Center Manifold Theorem for Integral Equations

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1 Introduction

In this paper we are concerned with the integral equation (with infinite delay)

$$x(t) = \int_{-\infty}^{t} K(t-s)x(s)ds + f(x_{t}),$$  \(E\)

where \(K\) is a measurable \(m \times m\) matrix valued function with complex components satisfying the condition \(\int_{0}^{\infty} \|K(t)\|e^{\rho t}dt < \infty\) and \(\text{ess sup}\{\|K(t)\|e^{\rho t} : t \geq 0\} < \infty\), and \(f\) is a nonlinear term belonging to the space \(C^{1}(X;\mathbb{C}^{m})\), the set of all continuously (Fréchet) differentiable functions mapping \(X\) into \(\mathbb{C}^{m}\), with the property that \(f(0) = 0\) and \(Df(0) = 0\); here, \(\rho\) is a positive constant which is fixed throughout the paper, and \(X := L_{\rho}^{1}(\mathbb{R}^{-};\mathbb{C}^{m})\), \(\mathbb{R}^{-} := (-\infty, 0]\), is a Banach space (employed throughout the paper as the phase space for Eq. \((E)\)) equipped with norm \(\|\phi\|_{X} := \int_{-\infty}^{0} |\phi(\theta)|e^{\rho \theta}d\theta\) \((\forall \phi \in X)\), and \(x_{t}\) is an element in \(X\) defined as \(x_{t}(\theta) = x(t + \theta)\) for \(\theta \in \mathbb{R}^{-}\). The linearized equation of Eq. \((E)\) (around the equilibrium point \(0\)) is given by

$$x(t) = \int_{-\infty}^{t} K(t-s)x(s)ds,$$  \(1\)

which possesses the characteristic matrix \(\Delta(\lambda) := E_m - \int_{0}^{\infty} K(t)e^{-\lambda t}dt\) \((\text{Re} \lambda < -\rho)\); here \(E_m\) is the \(m \times m\) unit matrix. Recently, Diekmann and Gyllenberg [3] have treated Eq. \((E)\), and established the principle of linearized stability for integral equations. In the paper, as a further development in the stability problem of Eq. \((E)\), we treat the case that the equilibrium point zero is nonhyperbolic (that is, the set \(\{\lambda \in \mathbb{C} : \det \Delta(\lambda) = 0 & \text{Re} \lambda = 0\}\) is nonempty), and establish center manifold theorem for Eq. \((E)\); and then we will investigate stability properties of the zero solution of Eq. \((E)\) in the critical case.
Several preparatory results for integral equations

In this section, following [6] we summarize several preliminary results necessary for our later arguments. Eq. (E) can be formulated as an abstract equation on the space $X$ of the form

$$x(t) = L(x_t) + f(x_t),$$

where $L : X \to C^m$ is a bounded linear operator defined by $L(\phi) := \int_{-\infty}^{0} K(-\theta)\phi(\theta)d\theta$ for $\phi \in X$. Let us consider Eq. (E) with the initial condition

$$x_{\sigma} = \phi, \quad \text{that is,} \quad x(\sigma + \theta) = \phi(\theta) \quad \text{for} \quad \theta \in \mathbb{R}^-,$$

where $(\sigma, \phi) \in \mathbb{R} \times X$ is given arbitrarily. A function $x : (-\infty, a) \to C^m$ is said to be a solution of the initial value problem (E)-(2) on the interval $(\sigma, a)$ if $x$ satisfies the following conditions: (i) $x_{\sigma} = \phi$, that is, $x(\sigma + \theta) = \phi(\theta)$ for $\theta \in \mathbb{R}^-$; (ii) $x \in L^1_{loc}[\sigma, a)$, $x$ is locally integrable on $[\sigma, a)$; (iii) $x(t) = L(x_t) + f(x_t)$ for $t \in (\sigma, a)$.

By virtue of [6, Proposition 1], the initial value problem (E)-(2) has a unique (local) solution which is denoted by $x(t; \sigma, \phi, f)$; in fact, $x(t; \sigma, \phi, f)$ is defined globally if, in particular, $f(\phi)$ is globally Lipschitz continuous in $\phi$. Moreover, we remark that if $x(t)$ is a solution of Eq. (E) on $(\sigma, a)$, then $x_t$ is an $X$-valued continuous function on $[\sigma, a)$. Now suppose that $\phi = \psi$ in $X$, that is, $\phi(\theta) = \psi(\theta)$ a.e. $\theta \in \mathbb{R}^-$. Then by the uniqueness of solutions of (E)-(2) it follows that $x(t; \sigma, \phi, f) = x(t; \sigma, \psi, f)$ for $t \in (\sigma, a)$, so that $x_t(\sigma, \phi, f) = x_t(\sigma, \psi, f)$ in $X$ for $t \in [\sigma, a)$. In particular, given $\sigma \in \mathbb{R}$, $x_t(\sigma, \cdot, f)$ induces a transformation on $X$ for each $t \in [\sigma, a)$ provided that $x(t; \sigma, \phi, f)$ is the solution of (E)-(2) on $(\sigma, a)$.

For any $t \geq 0$ and $\phi \in X$, we define $T(t)\phi \in X$ by

$$[T(t)\phi](\theta) := x_t(\theta; 0, \phi, 0) = \begin{cases} x(t + \theta; 0, \phi, 0), & -t < \theta \leq 0, \\ \phi(t + \theta), & \theta \leq -t. \end{cases}$$

Then $T(t)$ defines a bounded linear operator on $X$. In fact, $\{T(t)\}_{t \geq 0}$ is a strongly continuous semigroup of bounded linear operators on $X$, called the solution semigroup for Eq. (1). Denote by $A$ the generator of $\{T(t)\}_{t \geq 0}$, and let $\sigma(A)$ and $P_{\sigma}(A)$ be the spectrum and the point spectrum of the generator $A$, respectively. Between the spectrum of $A$ and the characteristic roots of Eq. (1), the relation $\sigma(A) \cap C_{-\rho} = P_{\sigma}(A) \cap C_{-\rho} = \{\lambda \in C_{-\rho} : \det \Delta(\lambda) = 0\} (= \Sigma)$ holds, where $C_{-\rho} := \{z \in C : \text{Re} z > -\rho\}$. Moreover, for ess $(A)$, the essential spectrum of $A$, we have the estimate $\sup_{\lambda \in \text{ess}(A)} \text{Re} \lambda \leq -\rho$. Now set

$$\Sigma^u := \{\lambda \in \sigma(A) : \text{Re} \lambda > 0\}, \Sigma^c := \{\lambda \in \sigma(A) : \text{Re} \lambda = 0\}, \text{and } \Sigma^s := \sigma(A) \backslash (\Sigma^c \cup \Sigma^u).$$

Then these observations, combined with the analyticity of $\det \Delta(\lambda)$ on the domain $C_{-\rho}$, yield the following result ([6, Theorem 2]):
Proposition 1. Let \( \{T(t)\}_{t \geq 0} \) be the solution semigroup of Eq. (1). Then \( X \) is decomposed as a direct sum of closed subspaces \( E^{u}, E^{c}, \) and \( E^{s} \)

\[ X = E^{u} \oplus E^{c} \oplus E^{s} \]

with the following properties:

(i) \( \dim (E^{u} \oplus E^{c}) < \infty \),

(ii) \( T(t)E^{u} \subset E^{u}, T(t)E^{c} \subset E^{c} \), and \( T(t)E^{s} \subset E^{s} \) for \( t \in \mathbb{R}^{+} := [0, \infty) \),

(iii) \( \sigma(A|_{E^{u}}) = \Sigma^{u}, \sigma(A|_{E^{c}}) = \Sigma^{c} \) and \( \sigma(A|_{E^{s} \cap \mathcal{D}(A)}) = \Sigma^{s} \),

(iv) \( T^{u}(t) := T(t)|_{E^{u}} \) and \( T^{c}(t) := T(t)|_{E^{c}} \) are extendable for \( t \in \mathbb{R} := (-\infty, \infty) \) as groups of bounded linear operators on \( E^{u} \) and \( E^{c} \), respectively,

(v) \( T^{s}(t) := T(t)|_{E^{s}} \) is a strongly continuous semigroup of bounded linear operators on \( E^{s} \), and its generator is identical with \( A|_{E^{s} \cap \mathcal{D}(A)} \),

(vi) there exist positive constants \( \alpha, \varepsilon \) with \( \alpha > \varepsilon \) and a constant \( C \geq 1 \) such that

\[
\begin{align*}
\|T^{s}(t)\|_{\mathcal{L}(X)} &\leq Ce^{-\alpha t}, \quad t \in \mathbb{R}^{+}, \\
\|T^{u}(t)\|_{\mathcal{L}(X)} &\leq Ce^{\alpha t}, \quad t \in \mathbb{R}^{-}, \\
\|T^{c}(t)\|_{\mathcal{L}(X)} &\leq Ce^{\varepsilon|t|}, \quad t \in \mathbb{R}.
\end{align*}
\]

In (vi) we note that \( C \) is a constant depending only on \( \alpha \) and \( \varepsilon \), and that the value of \( \varepsilon > 0 \) can be taken arbitrarily small. Also, we will use the notations \( E^{cu} = E^{c} \oplus E^{u} \), \( E^{su} = E^{s} \oplus E^{u} \) etc, and denote by \( \Pi^{s} \) the projection from \( X \) onto \( E^{s} \) along \( E^{cu} \), and similarly for \( \Pi^{u}, \Pi^{cu} \) etc.

We now introduce a continuous function \( \Gamma^{n} : \mathbb{R}^{-} \to \mathbb{R}^{+} \) for each natural number \( n \) which is of compact support with \textit{support} \( \Gamma^{n} \subset \left[-1/n, 0\right] \) and satisfies \( \int_{-\infty}^{0} \Gamma^{n}(\theta)d\theta = 1 \). Notice that \( \Gamma^{n}\beta \in X \) for any \( \beta \in \mathbb{C}^{m} \). Let us recall that \( x(\cdot; \sigma, \varphi, p) \) is the (unique) solution of the integral equation

\[ x(t) = \int_{-\infty}^{t} K(t-s)x(s)ds + p(t), \quad t > \sigma \]  

through \( (\sigma, \varphi) \); here \( \varphi \in X \). The following result ([6, Theorem 3]), which will often be referred to as VCF for short, gives a representation formula for \( x_{t}(\sigma, \varphi, p) \) in the space \( X \) by using \( T(t), \varphi \) and \( p \).

Proposition 2. Let \( p \in C([\sigma, \infty); \mathbb{C}^{m}) \). Then

\[ x_{t}(\sigma, \varphi, p) = T(t-\sigma)\varphi + \lim_{n \to \infty} \int_{\sigma}^{t} T(t-s)(\Gamma^{n} p(s))ds, \quad \forall t \geq \sigma \]

in \( X \).
Let us consider a subset $\overline{X}$ consisting of all elements $\phi \in X$ which are continuous on $[-\epsilon_\phi, 0]$ for some $\epsilon_\phi > 0$, and set

$$X_0 = \{ \varphi \in X \mid \varphi = \phi \text{ a.e. on } \mathbb{R}^- \text{ for some } \phi \in \overline{X} \}.$$  

For any $\varphi \in X_0$, we define the value of $\varphi$ at zero by $\varphi[0] = \phi(0)$, where $\phi$ is an element belonging to $\overline{X}$ satisfying $\phi = \varphi$ a.e. on $\mathbb{R}^-$. We note that the value $\varphi[0]$ is well-defined; that is, it does not depend on the particular choice of $\phi$ since $\phi(0) = \psi(0)$ for any other $\psi \in \overline{X}$ such that $\phi = \psi$ a.e. on $\mathbb{R}^-$. It is clear that $X_0$ is a normed space equipped with norm

$$\|\varphi\|_{X_0} := \|\varphi\|_X + |\varphi[0]|, \quad \forall \varphi \in X_0.$$  

We note that the solution $x(\cdot; \sigma, \psi, p)$ of Eq. (3) through $(\sigma, \psi) \in \mathbb{R} \times X$ satisfies the relation $x_t(\sigma, \psi, p) \in X_0$ with $(x_t(\sigma, \psi, p))[0] = x(t; \sigma, \psi, p)$ whenever $t > \sigma$.

