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INDECOMPOSABLE HILBERT REPRESENTATIONS OF THE KRONECKER QUIVER ON INFINITE-DIMENSIONAL HILBERT SPACES

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1. INTRODUCTION

This is a joint work with Yasuo Watatani. We aim to study relations between operator theory and Hilbert representations of quivers on infinite dimensional Hilbert spaces . Invariant subspace problem is the existence problem of simple representations of a loop in infinite dimensional Hilbert spaces. Three subspace problem is the existence problem of indecomposable representations of D_4 in infinite dimensional Hilbert spaces. We mainly report indecomposable Hilbert representations of the Kronecker quiver on infinite-dimensional Hilbert spaces.

2. FUNDAMENTAL CONCEPTS

At first we shall explain some notions to describe our results. A family $\Gamma=(V,E,s,r)$ is called a quiver if V is a vertex set and E is an edge set and s,r are mappings from E to V such that for $\alpha\in E$, $s(\alpha)\in V$ is the initial point of α and $r(\alpha)\in V$ is the end point of α. A quiver $\Gamma=(V,E,s,r)$ is called the Kronecker quiver if V is a two point set $\{0,1\}$ and E is a two point set $\{\alpha,\beta\}$ and $s(\alpha)=0, s(\beta)=0$, and $r(\alpha)=1, r(\beta)=1$. A pair (H,f) is called a Hilbert representation of a quiver Γ if $H=(H_v)_{v\in V}$ is a family of Hilbert spaces and $f=(f_\alpha)_{\alpha\in E}$ is a family of bounded linear operators f_α from $H_{s(\alpha)}$ to $H_{r(\alpha)}$. For Hilbert representations (K,g) and (K',g') of a quiver Γ, we define the direct sum (H,f) by $H_v=K_v\oplus K_v'$, $(v\in V)$, $f_\alpha=g_\alpha\oplus g_\alpha'$, $(\alpha\in E)$.

For Hilbert representations (H,f) and (K,g) of Γ, a homomorphism $\phi:(H,f)\to(K,g)$ is a family $\phi=(\phi_v)_{v\in V}$ of bounded operators $\phi_v: H_v\to K_v$ satisfying, for any arrow $\alpha\in E$, $\phi_{r(\alpha)}f_\alpha=g_\alpha\phi_{s(\alpha)}$.

Let $Hom((H,f),(K,g))$ be the set of homomorphisms from (H,f) to (K,g). Let $End(H,f)$ be the set $Hom((H,f),(H,f))$. Let $Idem(H,f)$ be the set of idempotents of $End(H,f)$.

Hilbert representations (H,f) and (K,g) of Γ are called isomorphic if there exists an isomorphism $\phi:(H,f)\to(K,g)$, that is, there exists a family $\phi=(\phi_v)_{v\in V}$ of bounded invertible operators $\phi_v\in B(H_v,K_v)$ such that, for any arrow $\alpha\in E$, $\phi_{r(\alpha)}f_\alpha=g_\alpha\phi_{s(\alpha)}$.

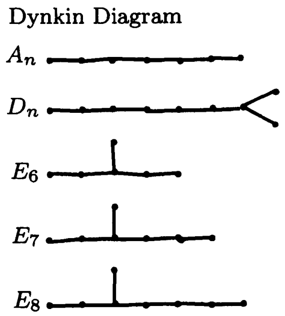
A Hilbert representation (H,f) of Γ is called indecomposable if it is not isomorphic to nontrivial direct sum of Hilbert representations of Γ.

A Hilbert representation (H, f) of Γ, is called transitive if $End(H, f) = \mathbb{C}$.

