C-2A}{C-B} \right\} \frac{a_2}{\pi^2} \right\}$$

*such that (3.1) has exactly two solutions  $(v(x), \tau)$  if and only if  $d_2 \in D$ .*

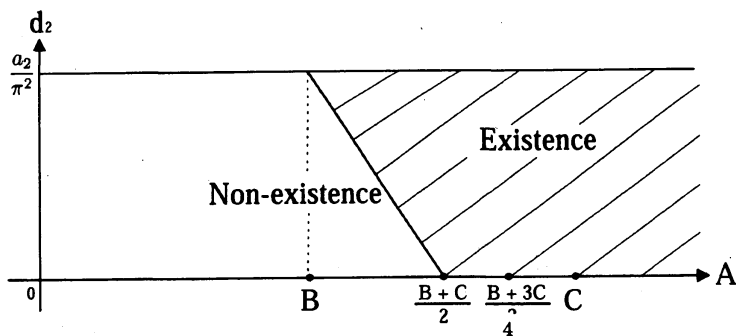


Figure 3.1: Case  $B < C \leq 7B/3$

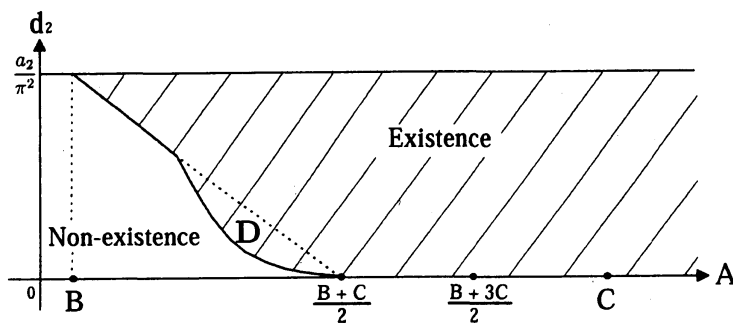


Figure 3.2: Case  $7B/3 < C$

The following theorems in [3] give the shape of solutions to (3.1) as  $d_2 \uparrow a_2/\pi^2$ .

**Theorem 3.5** (Shape of solutions as  $d_2 \uparrow a_2/\pi^2$ ) Suppose that  $B < C$ .

Let  $(v(x, d_2), \tau(d_2))$  be solutions of (3.1). If  $A \geq B$ , then

$$v(x; d_2) \rightarrow 0, \quad \frac{v(x; d_2) - v(0; d_2)}{v(1; d_2) - v(0; d_2)} \rightarrow \frac{1 - \cos(\pi x)}{2},$$

$$\frac{\tau(d_2)}{v(x; d_2)} \rightarrow \frac{a_2}{b_2} \cdot \frac{1}{1 - \sqrt{1 - \frac{B}{A}} \cos(\pi x)}$$

uniformly on  $[0, 1]$  as  $d_2 \uparrow a_2/\pi^2$ .

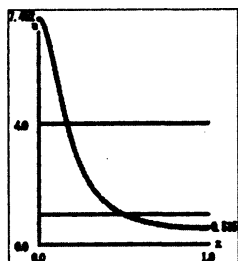


Figure 3.3:  $u$  as  $d_2 \uparrow a_2/\pi^2$

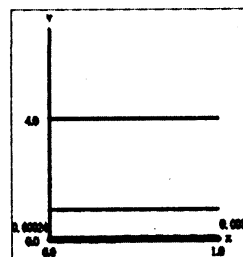


Figure 3.4:  $v$  as  $d_2 \uparrow a_2/\pi^2$

The following theorems in [3] give the shape of solutions to (3.1) as  $d_2 \downarrow 0$ . A new number  $(B + 3C)/4$  appears. The shape is drastically change at  $A = (B + 3C)/4$ .