The following lemma can be established by applying Proposition 2 and [6, Theorem 4]. We omit the proof.

**Lemma 1.** Let $f_* \in C(X; \mathbb{C}^m)$, and consider the equation

$$x(t) = \int_{-\infty}^{t} K(t-s)x(s)ds + f_*(x_t). \quad (E_*)$$

Moreover, let $\psi \in E^c$, and $\eta$ be a constant such that $\varepsilon < \eta < \alpha$. Then we have:

(i) If $x(t)$ is a solution of Eq. (E$_*$) defined on $\mathbb{R}$ with the properties that $\Pi^c x_0 = \psi$, $\sup_{t \in \mathbb{R}} \|x_t\|_{X} e^{-\eta|t|} < \infty$ and $\sup_{t \in \mathbb{R}} |f_*(x_t)| < \infty$, then the $X$-valued function $u(t) := x_t$ satisfies

$$u(t) = T^c(t)\psi + \lim_{n \to \infty} \int_{0}^{t} T^c(t-s)\Pi^n f_*(u(s))ds$$

$$- \lim_{n \to \infty} \int_{t}^{\infty} T^u(t-s)\Pi^u \Gamma^n f_*(u(s))ds + \lim_{n \to \infty} \int_{-\infty}^{t} T^s(t-s)\Pi^s \Gamma^n f_*(u(s))ds$$

for $t \in \mathbb{R}$, and moreover $u$ belongs to $C(\mathbb{R}; X_0)$.

(ii) Conversely, if $y \in C(\mathbb{R}; X)$ with $\sup_{t \in \mathbb{R}} \|y(t)\|_X e^{-\eta|t|} < \infty$ and $\sup_{t \in \mathbb{R}} |f_*(y(t))| < \infty$ satisfies

$$y(t) = T^c(t)\psi + \lim_{n \to \infty} \int_{0}^{t} T^c(t-s)\Pi^n f_*(y(s))ds$$

$$- \lim_{n \to \infty} \int_{t}^{\infty} T^u(t-s)\Pi^u \Gamma^n f_*(y(s))ds + \lim_{n \to \infty} \int_{-\infty}^{t} T^s(t-s)\Pi^s \Gamma^n f_*(y(s))ds$$

for $t \in \mathbb{R}$, then $y$ belongs to $C(\mathbb{R}; X_0)$ and the function $\xi(t)$ defined by

$$\xi(t) := (y(t))[0], \quad t \in \mathbb{R}$$

is a solution of Eq. (E$_*$) on $\mathbb{R}$ satisfying $\Pi^c \xi_0 = \psi$, $\sup_{t \in \mathbb{R}} \|\xi_t\|_{X} e^{-\eta|t|} < \infty$ and $\xi_t = y(t)$ for $t \in \mathbb{R}$. 

3 Center manifold and its exponential attractivity

In what follows we assume that $f \in C^1(X;\mathbb{C}^m)$ satisfies $f(0) = 0$ and $Df(0) = 0$. In this section we will establish the existence of local center manifolds of the equilibrium point $0$ of Eq. $(E)$ and study their properties. To do so, in parallel with Eq. $(E)$, we will consider a modified equation of $(E)$ of the form

$$x(t) = \int_{-\infty}^{t} K(t-s)x(s)ds + f_\delta(x_t), \quad (E_\delta)$$

where $f_\delta$ with $\delta > 0$ is a modification of the original nonlinear term $f$; more precisely let $\chi : \mathbb{R} \to [0,1]$ be a $C^\infty$-function such that $\chi(t) = 1 (|t| \leq 2)$ and $\chi(t) = 0 (|t| \geq 3)$, and define

$$f_\delta(\phi) := \chi(\|\Pi^{su}\phi\|_{X}/\delta)\chi(\|\Pi^{c}\phi\|_{X}/\delta)f(\phi), \quad \phi \in X.$$

The function $f_\delta : X \to \mathbb{C}^m$ is continuous on $X$, and is of class $C^1$ when restricted to the open set $S_\delta := \{\phi \in X : \|\Pi^{au}\phi\|_{X} < \delta\}$ since we may assume that $\|\Pi^{c}\phi\|_{X}$ is of class $C^1$ for $\phi \neq 0$ because of $\dim E^c < \infty$. Moreover, by the assumption $f(0) = Df(0) = 0$, there exist $\delta_1 > 0$ and a nondecreasing continuous function $\zeta_* : (0, \delta_1] \to \mathbb{R}^+$ such that $\zeta_*(+0) = 0,$

$$\|f_\delta(\phi)\|_{X} \leq \delta \zeta_*(\delta) \quad \text{and} \quad \|f_\delta(\phi) - f_\delta(\psi)\|_{X} \leq \zeta_*(\delta)\|\phi - \psi\|_{X}$$

for $\phi, \psi \in X$ and $\delta \in (0, \delta_1]$. Indeed, we may put

$$\zeta_*(\delta) = (\sup_{\|\phi\|_{X} \leq \delta} \|Df(\phi)\|_{\mathcal{L}(X;\mathbb{C}^m)}) \cdot (1 + 3 \sup_{0 \leq t \leq 3} |\chi'(t)|)$$

(cf. [2, Lemma 4.1]). Taking $\delta_1 > 0$ small, we may also assume that there exists a positive number $M_1(\delta_1) =: M_1$ such that

$$\|Df_\delta(\phi)\|_{\mathcal{L}(X;\mathbb{C}^m)} \leq M_1, \quad \phi \in S_\delta$$

for any $\delta \in (0, \delta_1]$. Fix a positive number $\eta$ such that

$$\epsilon < \eta < \alpha,$$

where $\epsilon$ and $\alpha$ are the constants in Proposition 1.

For the existence of center manifold for Eq. $(E_\delta)$ and its exponential attractivity, we have the following:

**Theorem 1.** There exist a positive number $\delta$ and a $C^1$-map $F_{*,\delta} : E^c \to E^{au}$ with $F_{*,\delta}(0) = 0$ such that the following properties hold:

(i) $W^c_{\delta} := \text{graph } F_{*,\delta}$ is tangent to $E^c$ at zero,
(ii) $W^c_\delta$ is invariant for Eq. $(E_\delta)$, that is, if $\xi \in W^c_\delta$, then $x_t(0, \xi, f) \in W^c_\delta$ for $t \in \mathbb{R}$.

(iii) Assume moreover that $\Sigma^u = \emptyset$. Then there exists a positive constant $\beta_0$ with the property that if $x$ is a solution of Eq. $(E_\delta)$ on an interval $J = [t_0, t_1]$, then the inequality

$$
\|\Pi^s x_t - F_\delta(x_t)\|_X \leq C\|\Pi^s x_{t_0} - F_\delta(x_{t_0})\|_X e^{-\beta_0(t-t_0)}, \quad t \in J
$$

holds true. In particular, if $x$ is a solution on an interval $[t_0, \infty)$, $x_t$ tends to $W^c_\delta$ exponentially as $t \to \infty$.

As will be shown in Proposition 4 given later, the map $F_{*,\delta} : E^c \to E^u$ in Theorem 1 is globally Lipschitz continuous with the Lipschitz constant $L(\delta) = 4C^2C_1\zeta_*(\delta)/\alpha$. Noticing that $L(\delta) \to 0$ as $\delta \to 0$, one can assume that the number $\delta$ satisfies $\delta \in (0, \delta_1]$ together with $L(\delta) \leq 1$. Let us take a small $r \in (0, \delta)$ so that $\|F_{*,\delta}(\psi)\|_X < \delta$ for any $\psi \in B_{E^c}(r) := \{\phi \in E^c : \|\phi\|_X < r\}$. Such a choice of $r$ is possible by the continuity of $F_{*,\delta}$. Set $F_* := F_{*,\delta}|_{B_{E^c}(r)}$ and consider an open neighborhood $\Omega_0$ of 0 in $X$ defined by

$$
\Omega_0 := \{\phi \in X : \|\Pi^u \phi\|_X < \delta, \|\Pi^c \phi\|_X < r\}.
$$

Observe that $f \equiv f_\delta$ on $\Omega_0$. Then the following theorem which yields a local center manifold for Eq. $(E)$ as the graph of $F_*$ immediately follows from Theorem 1.

**Theorem 2.** Assume that $f \in C^1(X; \mathbb{C}^m)$ with $f(0) = Df(0) = 0$. Then there exist positive numbers $r, \delta$, and a $C^1$-map $F_* : B_{E^c}(r) \to E^u$ with $F_*(0) = 0$, together with an open neighborhood $\Omega_0$ of 0 in $X$, such that the following properties hold:

(i) $W^c_{loc}(r, \delta) := \text{graph } F_*$ is tangent to $E^c$ at zero,

(ii) $W^c_{loc}(r, \delta)$ is locally invariant for Eq. $(E)$, that is,

(a) for any $\xi \in W^c_{loc}(r, \delta)$ there exists a $t_\xi > 0$ such that $x_t(0, \xi, f) \in W^c_{loc}(r, \delta)$ for $|t| \leq t_\xi$,

(b) if $\xi \in W^c_{loc}(r, \delta)$ and $x_t(0, \xi, f) \in \Omega_0$ for $0 \leq t \leq T$, then $x_t(0, \xi, f) \in W^c_{loc}(r, \delta)$ for $0 \leq t \leq T$.