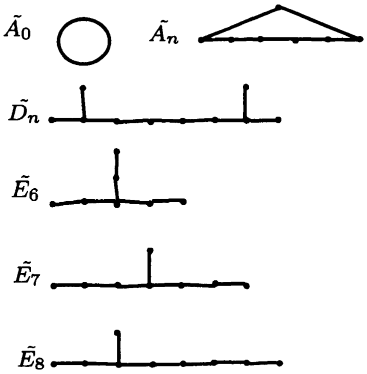
We note that a Hilbert representation (H, f) of Γ is indecomposable if and only if $Idem(H, f) = \{0, 1\}$.

3. Gabriel's theorem in infinite dimensional spaces

Gabriel's theorem says that a finite, connected quiver has only finitely many indecomposable representations if and only if the underlying undirected graph is one of Dynkin diagrams A_n, D_n, E_6, E_7, E_8.

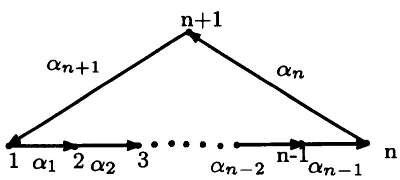


We succeeded in the establishment of a complement of Gabriel's theorem for Hilbert representations. We constructed some examples of indecomposable, infinite-dimensional representations of quivers with the underlying undirected graphs extended Dynkin diagrams $\tilde{A}_n$ $(n \geq 0)$,$\tilde{D}_n$ $(n \geq 4), \tilde{E}_6, \tilde{E}_7$ and $\tilde{E}_8$.



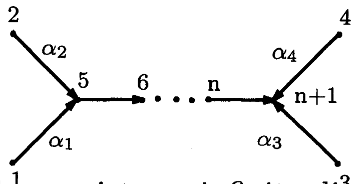
In order to do this, we considered the relative position of several subspaces along the quivers, where vertices are represented by a family of subspaces and arrows are represented by natural inclusion maps.

Let $\Gamma = (V, E, s, r)$ be a quiver whose underlying undirected graph is an extended Dynkin diagram $\tilde{A}_n$, $(n \geq 0)$. Then there exist uncountably many infinite-dimensional, indecomposable Hilbert representations of Γ. For example, consider



Define a Hilbert representation (H, f) of Γ by $H_1 = H_2 = \cdots = H_{n+1} = \ell^2(\mathbb{N})$, $f_{\alpha_2} = f_{\alpha_3} = \cdots = f_{\alpha_{n+1}} = I$ and $f_{\alpha_1} = S$, the unilateral shift. Then (H, f) is indecomposable.

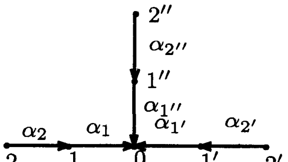
Lemma 3.1. *Let $\Gamma = (V, E, s, r)$ be the following quiver with the underlying undirected graph an extended Dynkin diagram $\tilde{D}_n$ for $n \geq 4$:*



Then there exists an infinite-dimensional, indecomposable Hilbert representation (H, f) of Γ.

Let $K = \ell^2(\mathbb{N})$ and S a unilateral shift on K. We define a Hilbert representation $(H, f) := ((H_v)_{v \in V}, (f_\alpha)_{\alpha \in E})$ of Γ as follows:
Define $H_1 = K \oplus 0$, $H_2 = 0 \oplus K$, $H_3 = \{(x, Sx) \in K \oplus K | x \in K\}$,
$H_4 = \{(x, x) \in K \oplus K | x \in K\}$, $H_5 = H_6 = \cdots = H_{n+1} = K \oplus K$.
Let $f_{\alpha_k} : H_{s(\alpha_k)} \to H_{r(\alpha_k)}$ be the inclusion map for any $\alpha_k \in E$ for $k = 1, 2, 3, 4$, and $f_\beta = id$ for other arrows $\beta \in E$. Then (H, f) is indecomposable. .