**Theorem 3.6** (Shape of solutions as  $d_2 \rightarrow 0$  for  $A < \frac{B+3C}{4}$  ) Suppose that  $B \neq C$ . Let  $(v(x, d_2), \tau(d_2))$  be solutions of (3.1). If  $A < \frac{B+3C}{4}$  and  $B < C$ , then

$$v(0; d_2) \rightarrow 2 \cdot \frac{a_2}{c_2} \cdot \frac{\frac{B+3C}{4} - A}{C - B}, \quad v(x; d_2) \rightarrow \frac{a_2}{c_2} \cdot \frac{A - B}{C - B} \quad \text{for } x > 0,$$

$$\frac{\tau(d_2)}{v(0; d_2)} \rightarrow \frac{a_2}{2c_2} \cdot \frac{C - A}{C - B} \cdot \frac{A - B}{\frac{B+3C}{4} - A}, \quad \frac{\tau(d_2)}{v(x; d_2)} \rightarrow \frac{a_2}{b_2} \cdot \frac{C - A}{C - B} \quad \text{for } x > 0.$$

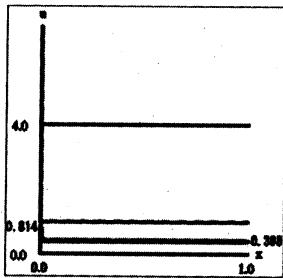


Figure 3.5: u for  $A \leq (B + 3C)/4$

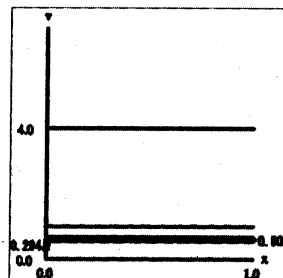


Figure 3.6: v for  $A \leq (B + 3C)/4$

**Theorem 3.7** (Shape of solutions as  $d_2 \rightarrow 0$  for  $A \geq \frac{B+3C}{4}$  ) Suppose that  $B \neq C$ . Let  $(v(x, d_2), \tau(d_2))$  be solutions of (3.1). If  $B < C$  and  $A \geq \frac{B+3C}{4}$ , then

$$v(0; d_2) \rightarrow 0, \quad v(x; d_2) \rightarrow \frac{3a_2}{4c_2} \quad \text{for } x > 0,$$

$$\frac{\tau(d_2)}{v(0; d_2)} \rightarrow \infty, \quad \frac{\tau(d_2)}{v(x; d_2)} \rightarrow \frac{a_2}{4c_2} \quad \text{for } x > 0, \text{ as } d_2 \rightarrow 0.$$

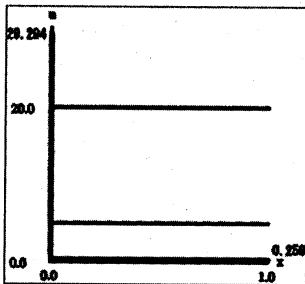


Figure 3.7: u for  $(B + 3C)/4 < A$

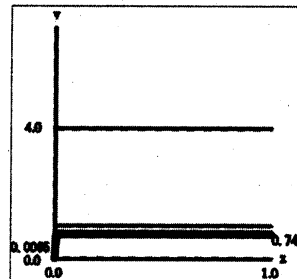


Figure 3.8: v for  $(B + 3C)/4 < A$

Let us comment on the stability of stationary solutions in 1-dim case.

The following Figure 3.9 shows numerical results for

$$d_1 = 1, \quad d_2 = *, \quad r = 700,000$$

$$a_2 = *, \quad b_2 = 1, \quad c_2 = 2.$$

$$a_2 = 1, \quad b_2 = 1, \quad c_2 = 1.$$

We note that  $C < 7B/3$ ,  $(B + C)/2 = 1.5$  and  $(B + 3C)/4 = 1.75$ .

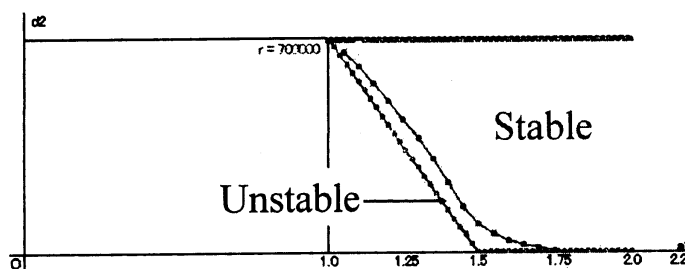


Figure 3.9: Stability and instability

Yaping Wu[8] gave a proof of instability for

$$d_2 \text{ sufficiently small with } (B + C)/2 < A < (B + 3C)/4.$$

Recently, she has also given proofs of the asymptotically stability for

$$d_2 (< a_2/\pi^2) \text{ sufficiently close to } a_2/\pi^2 \text{ with } (B + C)/2 < A < (B + 3C)/4,$$

$$d_2 \text{ sufficiently small with } (B + 3C)/4 < A.$$

## 4 Multi-dimensional problem

We have done various numerical computations for the case  $\Omega$  is rectangles in 2-dimensional space. It seems that the structure of stable stationary solutions is essentially very similar to 1-dimensional case, though there are much varieties of shape of solutions in 2-dimensional case than in one-dimensional case.