(iii) Assume moreover that $\Sigma^u = \emptyset$. Then there exists a positive constant $\beta_0$ with the property that if $x$ is a solution of Eq. $(E)$ on an interval $J = [t_0, t_1]$ satisfying $x_t \in \Omega_0$ on $J$, then the inequality

$$
\|\Pi^s x_t - F_*(x_t)\|_X \leq C\|\Pi^s x_{t_0} - F_*(x_{t_0})\|_X e^{-\beta_0(t-t_0)}, \quad t \in J
$$

holds true. In particular, if the solution $x(t)$ is defined on $[t_0, \infty)$ satisfying $x_t \in \Omega_0$ on $[t_0, \infty)$, then $x_t$ tends to $W^c_{loc}(r, \delta)$ exponentially as $t \to \infty$. 
In what follows, we will prove Theorem 1 by establishing several propositions. We now take a $\delta_1 > 0$ sufficiently small so that

$$
\zeta_*(\delta_1)CC_1 \left( \frac{1}{\eta - \varepsilon} + \frac{2}{\alpha + \eta} + \frac{2}{\alpha - \eta} \right) < \frac{1}{2}
$$

(7)

holds, and let $\delta \in (0, \delta_1]$. Also, let us consider the Banach space $Y_\eta$ defined by

$$
Y_\eta := \{ y \in C(\mathbb{R}; X) : \sup_{t \in \mathbb{R}} ||y(t)||_X e^{-\eta|t|} < \infty \}
$$

with norm $||y||_{Y_\eta} := \sup_{t \in \mathbb{R}} ||y(t)||_X e^{-\eta|t|}, y \in Y_\eta$. For any $(\psi, y) \in E^c \times Y_\eta$, we set

$$
\mathcal{F}_\delta(\psi, y)(t) := T^c(t)\psi + \lim_{narrow\rightarrow\infty} \int_{0}^{t} T^c(t-s)\Pi^n\Gamma^n f_\delta(y(s))ds
$$

(8)

$$
- \lim_{narrow\rightarrow\infty} \int_{t}^{\infty} T^u(t-s)\Pi^n\Gamma^n f_\delta(y(s))ds
$$

$$
+ \lim_{narrow\rightarrow\infty} \int_{-\infty}^{t} T^s(t-s)\Pi^n\Gamma^n f_\delta(y(s))ds
$$

for $t \in \mathbb{R}$. Notice that the right-hand side is well-defined and that $\mathcal{F}_\delta(\psi, y)$ is an $X$-valued function on $\mathbb{R}$ for each $(\psi, y) \in E^c \times Y_\eta$. It is straightforward to certify that $\mathcal{F}_\delta(\psi, y) \in Y_\eta$ by virtue of Proposition 1 and (5); in other words, $\mathcal{F}_\delta$ defines a map from $E^c \times Y_\eta$ to $Y_\eta$. In fact, for each $\psi \in E^c$, $\mathcal{F}_\delta(\psi, \cdot)$ is a contraction map from $Y_\eta$ into itself with Lipschitz constant $1/2$, because of the inequality

$$
||\mathcal{F}_\delta(\psi, y_1) - \mathcal{F}_\delta(\psi, y_2)||_{Y_\eta} \leq \sup_{t \in \mathbb{R}} e^{-\eta|t|} \left| \int_{0}^{t} CC_1 \zeta_*(\delta)e^{-\varepsilon(t-s)}||y_1 - y_2||_{Y_\eta}e^{\eta|s|}ds \right|
$$

$$
+ \sup_{t \in \mathbb{R}} e^{-\eta|t|} \int_{t}^{\infty} CC_1 \zeta_*(\delta)e^{\alpha(t-s)}||y_1 - y_2||_{Y_\eta}e^{\eta|s|}ds
$$

$$
+ \sup_{t \in \mathbb{R}} e^{-\eta|t|} \int_{-\infty}^{t} CC_1 \zeta_*(\delta)e^{-\alpha(t-s)}||y_1 - y_2||_{Y_\eta}e^{\eta|s|}ds
$$

$$
\leq \zeta_*(\delta_1)CC_1 \left( \frac{1}{\eta - \varepsilon} + \frac{2}{\alpha + \eta} + \frac{2}{\alpha - \eta} \right) ||y_1 - y_2||_{Y_\eta}
$$

$$
\leq (1/2)||y_1 - y_2||_{Y_\eta}
$$

for $y_1, y_2 \in Y_\eta$. Thus, the map $\mathcal{F}_\delta(\psi, \cdot)$ has a unique fixed point for each $\psi \in E^c$, say $\Lambda_{*,\delta}(\psi) \in Y_\eta$, i.e., we have

$$
\Lambda_{*,\delta}(\psi)(t) = T^c(t)\psi + \lim_{narrow\rightarrow\infty} \int_{0}^{t} T^c(t-s)\Pi^n\Gamma^n f_\delta(\Lambda_{*,\delta}(\psi)(s))ds
$$

(9)

$$
- \lim_{narrow\rightarrow\infty} \int_{t}^{\infty} T^u(t-s)\Pi^n\Gamma^n f_\delta(\Lambda_{*,\delta}(\psi)(s))ds
$$

$$
+ \lim_{narrow\rightarrow\infty} \int_{-\infty}^{t} T^s(t-s)\Pi^n\Gamma^n f_\delta(\Lambda_{*,\delta}(\psi)(s))ds
$$

for $t \in \mathbb{R}$, whenever $0 < \delta \leq \delta_1$. 

Proposition 3. $\Lambda_{*,\delta}(\psi)$ satisfies the following:

(i) $\|\Lambda_{*,\delta}(\psi_{1}) - \Lambda_{*,\delta}(\psi_{2})\|_{Y_{\eta}} \leq 2C\|\psi_{1} - \psi_{2}\|_{X}$ for $\psi_{1}, \psi_{2} \in E^{c}$.

(ii) $\Lambda_{*,\delta}(\psi)(t + \tau) = \Lambda_{*,\delta}(\Pi^{c}(\Lambda_{*,\delta}(\psi)(\tau)))(t)$ holds for $t, \tau \in \mathbb{R}$.

Proof. Since $\epsilon < \eta$, (i) immediately follows from the estimate

\[\|\Lambda_{*}(\psi_{1}) - \Lambda_{*}(\psi_{2})\|_{Y_{\eta}} = \|\mathcal{F}_{\delta}(\psi_{1}, \Lambda_{*,\delta}(\psi_{1})) - \mathcal{F}_{\delta}(\psi_{2}, \Lambda_{*,\delta}(\psi_{2}))\|_{Y_{\eta}}\]

\[\leq \|\overline{J^{-}}_{\delta}(\psi_{1}, \Lambda_{*,\delta}(\psi_{1})) - \mathcal{F}_{\delta}(\psi_{1}, \Lambda_{*,\delta}(\psi_{2}))\|_{Y_{\eta}} + \|\mathcal{F}_{\delta}(\psi_{1}, \Lambda_{*,\delta}(\psi_{2})) - \mathcal{F}_{\delta}(\psi_{2}, \Lambda_{*,\delta}(\psi_{2}))\|_{Y_{\eta}}\]

\[\leq (1/2)\|\Lambda_{*,\delta}(\psi_{1}) - \Lambda_{*,\delta}(\psi_{2})\|_{Y_{\eta}} + \sup_{t \in \mathbb{R}}(Ce^{\epsilon|t|}\|\psi_{1} - \psi_{2}\|_{X}e^{-\eta|t|}) \leq (1/2)\|\Lambda_{*,\delta}(\psi_{1}) - \Lambda_{*,\delta}(\psi_{2})\|_{Y_{\eta}} + \sup_{t \in \mathbb{R}}(Ce^{\epsilon|t|}\|\psi_{1} - \psi_{2}\|_{X}e^{-\eta|t|}).\]

Next, given $\tau \in \mathbb{R}$, let us consider the function $\tilde{\Lambda}(t)$ defined by $\tilde{\Lambda}(t) := \Lambda_{*,\delta}(\psi)(t + \tau)$, $t \in \mathbb{R}$. Obviously, $\tilde{\Lambda}(\cdot) \in Y_{\eta}$. Also, it is easy to check that $\tilde{\Lambda}(t) = \mathcal{F}_{\delta}(\Pi^{c}(\Lambda_{*,\delta}(\psi)(\tau)), \tilde{\Lambda})(t)$ for all $t \in \mathbb{R}$; that is, $\tilde{\Lambda}$ is a fixed point of $\mathcal{F}_{\delta}(\Pi^{c}(\Lambda_{*,\delta}(\psi)(\tau)), \cdot)$. The uniqueness of the fixed points yields $\tilde{\Lambda} = \Lambda_{*,\delta}(\Pi^{c}(\Lambda_{*,\delta}(\psi)(\tau)))$, and hence

\[\Lambda_{*,\delta}(\psi)(t + \tau) = \tilde{\Lambda}(t) = \Lambda_{*,\delta}(\Pi^{c}(\Lambda_{*,\delta}(\psi)(\tau)))(t), \quad t \in \mathbb{R},\]

which shows (ii). \[\square\]

For $\delta \in (0, \delta_{1}]$ let $F_{*,\delta} : E^{c} \to E^{su}$ be the map defined by $F_{*,\delta}(\psi) := \Pi^{su} \circ ev_{0} \circ \Lambda_{*,\delta}(\psi)$ for $\psi \in E^{c}$, where $ev_{0}$ is the evaluation map: $ev_{0}(y) := y(0)$ for $y \in C(\mathbb{R}; X)$. Then

\[F_{*,\delta}(\psi) = - \lim_{n \to \infty} \int_{0}^{\infty} T^{u}(-s)\Pi^{u}\Gamma^{n} f_{\delta}(\Lambda_{*,\delta}(\psi)(s)) ds + \lim_{n \to \infty} \int_{-\infty}^{0} T^{s}(-s)\Pi^{s}\Gamma^{n} f_{\delta}(\Lambda_{*,\delta}(\psi)(s)) ds, \quad \psi \in E^{c};\]

and in particular $\Lambda_{*,\delta}(\psi)(0) = \psi + F_{*,\delta}(\psi)$ for $\psi \in E^{c}$. Let us set

\[W_{\delta}^{c} := \text{graph } F_{*,\delta} = \{\psi + F_{*,\delta}(\psi) : \psi \in E^{c}\}.

Proposition 4. The map $F_{*,\delta}$ and its graph $W_{\delta}^{c}$ have the following properties:

(i) $F_{*,\delta}$ is (globally) Lipschitz continuous, i.e.,

\[\|F_{*,\delta}(\psi_{1}) - F_{*,\delta}(\psi_{2})\|_{X} \leq L(\delta)\|\psi_{1} - \psi_{2}\|_{X}, \quad \psi_{1}, \psi_{2} \in E^{c},\]

where $L(\delta) := 4C^{2}C_{1}\zeta_{*}(\delta)/{(\alpha - \eta)}$. 

(ii) Let \( \hat{\phi} \in W_{\delta}^s \) and \( \tau \in \mathbb{R} \). Then the solution of \((E_{\delta})\) through \((\tau, \hat{\phi})\), \(x(t; \tau, \hat{\phi}, f_{\delta})\), exists on \( \mathbb{R} \) and

\[
x_t(\tau, \hat{\phi}, f_\delta) = \Lambda_{*,\delta}(\hat{\psi})(t - \tau), \quad t \in \mathbb{R},
\]

where \( \hat{\psi} = \Pi^c \hat{\phi} \).

(iii) Moreover for \( \hat{\phi} \in W_{\delta}^s \) and \( \tau \in \mathbb{R} \),

\[
\Pi^{su} x_t(\tau, \hat{\phi}, f_\delta) = F_{*,\delta}(\Pi^c x_t(\tau, \hat{\phi}, f_\delta)), \quad t \in \mathbb{R}.
\]

In particular \( W_{\delta}^s \) is invariant for \((E_{\delta})\), that is, \( x_t(\tau, \hat{\phi}, f_\delta) \in W_{\delta}^s \) for \( t \in \mathbb{R} \), provided that \( \hat{\phi} \in W_{\delta^c} \).

**Proof.** (i) By (10) and Proposition 3 (i), we get

\[
\|F_{*,\delta}(\psi_1) - F_{*,\delta}(\psi_2)\|_X \leq \int_0^\infty CC_1 e^{-\alpha s} \zeta_* (\delta) \|\Lambda_{*,\delta}(\psi_1)(s) - \Lambda_{*,\delta}(\psi_2)(s)\|_X ds
\]

\[
+ \int_{-\infty}^0 CC_1 e^{\alpha s} \zeta_* (\delta) \|\Lambda_{*,\delta}(\psi_1)(s) - \Lambda_{*,\delta}(\psi_2)(s)\|_X ds
\]

\[
\leq \frac{2CC_1 \zeta_* (\delta)}{\alpha - \eta} \times 2C \|\psi_1 - \psi_2\|_X = L(\delta) \|\psi_1 - \psi_2\|_X,
\]

as required.