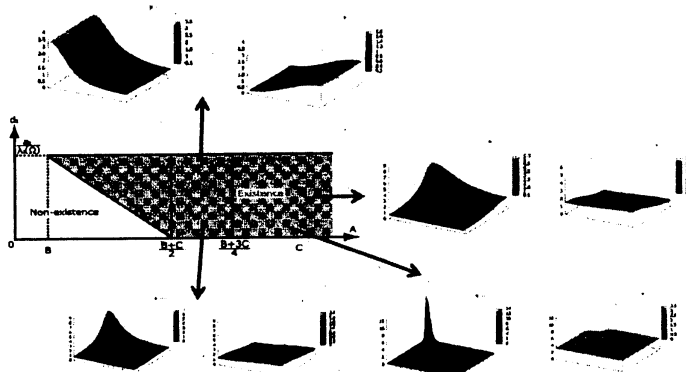


Figure 4.1: 2D global

Now, we will state some mathematical results. We prepare notations. Let

$$\lambda_0 = 0 < \lambda_1 \leq \lambda_2 \leq \dots$$

$$\varphi_0 = \text{const.}, \quad \varphi_1, \quad \varphi_2, \quad \dots$$

be eigen values and corresponding eigen functions of  $-\Delta$  in  $\Omega \subset R^N$  with Neumann boundary.

**Theorem 4.1** *Suppose that  $N \leq 3$  and  $\lambda_1$  be a simple eigen values with an eigen function  $\varphi_1$ . Then, there exists exactly two positive non-constant solutions  $(v_-, \tau_-)$  and  $(v_+, \tau_+)$  of (2.2) for  $d_2$  sufficiently close to  $a_2/\lambda_1$  with  $d_2 < a_2/\lambda_1$*

Moreover,

$$\begin{aligned} \tau &\rightarrow 0, \\ \frac{\tau_{\pm}(d_2)}{v_{\pm}(x; d_2)} &\rightarrow \frac{a_2}{b_2} \cdot \frac{1}{1 + \mu_{\pm} \varphi_1(x)} \end{aligned}$$

as  $d_2 \uparrow a_2/\lambda_1$ , where  $\mu_-, \mu_+$  ( $\mu_- < 0 < \mu_+$ ) are solutions of

$$\frac{\int_{\Omega} (1 + \mu \varphi_1(x))^{-2} dx}{\int_{\Omega} (1 + \mu \varphi_1(x))^{-1} dx} = \frac{A}{B}.$$

**Remark.** *The set  $\{(v_-, \tau_-), (v_+, \tau_+)\}$  is uniquely determined though there is a freedom to pick up  $\varphi_1$ . The condition  $N \leq 3$  comes from Harnack's inequality in our proof.*

**Remark.** *For  $N = 1$ ,  $\Omega = (0, 1)$ , it is easy to see that*

$$\lambda_1 = \pi^2, \quad \varphi_1(x) = \cos \pi x, \quad \frac{1}{1 - \mu^2} = \frac{A}{B}, \quad \mu_{\pm} = \pm \sqrt{1 - \frac{B}{A}}.$$

**Remark.** *For  $N = 2$ ,  $\Omega = (0, 1) \times (0, \ell)$  with  $0 < \ell < 1$ , it is easy to see that*

$$\lambda_1 = \pi^2, \quad \varphi_1(x, y) = \cos \pi x, \quad \frac{1}{1 - \mu^2} = \frac{A}{B}, \quad \mu_{\pm} = \pm \sqrt{1 - \frac{B}{A}}.$$

**Remark.** *A similar theorem holds for the equation (1.3) with sufficiently large  $r$ .*

**Remark.** *Suppose that  $N \leq 3$  and  $\lambda_1$  be a simple eigen values. Then,  $(v_-, \tau_-)$  and  $(v_+, \tau_+)$  defined by Theorem 4.1 are asymptotically stable for  $d_2$  sufficiently close to  $a_2/\lambda_1$  with  $d_2 < a_2/\lambda_1$ .*

The following general lemma plays crucial role to prove Theorems 4.1.



**Lemma 4.2** Suppose that  $N \geq 1$  and  $\varphi_1$  be eigen values corresponding to  $\lambda_1$ . Let  $g(\mu)$  be defined by

$$g(\mu) := \frac{\int_{\Omega} (1 + \mu \varphi_1(x))^{-2} dx}{\int_{\Omega} (1 + \mu \varphi_1(x))^{-1} dx}$$

for  $\mu \in (-1/\max_{\bar{\Omega}} \varphi_1, -1/\min_{\bar{\Omega}} \varphi_1)$ . Then

$$\frac{dg(\mu)}{d\mu} = \begin{cases} + & \text{for } \mu > 0, \\ 0 & \text{for } \mu = 0, \\ - & \text{for } \mu < 0. \end{cases}$$

Moreover, for  $N \leq 4$ ,

$$\begin{cases} g(\mu) \rightarrow \infty & \text{as } \mu \uparrow -1/\min_{\bar{\Omega}} \varphi_1, \\ g(\mu) \rightarrow \infty & \text{as } \mu \downarrow -1/\max_{\bar{\Omega}} \varphi_1. \end{cases}$$

#### Idea of a proof of Theorem 4.1.

Step 1:  $\{\tau(d_2)\}$  is bounded, and  $\{v(x; d_2)\}$  is  $L^\infty$ -bounded as  $d_2 \uparrow a_2/\lambda_1$ .

Step 2: There exists a sequence  $\{d_{2,j}\}$  such that  $\tau(d_{2,j}) \rightarrow \hat{\tau}$  and  $v(x; d_{2,j}) \rightarrow \hat{v}(x)$ .

Step 3:  $\hat{v}(x) \equiv 0$  and  $\hat{\tau} = 0$ .

Step 4:  $\{\tau(d_2)/v(x; d_2)\}$  is  $L^2$ -bounded.  $\{v(x; d_2)/\tau(d_2)\}$  is  $L^\infty$ -bounded. We use the assumption  $N \leq 3$  to apply Harnach's inequality to  $\{v(x; d_2)/\tau(d_2)\}$ .

Step 5: There exists a sequence  $\{d_{2,j}\}$  such that  $v(x; d_{2,j})/\tau(d_{2,j}) \rightarrow w(x)$ , and  $w(x) = a_2^{-1}b_2 \cdot (1 + \mu\varphi_1(x))$  for some  $\mu$ . Moreover,  $w(x) > 0$ .

Step 6: Substitute  $w(x)$  into the first equation, we obtain the equation for  $\mu$ .

Step 7: Apply the implicit function theorem to  $w := v/\tau$  as follows. Let us take  $p$  with  $p > N$ . Define

$$F : W_v^{2,p} \times R^1 \times (0, \infty) \rightarrow L^p \times R^1$$

by

$$F(w, \tau, d_2) := \left( d_2 \Delta w + w(a_2 - c_2 \tau w) - b_2, \int_{\Omega} \frac{1}{w} \left( a_1 - \frac{b_1}{w} \right) dx - c_1 \tau |\Omega| \right).$$

We see that

$$F\left(\frac{a_2}{b_2} \cdot (1 + \mu_{\pm} \varphi_1(x)), 0, \frac{a_2}{\lambda}\right) = (0, 0),$$

and

$$\text{Ker of } DF_{(w,\tau)}\left(\frac{a_2}{b_2} \cdot (1 + \mu_{\pm} \varphi_1(x)), 0, \frac{a_2}{\lambda_1}\right) = (0, 0).$$

Thus we can apply the implicit function theorem. Moreover, we obtain

$$\tau' \left( \frac{a_2}{\lambda_1} \right) < 0, \quad \tau \left( \frac{a_2}{\lambda_1} \right) = 0.$$

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