(ii) Applying Lemma 1 (i), we deduce that \( \Lambda_{*,\delta}(\hat{\psi}) \in C(\mathbb{R}; X_0) \) and that the \( X \)-valued function \( \xi(t) := (\Lambda_{*,\delta}(\hat{\psi})(t))[0] \) \((t \in \mathbb{R})\) satisfies \( \xi_t = \Lambda_{*,\delta}(\hat{\psi})(t) \) for \( t \in \mathbb{R} \) and is a solution of \((E_\delta)\) on \( \mathbb{R} \) with \( \xi_0 = \Lambda_{*,\delta}(\hat{\psi})(0) = \hat{\psi} + F_{*,\delta}(\hat{\psi}) = \hat{\phi} \). Let \( x(t) := \xi(t - \tau) \). Then \( x(t) \) is a solution of \((E_{\delta})\) on \( \mathbb{R} \) with \( x_\tau = \hat{\phi} \), so that \( x(t) = x(t; \tau, \hat{\phi}, f_\delta) \) for \( t \in \mathbb{R} \). Consequently,

\[
x_t(\tau, \hat{\phi}, f_\delta) = \xi_{t-\tau} = \Lambda_{*,\delta}(\hat{\psi})(t - \tau), \quad t \in \mathbb{R}.
\]

(iii) Notice from Proposition 3 (ii) that \( \Lambda_{*,\delta}(\hat{\psi})(t - \tau) = \Lambda_{*,\delta}(\Pi^c(\Lambda_{*,\delta}(\hat{\psi})(t - \tau)))(0) \) for \( \hat{\psi} := \Pi^c \hat{\phi} \), which, combined with (ii), yields that

\[
\Pi^{su} x_t(\tau, \hat{\phi}, f_\delta) = \Pi^{su}(\Lambda_{*,\delta}(\Pi^c(\Lambda_{*,\delta}(\hat{\psi})(t - \tau)))(0))
\]

\[
= \Pi^{su}(\Lambda_{*,\delta}(\Pi^c x_t(\tau, \hat{\phi}, f_\delta)))(0) = F_{*,\delta}(\Pi^c x_t(\tau, \hat{\phi}, f_\delta));
\]

which is the desired one. The latter part of (iii) is obvious. \( \square \)

Now assume that \( \Sigma^u = \emptyset \), i.e., \( E^u = \{0\} \). Fix a \( \delta \in (0, \delta_1] \) and let

\[
K := CC_1 \zeta_* (\delta), \quad \mu := K + \varepsilon.
\]

**Proposition 5.** Let \( x(t) \) be a solution of \((E_\delta)\) on an interval \( J := [t_0, t_1] \). Given \( \tau \in J \), put \( \hat{\phi} := \Pi^c x_{\tau} + F_{*,\delta}(\Pi^c x_{\tau}) \). Then the following inequalities hold:
\[(i) \quad \text{For } t_0 \leq t \leq \tau \]
\[
\|\Pi^c x_t - \Pi^c x_t(\tau, \hat{\phi}, f_\delta)\|_X \leq K \int_t^\tau e^{\mu(s-t)}\|\Pi^s x_s - \Pi^s x_s(\tau, \hat{\phi}, f_\delta)\|_X ds.
\]

\[(ii) \quad \text{Moreover for } t_0 \leq t \leq \tau \]
\[
\|\Pi^c x_t - \Pi^c x_t(\tau, \hat{\phi}, f_\delta)\|_X \leq K \int_t^\tau e^{\mu(s-t)}\|\Pi^s x_s - \Pi^s x_s(\tau, \hat{\phi}, f_\delta)\|_X ds,
\]

where \( \mu' := \mu + KL(\delta) \) and \( \xi(t) := \Pi^s x_t - F_{*,\delta}(\Pi^c x_t) \) for \( t \in \mathbb{R} \).

Proof. By virtue of Proposition 4 (ii) and (iii), the solution \( x(t; \tau, \hat{\phi}, f_\delta) \) exists on \( \mathbb{R} \) and \( \Pi^c x_t(\tau, \hat{\phi}, f_\delta) = F_{*,\delta}(\Pi^c x_t(\tau, \hat{\phi}, f_\delta)) \) for \( t \in \mathbb{R} \). Let \( t_0 \leq t \leq \tau \). VCF gives
\[
x_{\tau}(\tau, \hat{\phi}, f_\delta) = T(\tau - t)x_t(\tau, \hat{\emptyset}, f_\delta) + \lim_{narrow \infty} \int_t^\tau T(\tau - s)\Gamma^n f_\delta(x_s(\tau, \hat{\phi}, f_\delta))ds,
\]
in particular
\[
\Pi^c x_{\tau}(\tau, \hat{\phi}, f_\delta) = T^c(\tau - t)\Pi^c x_t(\tau, \hat{\phi}, f_\delta) + \lim_{narrow \infty} \int_t^\tau T^c(\tau - s)\Pi^c \Gamma^n f_\delta(x_s(\tau, \hat{\phi}, f_\delta))ds.
\]
By the group property of \( \{T^c(t)\}_{t \in \mathbb{R}} \), we get
\[
\Pi^c x_t(\tau, \hat{\phi}, f_\delta) = T^c(t - \tau)\Pi^c x_{\tau}(\tau, \hat{\phi}, f_\delta) - \lim_{narrow \infty} \int_t^\tau T^c(t - s)\Pi^c \Gamma^n f_\delta(x_s(\tau, \hat{\phi}, f_\delta))ds.
\]
Similarly for the solution \( x(t) \)
\[
\Pi^c x_t = T^c(t - \tau)\Pi^c x_{\tau} - \lim_{narrow \infty} \int_t^\tau T^c(t - s)\Pi^c \Gamma^n f_\delta(x_s)ds.
\]
Then, since \( \Pi^c x_{\tau}(\tau, \hat{\phi}, f_\delta) = \Pi^c \hat{\emptyset} = \Pi^c x_{\mathcal{T}} \), it follows that
\[
e^{\epsilon t}\|\Pi^c x_t - \Pi^c x_t(\tau, \hat{\phi}, f_\delta)\|_X \leq \int_t^\tau K \int_t^\tau e^{\epsilon(s-t)}\|\Pi^c x_s - \Pi^c x_s(\tau, \hat{\phi}, f_\delta)\|_X ds + \int_t^\tau KL(\delta)e^{\epsilon s}\|\Pi^c x_s - \Pi^c x_s(\tau, \hat{\phi}, f_\delta)\|_X ds
\]
for \( t_0 \leq t \leq \tau \). Hence we get
\[
e^{\epsilon t}\|\Pi^c x_t - \Pi^c x_t(\tau, \hat{\phi}, f_\delta)\|_X \leq \int_t^\tau K e^{K(s-t)}e^{\epsilon s}\|\Pi^c x_s - \Pi^c x_s(\tau, \hat{\phi}, f_\delta)\|_X ds,
\]
which implies (i).

Next we will verify (ii). By Proposition 4 (iii) and (i), we get \( \|\Pi^s x_s - \Pi^s x_s(\tau, \hat{\phi}, f_\delta)\|_X \leq \|\xi(s)\|_X + L(\delta)\|\Pi^c x_s - \Pi^c x_s(\tau, \hat{\phi}, f_\delta)\|_X \) for \( s \in J \). Hence it follows from (i) that
\[
e^{\mu t}\|\Pi^c x_t - \Pi^c x_t(\tau, \hat{\phi}, f_\delta)\|_X \leq \int_t^\tau K e^{\mu(s-t)}\|\xi(s)\|_X ds + \int_t^\tau KL(\delta)e^{\mu s}\|\Pi^c x_s - \Pi^c x_s(\tau, \hat{\phi}, f_\delta)\|_X ds;
then
\[ e^{\mu t} \| \Pi^c x_t - \Pi^c x_t(\tau, \hat{\phi}, f_\delta) \|_X \leq \int_t^T K e^{KL(\delta)(s-t)} e^{\mu s} \| \xi(s) \|_X ds, \]
which implies (ii).

Recall that
\[ K := CC_1 \zeta_*(\delta), \quad \mu := K + \epsilon, \quad \mu' := \mu + KL(\delta) = K(1 + L(\delta)) + \epsilon. \quad (12) \]

**Proposition 6.** Assume that \( \Sigma^u = \emptyset \) and \( x(t) \) is a solution of \((E_\delta)\) on \( J = [t_0, t_1] \). Define \( \hat{x}_t \in W^c \) by \( \hat{x}_t := \Pi^c x_t + F_{*,\delta}(\Pi^c x_t) \) for \( t \in J \), and set \( y(s; t) := \Pi^c x_s(t, \hat{x}_t, f_\delta) \) for \( t \in J \) and \( s \leq t \). Then the following inequality holds:

\[ \| y(s; t) - y(s; t_0) \|_X \leq K \int_{t_0}^{t} e^{\mu'(\theta-s)} \| \xi(\theta) \|_X d\theta, \quad s \leq t_0, \]

where \( \xi(\theta) := \Pi^s x_\theta - F_{*,\delta}(\Pi^c x_\theta) \) for \( \theta \in [t_0, t] \).

**Proof.** Suppose that \( s \leq t_0 \). By the same reasoning as (11)

\[ \Pi^c x_s(t, \hat{x}_t, f_\delta) = T^c(s - t) \Pi^c \hat{x}_t - \lim_{n \to \infty} l^t T^c(s - \sigma) \Pi^c \Gamma^nf_\delta(x_\sigma(t, \hat{x}_t, f_\delta)) d\sigma. \quad (13) \]

Applying VCF to \( x_t \) and using \( \Pi^c \hat{x}_\tau = \Pi^c x_\tau \) (\( \tau \in J \)), we deduce that

\[ \Pi^c \hat{x}_t = T^c(t - t_0) \Pi^c \hat{x}_{t_0} + \lim_{n \to \infty} \int_{t_0}^{t} T^c(t - \sigma) \Pi^c \Gamma^nf_\delta(x_\sigma) d\sigma, \]

and thus, (13) becomes

\[ \Pi^c x_s(t, \hat{x}_t, f_\delta) = T^c(s - t_0) \Pi^c \hat{x}_{t_0} + \lim_{n \to \infty} \int_{t_0}^{t} T^c(s - \sigma) \Pi^c \Gamma^nf_\delta(x_\sigma) d\sigma \]
\[ - \lim_{n \to \infty} \int_{s}^{t} T^c(s - \sigma) \Pi^c \Gamma^nf_\delta(x_\sigma(t, \hat{x}_t, f_\delta)) d\sigma, \quad t \in J. \]

Therefore

\[ \| y(s; t) - y(s; t_0) \|_X = \| \Pi^c x_s(t, \hat{x}_t, f_\delta) - \Pi^c x_s(t_0, \hat{x}_{t_0}, f_\delta) \|_X \]
\[ = \left\| \lim_{n \to \infty} \int_{t_0}^{t} T^c(s - \sigma) \Pi^c \Gamma^nf_\delta(x_\sigma) d\sigma \right. \]
\[ - \lim_{n \to \infty} \int_{s}^{t} T^c(s - \sigma) \Pi^c \Gamma^nf_\delta(x_\sigma(t, \hat{x}_t, f_\delta)) d\sigma \]
\[ + \lim_{n \to \infty} \int_{s}^{t} T^c(s - \sigma) \Pi^c \Gamma^nf_\delta(x_\sigma(t_0, \hat{x}_{t_0}, f_\delta)) d\sigma \right\|_X \]
\[ \leq \int_{t_0}^{t} CC_1 e^{\epsilon|s-\sigma|} \zeta_*(\delta) \| x_\sigma - x_\sigma(t, \hat{x}_t, f_\delta) \|_X d\sigma \]
\[ + \int_{s}^{t} CC_1 e^{\epsilon|s-\sigma|} \zeta_*(\delta) \| x_\sigma(t_0, \hat{x}_{t_0}, f_\delta) - x_\sigma(t, \hat{x}_t, f_\delta) \|_X d\sigma. \quad (14) \]
Observe that
\begin{align}
\|x_\sigma - x_\sigma(t, \hat{x}_t, f_\delta)\|_X &\leq \|\Pi^c x_\sigma - F_{*, \delta}(\Pi^c x_\sigma)\|_X + \|F_{*, \delta}(\Pi^c x_\sigma(t, \hat{x}_t, f_\delta))\|_X \\
&\quad + \|\Pi^c x_\sigma - \Pi^c x_\sigma(t, \hat{x}_t, f_\delta)\|_X \\
&\leq \|\xi(\sigma)\|_X + (1 + L(\delta))\|\Pi^c x_\sigma - \Pi^c x_\sigma(t, \hat{x}_t, f_\delta)\|_X. \tag{15}
\end{align}

where we used Proposition 4 (i) and (iii). Note also that
\begin{align}
\|x_\sigma(t_0, \hat{x}_{t_0}, f_\delta) - x_\sigma(t, \hat{x}_t, f_\delta)\|_X &\leq \|F_{*, \delta}(\Pi^c x_\sigma(t_0, \hat{x}_{t_0}, f_\delta)) - F_{*, \delta}(\Pi^c x_\sigma(t, \hat{x}_t, f_\delta))\|_X \\
&\quad + \|\Pi^c x_\sigma(t_0, \hat{x}_{t_0}, f_\delta) - \Pi^c x_\sigma(t, \hat{x}_t, f_\delta)\|_X \\
&\leq (1 + L(\delta))\|y(\sigma; t) - y(\sigma; t_0)\|_X. \tag{16}
\end{align}

In view of (14), (15) and (16), combined with Proposition 5 (ii), we deduce
\begin{align}
\|y(s; t) - y(s; t_0)\|_X &\leq \int_{t_0}^{t} Ke^{(\epsilon - \mu')(t_{0} - \sigma)}(\|\xi(\sigma)\|_X + (1 + L(\delta))\|\Pi^c x_\sigma - \Pi^c x_\sigma(t, \hat{x}_t, f_\delta)\|_X)d\sigma \\
&\quad + \int_{t_0}^{t} Ke^{(\sigma-s)}(1 + L(\delta))\|y(\sigma; t) - y(\sigma; t_0)\|_X d\sigma \\
&\quad + \int_{t_0}^{t} Ke^{(\sigma-s)}(1 + L(\delta))\|y(\sigma; t) - y(\sigma; t_0)\|_X d\sigma. \tag{17}
\end{align}

Notice that the second term of the right-hand side becomes
\begin{align}
K \int_{t_0}^{t} (e^{(t_0-s)+\mu'(t_0-t_0)} - e^{(t_0-s)})\|\xi(\sigma)\|_X d\sigma
end{align}

because of (12). So we see from (17) that for \(s \leq t_0\)
\begin{align}
e^{\epsilon s}\|y(s; t) - y(s; t_0)\|_X &\leq K \int_{t_0}^{t} e^{(\epsilon - \mu')(t_{0} - \sigma)}\|\xi(\sigma)\|_X d\sigma \\
&\quad + K(1 + L(\delta)) \int_{s}^{t} e^{\epsilon \sigma}\|y(\sigma; t) - y(\sigma; t_0)\|_X d\sigma.
\end{align}

By Gronwall’s inequality and (12)
\begin{align}
e^{\epsilon s}\|y(s; t) - y(s; t_0)\|_X &\leq \left(K \int_{t_0}^{t} e^{(\epsilon - \mu')(t_{0} - \sigma)}\|\xi(\sigma)\|_X d\sigma\right) e^{K(1 + L(\delta))(t_0 - s)} \\
&= Ke^{-(\mu' - \epsilon)s} \int_{t_0}^{t} e^{\mu' \sigma}\|\xi(\sigma)\|_X d\sigma,
\end{align}

which yields the desired one. \(\square\)
Proposition 7. Assume that $\Sigma^u = \emptyset$, and let $\delta \in (0, \delta_1]$ be a sufficiently small number satisfying

$$\max \left( \mu', \frac{K(\alpha - \varepsilon)}{\alpha - \mu'} \right) < \alpha. \tag{18}$$

If $x(t)$ is a solution of $(E_{\delta})$ on $J = [t_0, t_1]$, then the function $\xi(t) := \Pi^c x_t - F_{*,\delta}(\Pi^c x_t)$ satisfies the inequality

$$\|\xi(t)\|_X \leq C\|\xi(t_0)\|_X e^{-\beta_0(t-t_0)}, \quad t \in J,$$

where $\beta_0 := \alpha - K(\alpha - \varepsilon)/(\alpha - \mu') > 0$. If in particular $J = [t_0, \infty)$, dist $(x_t, W^c_{\delta})$ tends to 0 exponentially as $t \to \infty$.

Proof. By applying VCF, one can easily deduce the relation

$$\xi(t) = T^s(t-t_0)\xi(t_0) + \lim_{n \to \infty} \int_{t_0-t}^{0} T^s(-s)\Pi^s \Gamma^n (f_{\delta}(x_{s+t}) - f_{\delta}(\Lambda_{*,\delta}(\Pi^c x_t)(s))) ds$$

$$+ \lim_{n \to \infty} \int_{-\infty}^{t_0-t} T^s(-s)\Pi^s \Gamma^n (f_{\delta}(\Lambda_{*,\delta}(\Pi^c x_{t_0})(t-t_0+s)) - f_{\delta}(\Lambda_{*,\delta}(\Pi^c x_t)(s))) ds,$$

$t \in J$.

If we set $\hat{x}_t := \Pi^c x_t + F_{*,\delta}(\Pi^c x_t)$ for $t \in J$, by Proposition 4 (ii)

$$\Lambda_{*,\delta}(\Pi^c x_t)(s) = x_{\delta}(0, \hat{x}_t, f_{\delta}) = x_{\epsilon+t}(t, \hat{x}_t, f_{\delta})$$

and

$$\Lambda_{*,\delta}(\Pi^c x_{t_0})(t-t_0+s) = x_{t-t_0+s}(0, \hat{x}_{t_0}, f_{\delta}) = x_{\delta+t_0}(t, \hat{x}_{t_0}, f_{\delta})$$

in particular for $s \in \mathbb{R}^-$. So

$$\xi(t) = T^s(t-t_0)\xi(t_0) + \lim_{n \to \infty} \int_{t_0-t}^{0} T^s(-s)\Pi^s \Gamma^n (f_{\delta}(x_{s+t}) - f_{\delta}(x_{s+t}(t, \hat{x}_t, f_{\delta}))) ds$$

$$+ \lim_{n \to \infty} \int_{-\infty}^{t_0-t} T^s(-s)\Pi^s \Gamma^n (f_{\delta}(x_{s+t}(t_0, \hat{x}_{t_0}, f_{\delta})) - f_{\delta}(x_{s+t}(t, \hat{x}_t, f_{\delta}))) ds,$$

and thus

$$\|\xi(t)\|_X \leq C e^{-\alpha(t-t_0)}\|\xi(t_0)\|_X + \int_{t_0}^{t} Ke^{\alpha(\theta-t)}\|x_\theta(t, \hat{x}_t, f_{\delta})\|_X d\theta$$

$$+ \int_{-\infty}^{t_0} Ke^{\alpha(\theta-t)}\|x_\theta(t_0, \hat{x}_{t_0}, f_{\delta}) - x_\theta(t, \hat{x}_t, f_{\delta})\|_X d\theta.$$

Since $x_\theta(t, \hat{x}_t, f_{\delta})$ ($t \in J, \theta \in \mathbb{R}$) can be written as

$$x_\theta(t, \hat{x}_t, f_{\delta}) = \Pi^c x_\theta(t, \hat{x}_t, f_{\delta}) + \Pi^s x_\theta(t, \hat{x}_t, f_{\delta})$$

$$= \Pi^c x_\theta(t, \hat{x}_t, f_{\delta}) + F_{*,\delta}(\Pi^c x_\theta(t, \hat{x}_t, f_{\delta}))$$
by Proposition 4 (iii), it follows from Proposition 4 (i) and Proposition 6 that for \( \theta \leq t_0 \)

\[
\|x_{\theta}(t_0, \hat{x}_{t_0}, f_\delta) - x_{\theta}(t, \hat{x}_{t}, f_\delta)\|_X \leq \|\Pi x_{\theta}(t_0, \hat{x}_{t_0}, f_\delta) - \Pi x_{\theta}(t, \hat{x}_{t}, f_\delta)\|_X \\
+ \|F_{*,\delta}(\Pi x_{\theta}(t_0, \hat{x}_{t_0}, f_\delta)) - F_{*,\delta}(\Pi x_{\theta}(t, \hat{x}_{t}, f_\delta))\|_X \\
\leq (1 + L(\delta))\|y(\theta; t) - y(\theta; t_0)\|_X
\]

where \( y(\theta; t) \) is the one in Proposition 6. On the other hand, for \( t_0 \leq \theta \leq t \)

\[
\|x_{\theta} - x_{\theta}(t_0, \hat{x}_{t_0}, f_\delta)\|_X \leq \|\Pi x_{\theta} - F_{*,\delta}(\Pi x_{\theta})\|_X \\
+ \|F_{*,\delta}(\Pi x_{\theta}) - F_{*,\delta}(\Pi x_{\theta}(t, \hat{x}_{t}, f_\delta))\|_X \\
\leq \|\xi(\theta)\|_X + (1 + L(\delta))\|\Pi x_{\theta} - \Pi x_{\theta}(t, \hat{x}_{t}, f_\delta)\|_X
\]

where we used Proposition 4 (i), (iii) and Proposition 5 (ii). Thus we have

\[
\|\xi(t)\|_X \leq Ce^{-\alpha(t-t_0)}\|\xi(t_0)\|_X \\
+ \int_{t_0}^{t} Ke^{\alpha(t-\tau)}\left(\|\xi(\tau)\|_X + (1 + L(\delta))\int_{\theta}^{t} e^{\mu'(\tau-\theta)}\|\xi(\tau)\|_X d\tau\right) d\theta \\
+ \int_{-\infty}^{t_0} Ke^{\alpha(t-\tau)}(1 + L(\delta))K\left(\int_{t_0}^{t} e^{\mu'(\tau-\theta)}\|\xi(\tau)\|_X d\tau\right) d\theta
\]

so that

\[
e^{\alpha t}\|\xi(t)\|_X \leq Ce^{\alpha t_0}\|\xi(t_0)\|_X + \hat{K} \int_{t_0}^{t} e^{\alpha \sigma}\|\xi(\sigma)\|_X d\sigma,
\]

where \( \hat{K} := K + K^2(1 + L(\delta))/(\alpha - \mu') \). An application of Gronwall's inequality gives

\[
e^{\alpha t}\|\xi(t)\|_X \leq Ce^{\alpha t_0}\|\xi(t_0)\|_X e^{\hat{K}(t-t_0)}, \quad t \in J,
\]

which is the desired one because of \( \hat{K} = K(\alpha - \epsilon)/(\alpha - \mu') = \alpha - \beta_0 \).

The latter part of the proposition is evident. This completes the proof. \(\square\)

**Proof of Theorem 1.** The properties (ii) and (iii) of Theorem 1 are now immediate consequences of Propositions 4 and 7, respectively. We verify the property (i). Observe that \( Y_{\eta} \) is a subspace of \( Y_{\eta'} \) if \( \eta < \eta' < \alpha \), and denote the inclusion map by \( J : Y_{\eta} \to Y_{\eta'} \). By
almost the same reasoning as in [8], we see that $J\Lambda_{*,\delta}$ is $C^1$ smooth as a map from $E^c$ to $Y_{\eta'}$; and hence $F_{*,\delta} = \Pi^{su} \circ ev_0 \circ J\Lambda_{*,\delta}$ is also $C^1$ smooth. Moreover, since

$$[[D(J\Lambda_{*,\delta})(0)](t)]\psi = T^c(t)\psi, \quad \psi \in E^c, \quad t \in \mathbb{R}$$

holds by virtue of $Df_\delta(0) = Df(0) = 0$, it follows that

$$DF_{*,\delta}(0)\psi = D(\Pi^{su} \circ ev_0 \circ J\Lambda_{*,\delta})(0)\psi = \Pi^{su}T^c(0)\psi = \Pi^{su}\psi = 0, \quad \psi \in E^c;$$

hence $DF_{*,\delta}(0) = 0$, which implies (i).

4 Stability analysis of integral equations via central equations

Center manifolds play a crucial role in the stability analysis of systems around non-hyperbolic equilibria. Indeed, center manifolds for several kinds of equations allow us to reduce the stability analysis of an original system to that of its restriction to a center manifold; see e.g., [1, 4, 5, 9]. In this section, introducing an ordinary differential equation (called the "central equation" of Eq. (1)) which is expressed by using the explicit formula of the projection $\Pi^c$, we will establish the reduction principle for integral equations that the stability properties for the central equation imply those of Eq. (1) in the neighborhood of its zero solution.

Assume that $\Sigma^c \neq \emptyset$. Let \{\phi_1, \ldots, \phi_{d_c}\} be a basis for $E^c$, where $d_c$ is the dimension of $E^c$. Then based on the formal adjoint theory for Eq. (1) developed in [7], one can consider its dual basis as elements in the Banach space

$$X^\sharp := L^\rho_{\rho}(\mathbb{R}^+; (\mathbb{C}^*)^m) = \{\psi : \mathbb{R}^+ \rightarrow (\mathbb{C}^*)^m : \psi(\tau)e^{-\rho\tau} \text{ is integrable on } \mathbb{R}^+\}$$

with norm

$$||\psi||_{X^\sharp} := \int_0^\infty |\psi(\tau)|e^{-\rho\tau}d\tau, \quad \psi \in X^\sharp,$$

where $(\mathbb{C}^*)^m$ is the space of $m$-dimensional row vectors with complex components equipped with the norm which is compatible with the one in $\mathbb{C}^m$, that is, $|z^*z| \leq |z^*||z|$ for $z^* \in (\mathbb{C}^*)^m$ and $z \in \mathbb{C}^m$. To be more precise, if we set

$$\langle\psi, \phi\rangle := \int_{-\infty}^{0} \left( \int_{\theta}^{0} \psi(\xi - \theta)K(-\theta)\phi(\xi)d\xi \right)d\theta, \quad (\psi, \phi) \in X^\sharp \times X,$$

then this pairing defines a bounded bilinear form on $X^\sharp \times X$ with the property

$$|\langle\psi, \phi\rangle| \leq ||K||_{\infty,\rho}||\psi||_{X^\sharp}||\phi||_X, \quad (\psi, \phi) \in X^\sharp \times X;$$
here we recall that $\|K\|_{\infty,0} = \text{ess sup}\{\|K(t)\|e^{\sigma t} : t \geq 0\}$. Then there exist $\{\psi_1, \ldots, \psi_{d_c}\}$, elements of $X^d$, such that $\langle \psi_i, \phi_j \rangle = 1$ if $i = j$ and 0 otherwise, and $\langle \psi_i, \phi \rangle = 0$ for $\phi \in E^s$ and $i = 1, 2, \ldots, d_c$; we call $\{\psi_1, \ldots, \psi_{d_c}\}$ the dual basis of $\{\phi_1, \ldots, \phi_{d_c}\}$; see [7] for details. Denote by $\Phi_c$ and $\Psi_c$, $(\phi_1, \ldots, \phi_{d_c})$ and $t'(\psi_1, \ldots, \psi_{d_c})$, the transpose of $(\psi_1, \ldots, \psi_{d_c})$, respectively. Then, for any $\phi \in X$ the coordinate of its $E^c$-component with respect to the basis $\{\phi_1, \ldots, \phi_{d_c}\}$, or $\Phi_c$ for short, is given by $\langle \Psi_c, \phi \rangle := t'((\psi_1, \phi), \ldots, (\psi_{d_c}, \phi)) \in \mathbb{C}^{d_c}$, and therefore the projection $\Pi^c$ is expressed, in terms of the basis $\Phi_c$ and its dual basis $\Psi_c$, by

$$\Pi^c \phi = \Phi_c \langle \Psi_c, \phi \rangle, \quad \phi \in X. \quad (19)$$

Since $\{T^c(t)\}_{t \geq 0}$ is a strongly continuous semigroup on the finite dimensional space $E^c$, there exists a $d_c \times d_c$ matrix $G_c$ such that

$$T^c(t)\Phi_c = \Phi_c e^{tG_c}, \quad t \geq 0, \quad (20)$$

and $\sigma(G_c)$, the spectrum of $G_c$, is identical with $\Sigma^c$. The $E^c$-components of solutions of Eq. $(E_\delta)$ can be described by a certain ordinary differential equation in $\mathbb{C}^{d_c}$. More precisely, let $x(t)$ be a solution of Eq. $(E_\delta)$ through $(\sigma, \phi)$, that is, $x(t) = x(t; \sigma, \phi, f)$. If we denote by $z_c(t)$ the component of $\Pi^c x_t$ with respect to the basis $\Phi_c$, that is, $\Phi_c z_c(t) := \Pi^c x_t$, or $z_c(t) := \langle \Psi_c, x_t \rangle$, then by virtue of [6, Theorem 7] $z_c(t)$ satisfies the ordinary differential equation

$$\dot{z}_c(t) = G_c z_c(t) + H_c f_\delta(\Phi_c z_c(t) + \Pi^m x_t), \quad (21)$$

where $H_c$ is the $d_c \times m$ matrix such that $H_c x := \lim_{n \to \infty} \langle \Psi_c, \Gamma^n x \rangle$ for $x \in \mathbb{C}^m$.

In connection with Eq. (21), let us consider the ordinary differential equations on $\mathbb{C}^{d_c}$

$$\dot{z}(t) = G_c z(t) + H_c f_\delta(\Phi_c z(t) + F_\star(f(\Phi_c z(t)))) \quad (CE_{\delta})$$

and

$$\dot{z}(t) = G_c z(t) + H_c f(\Phi_c z(t) + F(\Phi_c z(t))). \quad (CE)$$

We call Eq. $(CE)$ (resp. Eq. $(CE_{\delta})$) the central equation of $(E)$ (resp. $(E_\delta)$). Applying Proposition 4 (iii), one can easily derive the following result on relationships among solutions of Eq. $(E_\delta)$ (resp. Eq. $(E)$) and $(CE_{\delta})$ (resp. $(CE)$).

**Proposition 8.** The following statements hold true:

(i) Let $x$ be a solution of Eq. $(E_\delta)$ on an interval $J$ such that $x_t \in W^c_\delta$ ($t \in J$). Then the function $z_c(t) := \langle \Psi_c, x_t \rangle$ satisfies the equation $(CE_{\delta})$ on $J$.

Conversely, if $z(t)$ satisfies the equation $(CE_{\delta})$ on an interval $J$, then there exists a unique solution $x$ of Eq. $(E_\delta)$ on $J$ such that $x_t \in W^c_\delta$ and $\Pi^c x_t = \Phi_c z(t)$ on $J$. 

(ii) Let $x$ be a solution of Eq. (E) on an interval $J$ such that $x_t \in \mathcal{W}^{c}(r, \delta)$ ($t \in J$).
Then the function $z_c(t) := \langle \psi_c, x_t \rangle$ satisfies the equation (CE) on $J$, together with the inequality
\[ \sup_{t \in J} \| \Phi_c z_c(t) \| \leq r. \]
Conversely, if $z(t)$ satisfies the equation (CE) on an interval $J$ together with the inequality
\[ \sup_{t \in J} \| \Phi_c z(t) \| \leq r, \] then there exists a unique solution $x$ of Eq. (E) on $J$ such that $x_t \in \mathcal{W}^{c}(r, \delta)$ and $\Pi^c x_t = \Phi_c z(t)$ on $J$.

Since $f(0) = f_\delta(0) = 0$, both equations (CE) and (CE$\delta$) (as well as (E) and (E$\delta$)) possess the zero solution. Notice that the zero solution of (CE) (resp. (E)) is uniformly asymptotically stable if and only if the zero solution of (CE$\delta$) (resp. (E$\delta$)) is uniformly asymptotically stable. Likewise, the zero solution of (CE) (resp. (E)) is unstable if and only if the zero solution of (CE$\delta$) (resp. (E$\delta$)) is unstable. Here, for the definition of several stability properties utilized in this paper, we refer readers to the books [10, 5].

Now suppose that $\Sigma^u = \emptyset$. Then the dynamics near the zero solution of (E) is determined by the dynamics near $z_c = 0$ of (CE) in the following sense.

**Theorem 3.** Assume that $\Sigma^u = \emptyset$. If the zero solution of (CE) is uniformly asymptotically stable (resp. unstable), then the zero solution of (E) is also uniformly asymptotically stable (resp. unstable).

**Proof.** By the fact stated in the preceding paragraph of the theorem, it is sufficient to establish that the uniform asymptotic stability (resp. instability) of the zero solution of (CE$\delta$) implies the uniform asymptotic stability (resp. instability) of the zero solution of (E$\delta$).

If the zero solution of (CE$\delta$) is unstable, the instability of the zero solution of (E$\delta$) immediately follows from the invariance of $\mathcal{W}^{\delta}$ (Proposition 4 (iii)). In what follows, under the assumption that the the zero solution of (CE$\delta$) is uniformly asymptotically stable, we will establish the uniform asymptotic stability of the zero solution of (E$\delta$). By virtue of [5, Theorem 4.2.1], there exist positive constants $a$, $\bar{K}$ and a Liapunov function $V$ defined on $S_a := \{ y \in \mathbb{C}^d : |y| \leq a \}$ satisfying the following properties:

(i) There exists a $b \in C(\mathbb{R}^+; \mathbb{R}^+)$ which is strictly increasing with $b(0) = 0$ and
\[ b(|y|) \leq V(y) \leq |y| \text{ for } y \in S_a. \]

(ii) $|V(y) - V(z)| \leq \bar{K}|y - z|$ for $y, z \in S_a$.

(iii) $\dot{V}(z) \leq -V(z)$ for $z \in S_a$, where $\dot{V}(z) := \lim_{h \to +0} \sup_{h \to +0} (1/h)\{V(y(h)) - V(z)\}$, and $y(h)$ is the solution of (CE$\delta$) with $y(0) = z$. 

Choose a positive number $\tau_0$ such that
\[ e^{-\tau_0} \leq \frac{1}{2} \quad \text{and} \quad Ce^{-\beta_0\tau_0} \leq \frac{1}{4}, \]
where $\beta_0$ is the one in Proposition 7, and we may assume that $\beta_0 > \mu'$, taking $\delta$ so small if necessary. Put $K_\infty := \|K\|_{\infty, \rho}$ and take a positive number $P$ in such a way that
\[ P > \max \left( 1, \frac{4C}{\beta_0 - \mu} \frac{\bar{K}KK_\infty \|\Psi_c\|}{\beta_0 - \mu} \right), \]
and set $a_0 := ae^{-\eta\tau_0}/(4CK_\infty \|\Psi_c\|)$, where $\|\Psi_c\| := \left( \sum_{j=1}^{d_c} \|\psi_j\|_{X\#}^2 \right)^{1/2}$. Let $\Omega$ be a neighborhood of 0 in $X$ such that
\[ \langle \Psi_c, \phi \rangle \in S_a, \quad \|\Pi^c\phi\|_X \leq a_0, \quad \text{and} \quad Q \leq b(a) \]
for $\phi \in \Omega$, where
\[ Q := V(\langle \Psi_c, \phi \rangle) + \left( PC + \frac{\bar{K}KK_\infty \|\Psi_c\|KC}{\beta_0 - \mu} \right) \left( \|\Pi^s\phi\|_X + \|F_{*,\delta}(\Pi^c\phi)\|_X \right), \]
and consider the function $W(\phi)$ on $\Omega$ defined by
\[ W(\phi) := V(\langle \Psi_c, \phi \rangle) + P\|\Pi^s\phi - F_{*,\delta}(\Pi^c\phi)\|_X, \quad \phi \in \Omega. \]
$W$ is continuous in $\Omega$ with $W(0) = 0$ and is positive in $\Omega \setminus \{0\}$ because of (i) and (ii).

We will first certify the following claim.

**Claim 1.** There exists a positive number $c_0$ such that, for any $t_0 \in \mathbb{R}^+$ and $\phi \in X$ with $W(\phi) \leq c_0$, the solution $x(t; t_0, \phi, f_{\delta})$ exists on $[t_0, t_0 + \tau_0]$ and satisfies $x_t(t_0, \phi, f_{\delta}) \in \Omega$ for $t \in [t_0, t_0 + \tau_0]$; in particular, $\|\Pi^c x_t(t_0, \phi, f_{\delta})\|_X \leq a_0$ in this interval.

Indeed, suppose that $x_t(t_0, \phi, f_{\delta})$ is defined on the interval $[t_0, t_0 + t_*]$ with $t_* \leq \tau_0$. Applying VCF, we get
\[ \|x_t(t_0, \phi, f_{\delta})\|_X \leq M\|\phi\|_X + \int_{t_0}^{t} M\zeta_*(\delta)\|x_s(t_0, \phi, f_{\delta})\|_X ds \]
for $t \in [t_0, t_0 + t_*]$, where $M := \sup_{0\leq s\leq \tau_0} \|T(t)\|_{C(X)}$. Then Gronwall’s inequality yields that $\|x_t(t_0, \phi, f_{\delta})\|_X \leq M\|\phi\|_X e^{M\zeta_*(\delta)(t-t_0)} \leq M\|\phi\|_X e^{M\zeta_*(\delta)\tau_0}$ for $t \in [t_0, t_0 + t_*]$; which means that $x_t(t_0, \phi, f_{\delta})$ can be defined on the interval $[t_0, t_0 + t_*]$ and therefore on $[t_0, t_0 + \tau_0]$ (cf. [6, Corollary 1]). Thus it turns out that if $\|\phi\|_X$ is small enough, $x_t(t_0, \phi, f_{\delta})$ exists on $[t_0, t_0 + \tau_0]$ and moreover belongs to $\Omega$ in this interval. The claim readily follows from the fact that $\inf\{W(\phi) : \phi \in \Omega, \|\phi\|_X \geq r\} > 0$ for small $r > 0$, together with the property of $\Omega$. 

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Now given \( t_0 \in \mathbb{R}^+ \) and \( \phi \in X \) with \( W(\phi) \leq c_0 \), let us consider the solution \( x(t) := x(t; t_0, \phi, f_\delta) \). By Proposition 3 (i)

\[
\|\Lambda_{*,\delta}(\Pi^c x_t)(s)\|_X \leq \|\Lambda_{*,\delta}(\Pi^c x_t)\|_{Y_\eta} e^{\eta|s|} \leq e^{\eta|s|}2C\|\Pi^c x_t\|_X, \quad s \in \mathbb{R};
\]

hence taking account of \( \Lambda_{*,\delta}(\Pi^c x_t)(s) = x_{t+s}(t, \hat{x}_t, f_\delta) \) for \( s \in \mathbb{R} \) (Proposition 4 (ii)), we get \( \|x_{t+s}(t, \hat{x}_t, f_\delta)\|_X \leq e^{\eta t_0}2C\|\Pi^c x_t\|_X \) for \( s \in [-\tau_0, 0] \), where \( \hat{x}_t := \Pi^c x_t + F_{*,\delta}(\Pi^c x_t) \).

Set \( y^{O}(t+s;t) := \langle \Psi_c, x_{t+s}(t, \hat{x}_t, f_\delta) \rangle \).

Then

\[
|y^{O}(t+s;t)| \leq K_{\infty}\|\Psi_c\|\|x_{t+s}(t, \hat{x}_t, f_\delta)\|_X \leq 2CK_{\infty}\|\Psi_c\| e^{\eta t_0}\|\Pi^c x_t\|_X \leq 2CK_{\infty}\|\Psi_c\| e^{\eta_{\mathcal{T}0}} a_0 = a/2
\]

for \( s \in [-\tau_0, 0] \);

hence \( y^{O}(s;t) \in S_{a/2} \) and thus \( V(y^{O}(s;t)) \) is well-defined for \( s \in [t_0, t] \) with \( t \in [t_0, t_0 + \tau_0] \).

We next confirm:

**Claim 2.** \( \sup\{W(x_t) : t \in [t_0, t_0 + \tau_0]\} \leq Q \) and \( W(x_{t_0+\tau_0}(t_0, \phi, f_\delta)) \leq c_0/2 \).

Indeed, fix a \( t \in [t_0, t_0 + \tau_0] \) and set \( z(s) := y^{O}(s;t) \) for \( s \in [t_0, t] \). Since \( y^{O}(s;t) = \langle \Psi_c, x_s(t, \hat{x}_t, f_\delta) \rangle = \langle \Psi_c, \Pi^c x_s(t, \hat{x}_t, f_\delta) \rangle \) for \( s \in [t_0, t] \), \( z(s) \) is a solution of \((CE_{\delta})\) on \([t_0, t]\) with \( z(t) = y^{O}(t;t) = \langle \Psi_c, \Pi^c x_t \rangle \).

By the property (i), we see that \( \dot{V}(z(\mathcal{S})) \leq -V(z(s)) \) for \( s \in [t_0, t] \), which implies that \( (d/ds)(e^{s-t}V(z(s))) = e^{s-t}(V(z(s)) + \dot{V}(z(s))) \leq 0 \), so that

\[
V(\langle \Psi_c, \Pi^c x_t \rangle) - e^{t-t_0}V(y^{O}(t_0;t)) = V(z(t)) - e^{t-t_0}V(z(t_0)) \leq \int_{t_0}^{t} \frac{d}{d\mathcal{S}}(e^{s-t}V(z(s))) ds \leq 0;
\]

consequently,

\[
V(\langle \Psi_c, \Pi^c x_t \rangle) \leq e^{t-t_0}V(\langle \Psi_c, \Pi^c x_{t_0} \rangle) + e^{t-t_0}(V(y^{O}(t_0;t)) - V(\langle \Psi_c, \Pi^c x_{t_0} \rangle))
\]

\[
\leq e^{t-t_0}V(\langle \Psi_c, \Pi^c x_{t_0} \rangle) + e^{t-t_0}\bar{K}\|y^{O}(t_0;t) - \langle \Psi_c, \Pi^c x_{t_0} \rangle\|
\]

\[
\leq e^{t-t_0}V(\langle \Psi_c, \Pi^c \phi \rangle) + e^{t-t_0}\bar{K}K_{\infty}\|\Psi_c\|\|\Pi^c x_{t_0}(t, \hat{x}_t, f_\delta) - \Pi^c x_{t_0}\|_X
\]

\[
\leq e^{t-t_0}V(\langle \Psi_c, \Pi^c \phi \rangle) + e^{t-t_0}\bar{K}K_{\infty}\|\Psi_c\|K \int_{t_0}^{t} e^{\mu'(\theta-t_0)}\|\xi(\theta)\|_X d\theta,
\]

where the last inequality is due to Proposition 5 (ii). Therefore, applying Proposition 7,

\[
W(x_t) = V(\langle \Psi_c, \Pi^c x_t \rangle) + P\|\xi(t)\|_X
\]

\[
\leq e^{t-t_0}V(\langle \Psi_c, \Pi^c \phi \rangle) + e^{t-t_0}\bar{K}K_{\infty}\|\Psi_c\|K \int_{t_0}^{t} e^{\mu'(-\theta-t_0)}(C\|\xi(t_0)\|_X e^{-\beta_0(\theta-t_0)}) d\theta
\]

\[
+ PC\|\xi(t_0)\|_X e^{-\beta_0(-t-t_0)}
\]

\[
\leq e^{t-t_0}V(\langle \Psi_c, \Pi^c \phi \rangle) + C\|\xi(t_0)\|_X \left( \frac{\bar{K}K_{\infty}K\|\Psi_c\|}{\beta_0 - \mu'} e^{t-t_0} + Pe^{-\beta_0(t-t_0)} \right).
\]

(24)

In particular,

\[
W(x_{t_0+\tau_0}) \leq e^{-\tau_0}V(\langle \Psi_c, \Pi^c \phi \rangle) + C\|\xi(t_0)\|_X \left( \frac{\bar{K}K_{\infty}K\|\Psi_c\|}{\beta_0 - \mu'} e^{-\tau_0} + Pe^{-\beta_0\tau_0} \right)
\]

\[
\leq (1/2)V(\langle \Psi_c, \Pi^c \phi \rangle) + (1/2)P\|\xi(t_0)\|_X
\]

\[
= (1/2)W(x_{t_0}) = (1/2)W(\phi) \leq (1/2)c_0.
\]
Since $\|\xi(t_0)\|_X \leq \|\Pi^s\phi\|_X + \|F_{*,\delta}(\Pi^c\phi)\|_X$, (24) implies also
\[
\sup\{W(x_t) : t \in [t_0, t_0 + t_0]\} \leq V((\Psi_c, \Pi^c\phi)) + C\|\xi(t_0)\|_X \left(\frac{\bar{K}K_{\infty}K\|\Psi_c\|}{\beta_0 - \mu'} + P\right) \leq Q,
\]
as required.

By Claim 2, combined with Claim 1, $x(t) = x(t; t_0, \phi, f_{\delta})$ is defined on $[t_0, t_0 + 2\tau_0]$, and $y^\phi(s; t) \in S_{a/2}$ still holds for $s \in [t_0, t]$ with $t \in [t_0, t_0 + 2\tau_0]$. More generally, one can deduce that $x(t) = x(t; t_0, \phi, f_{\delta})$ is defined on $[t_0, t_0 + n\tau_0]$, and $y^\phi(s; t) \in S_{a/2}$ holds for $s \in [t_0, t]$ with $t \in [t_0, t_0 + n\tau_0]$ for any $n \in \mathbb{N}$, together with the relations
\[
\sup\{W(x_t) : t \in [t_0 + (n - 1)\tau_0, t_0 + n\tau_0]\} \leq \frac{Q}{2^n-1} \quad \text{and} \quad W(x_{t_0+n\tau_0}) \leq \frac{c_0}{2^n}
\]
for $n \in \mathbb{N}$. This means that $x(t) = x(t; t_0, \phi, f_{\delta})$ is actually defined on $[t_0, \infty)$ and that
\[
V((\Psi_c, x_t(t_0, \phi, f_{\delta}))) + P\|\Pi^s x_t - F_{*,\delta}(\Pi^c x_t)\|_X \leq Q 2^{-\frac{(t-t_0)}{\tau_0}}, \quad t \in [t_0, \infty).
\]
In view of (i) and $P > 1$, it follows that $b((\Psi_c, x_t(t_0, \phi, f_{\delta})) \leq Q 2^{-\frac{(t-t_0)}{\tau_0}} \leq b(a)$ and $\|\Pi^s x_t - F_{*,\delta}(\Pi^c x_t)\|_X \leq Q 2^{-\frac{(t-t_0)}{\tau_0}}$. Since $\|\Pi^c x_t(t_0, \phi, f_{\delta})\|_X = \|\Phi_c\langle\Psi_c, x_t(t_0, \phi, f_{\delta})\rangle\|_X \leq \|\Phi_c\| b^{-1}(Q 2^{-\frac{(t-t_0)}{\tau_0}})$ with $\|\Phi_c\| :=(\sum_{j=1}^{d_c} \|\phi_j\|_X^2)^{1/2}$ and $\|\Pi^s x_t - F_{*,\delta}(\Pi^c x_t)\|_X + \|F_{*,\delta}(\Pi^c x_t)\|_X \leq Q 2^{-\frac{(t-t_0)}{\tau_0}} + L(\delta)\|\Pi^c x_t\|_X$, we obtain that for any $\phi \in \Omega$ and $t \in [t_0, \infty)$
\[
|x_t(t_0, \phi, f_{\delta})|_X \leq \|\Pi^c x_t(t_0, \phi, f_{\delta})\|_X \leq \|\Pi^s x_t(t_0, \phi, f_{\delta})\|_X \leq Q 2^{-\frac{(t-t_0)}{\tau_0}} + (1 + L(\delta))\|\Phi_c\| b^{-1}(Q 2^{-\frac{(t-t_0)}{\tau_0}}),
\]
which shows that the zero solution of $(E_{\delta})$ is uniformly asymptotically stable. \hfill \Box

Before concluding this section, we will provide an example to illustrate how our Theorem 3 is available for stability analysis of some concrete equations. Let us consider nonlinear (scalar) integral equation
\[
x(t) = \int_{-\infty}^{t} P(t-s)x(s)ds + f(x_t), \quad (25)
\]
where $P$ is a nonnegative continuous function on $\mathbb{R}^+$ satisfying $\int_{0}^{\infty} P(t)dt = 1$ together with the condition $\|P\|_{1,\rho} := \int_{0}^{\infty} P(t)e^{\rho t}dt < \infty$ and $\|P\|_{\infty,\rho} := \text{ess sup}\{P(t)e^{\rho t} : t \geq 0\} < \infty$ for some positive constant $\rho$, and $\phi \in C^1(X; \mathbb{C})$, $X := L^1_1(\mathbb{R}^{-}; \mathbb{C})$, satisfies $f(0) = 0$ and $Df(0) = 0$. Eq. (25) is written as Eq. (E) with $m = 1$ and $K \equiv P$. The characteristic operator $\Delta(\lambda)$ of Eq. (25) is given by $\Delta(\lambda) = 1 - \int_{0}^{\infty} P(t)e^{-\lambda t}dt$. We thus get $\Sigma^u = \emptyset$ and $\Sigma^c = \{0\}$. Indeed, in this case, 0 is a simple root of the equation $\Delta(\lambda) = 0$, and $E^c$ is 1-dimensional space with a basis $\{\phi_1\}, \phi_1 \equiv 1$, together with $\{\psi_1\}, \psi_1 \equiv 1/r$ (here
\[ r := \int_{0}^{\infty} \tau P(\tau) d\tau, \] as the dual basis of \( \{ \phi_1 \} \); see [7] for details. The projection \( \Pi^c \) is given by the formula \( \Pi^c \phi = \Phi_c(\psi_1, \phi) \), \( \forall \phi \in X \), and hence

\[
\Pi^c \phi = \phi_1(\psi_1, \phi) = \phi_1 \left( \int_{-\infty}^{0} \int_{\theta}^{0} \psi_1(\xi - \theta) P(-\theta) \phi(\xi) d\xi d\theta \right) = \Phi_c \left( \frac{1}{r} \int_{-\infty}^{0} P(-\theta) \left( \int_{\theta}^{0} \phi(\xi) d\xi \right) d\theta \right).
\]

Thus, for a solution \( x(t) \) of Eq. (25), the component \( z_c(t) \) of \( \Pi^c x_t \) with respect to \( \Phi_c \) is given by

\[
z_c(t) = \frac{1}{r} \int_{-\infty}^{t} \hat{P}(t - s) x(s) ds
\]

with \( \hat{P}(t) := \int_{t}^{\infty} P(\tau) d\tau \), because of

\[
rz_c(t) = \int_{-\infty}^{0} P(-\theta) \left( \int_{\theta}^{0} x(t + \xi) d\xi \right) d\theta = \int_{-\infty}^{0} P(-\theta) \left( \int_{t+\theta}^{t} x(s) ds \right) d\theta
\]

Observe that \( z_c(t) \) satisfies the ordinary equation

\[
r \dot{z}_c(t) = \hat{P}(0) x(t) + \int_{-\infty}^{t} (-P(t-s)) x(s) ds = x(t) - \int_{-\infty}^{t} P(t-s) x(s) ds,
\]

that is, \( r \dot{z}_c(t) = f(x_t) = f(\Phi_c z_c(t) + \Pi^s x_t) \). In particular, if \( x \) is a solution of Eq. (25) satisfying \( x_t \in W^{1oc}_c(r, \delta) \) on an interval \( J \), then \( \Pi^s x_t = F_*(\Phi_c z_c(t)) \) on \( J \); hence we get

\[
\dot{z}_c(t) = (1/r)f(\Phi_c z_c(t) + F_*(\Phi_c z_c(t)))
\]

on \( J \). This observation leads to that \( G_c = 0 \) and \( H_c = 1/r \) in the central equation \((CE)\); in fact, by noticing that \( \Sigma^c = \{ 0 \} \) and \( \Pi^s x = \lim_{n \to \infty} (\psi_1, \Gamma^n x) = (1/r) x \), \( \forall x \in \mathbb{C} \), one can also certify this fact. Consequently, the central equation of Eq. (25) is identical with the scalar equation \( \dot{z} = H(z) \); here

\[
H(w) := (1/r)f(\Phi_c w + F_*(\Phi_c w)), \quad (w \in \mathbb{C} \text{ and } |w| \text{ is small}).
\]

In what follows, we will determine the function \( H \) for some special functions \( f \).

Let us assume that \( f \) is of the form

\[
f(\phi) = \varepsilon \left( \int_{-\infty}^{0} Q(-\theta) \phi(\theta) d\theta \right)^m + g(\phi), \quad \forall \phi \in X,
\]

where \( m \) is a natural number such that \( m \geq 2 \), \( \varepsilon \) is a nonzero real number, \( Q \) is a function satisfying \( \|Q\|_{1,\rho} < \infty \) and \( \|Q\|_{\infty,\rho} < \infty \) and \( c_0 := \int_{-\infty}^{0} Q(-\theta) d\theta > 0 \), and
$g \in C^1(X; \mathbb{C})$ satisfies $|g(\phi)| = o(\|\phi\|_X^m)$ as $\|\phi\|_X \to 0$ (here, $o$ means Landau's notation "small oh"). One can easily see that the function $f$ given by (26) satisfies $f \in C^1(X; \mathbb{C})$ and $f(0) = Df(0) = 0$. For any $w$ with small $|w|$, we get
\[
f(\Phi_c w) = \epsilon \left( \int_{-\infty}^{0} Q(-\theta)(\Phi_c w)(\theta) d\theta \right)^m + g(\Phi_c w)
= \epsilon \left( w \int_{-\infty}^{0} Q(-\theta) d\theta \right)^m + o(w^m) = \epsilon (c_0 w)^m + o(w^m);
\]
hence,
\[
rH(w) = f(\Phi_c w) + F_\epsilon(\phi_c w)
= f(\Phi_c w) + f(\Phi_c w + F_\epsilon(\phi_c w)))
= f(\Phi_c w) + \epsilon \{ [L_1(\Phi_c w + F_\epsilon(\phi_c w))]^m - [L_1(\Phi_c w)]^m \} + o(w^m)
= \epsilon (c_0 w)^m + o(w^m) + \epsilon \sum_{k=0}^{m-1} \binom{m}{k} [L_1(\Phi_c w)]^k [F_\epsilon(\Phi_c w)]^{m-k},
\]
here $L_1$ is a bounded linear functional on $L^1_\rho$ defined by $L_1(\phi) := \int_{-\infty}^{0} Q(-\theta) \phi(\theta) d\theta$.
Recall that $L_1(F_\epsilon(\Phi_c w)) = o(w)$ as $w \to 0$; hence
\[
\sum_{k=0}^{m-1} \binom{m}{k} [L_1(\Phi_c w)]^k [F_\epsilon(\Phi_c w)]^{m-k} = o(w^m) \text{ as } w \to 0.
\]
Thus $rH(w) = \epsilon (c_0 w)^m + o(w^m)$ as $w \to 0$. Hence it follows that
\[
H(w) = \left( \frac{\epsilon}{r} \right) c_0^m w^m + o(w^m) \text{ as } w \to 0.
\]
Consequently, one can easily see that the zero solution of the central equation of Eq. (25)
is uniformly asymptotically stable if $\epsilon < 0$ and if $m$ is an odd natural number; while it
is unstable if $\epsilon > 0$ and if $m$ is an odd natural number, or if $\epsilon \neq 0$ and if $m$ is an even
natural number. Therefore, by virtue of Theorem 3, we get the following result:

**Proposition 9.** Assume that
\[
f(\phi) = \epsilon \left( \int_{-\infty}^{0} Q(-\theta) \phi(\theta) d\theta \right)^m + g(\phi), \quad \forall \phi \in X,
\]
here $\epsilon$ is a nonzero constant, $m$ is a natural number such that $m \geq 2$, $Q$ is a function
satisfying $\|Q\|_{1,\rho} < \infty$, $\|Q\|_{\infty,\rho} < \infty$ and $\int_{-\infty}^{\infty} Q(t) dt > 0$ and $g(\phi) = o(\|\phi\|_X^m)$ as $\|\phi\|_X \to 0$
with $g \in C^1(X; \mathbb{C})$. Then the following statements hold true;

(i) if $m$ is odd and $\epsilon < 0$, then the zero solution of Eq. (25) is uniformly asymptotically
stable (in $L^1_\rho$);

(ii) if $m$ is odd and $\epsilon > 0$, then the zero solution of Eq. (25) is unstable (in $L^1_\rho$);

(iii) if $m$ is even and $\epsilon \neq 0$, then the zero solution of Eq. (25) is unstable (in $L^1_\rho$).
References