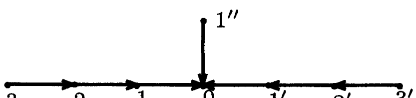
Consider the following quiver $\Gamma = (V, E, s, r)$



Then underlying undirected graph is an extended Dynkin diagram $\tilde{E}_6$. Let $K = \ell^2(\mathbb{N})$ and S a unilateral shift on K. We define a Hilbert representation $(H, f) := ((H_v)_{v \in V}, (f_\alpha)_{\alpha \in E})$ of Γ as follows:

Put $H_0 = K \oplus K \oplus K$, $H_1 = K \oplus 0 \oplus K$, $H_2 = 0 \oplus 0 \oplus K$,
$H_{1'} = K \oplus K \oplus 0$, $H_{2'} = 0 \oplus K \oplus 0$,
$H_{1''} = \{(x, x, x) + (y, Sy, 0) \in K^3 \mid x, y \in K\}$ and
$H_{2''} = \{(x, x, x) \in K^3 \mid x \in K\}$.
Then (H, f) is indecomposable.

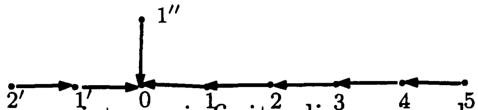
Lemma 3.2. *Let $\Gamma = (V, E, s, r)$ be the following quiver with the underlying undirected graph an extended Dynkin diagram $\tilde{E}_7$:*



Then there exists an infinite-dimensional, indecomposable Hilbert representation (H, f) of Γ.

Let $K = \ell^2(\mathbb{N})$ and S a unilateral shift on K. We define a Hilbert representation $(H, f) := ((H_v)_{v \in V}, (f_\alpha)_{\alpha \in E})$ of Γ as follows:
Let $H_0 = K \oplus K \oplus K \oplus K$, $H_1 = K \oplus 0 \oplus K \oplus K$,
$H_2 = K \oplus 0 \oplus \{(x, x); x \in K\}$, $H_3 = K \oplus 0 \oplus 0 \oplus 0$,
$H_{1'} = 0 \oplus K \oplus K \oplus K$, $H_{2'} = 0 \oplus K \oplus \{(y, Sy) \in K^2 \mid y \in K\}$,
$H_{3'} = 0 \oplus K \oplus 0 \oplus 0$ and $H_{1''} = \{(x, y, x, y) \in K^4 \mid x, y \in K\}$. For any arrow $\alpha \in E$, let $f_\alpha : H_{s(\alpha)} \to H_{r(\alpha)}$ be the canonical inclusion map. Then (H, f) is indecomposable.

Lemma 3.3. *Let $\Gamma = (V, E, s, r)$ be the following quiver with the underlying undirected graph an extended Dynkin diagram $\tilde{E}_8$:*



Then there exists an infinite-dimensional, indecomposable Hilbert representation (H, f) of Γ.

Let $K = \ell^2(\mathbb{N})$ and S a unilateral shift on K. We define a Hilbert representation $(H, f) := ((H_v)_{v \in V}, (f_\alpha)_{\alpha \in E})$ of Γ as follows:
Let $H_0 = K \oplus K \oplus K \oplus K \oplus K \oplus K$,
$H_1 = \{(x, x) \in K^2 \mid x \in K\} \oplus K \oplus K \oplus K \oplus K$,
$H_2 = 0 \oplus 0 \oplus K \oplus K \oplus K \oplus K$, $H_3 = 0 \oplus 0 \oplus 0 \oplus K \oplus K \oplus K$,
$H_4 = 0 \oplus 0 \oplus 0 \oplus K \oplus \{(y, Sy) \in K^2 \mid y \in K\}$, $H_5 = 0 \oplus 0 \oplus 0 \oplus K \oplus 0 \oplus 0$,
$H_{1'} = K \oplus K \oplus \{(x, y, x, y) \in K^4 \mid x, y \in K\}$, $H_{2'} = K \oplus K \oplus 0 \oplus 0 \oplus 0 \oplus 0$,
$H_{1''} = \{(y, z, x, 0, y, z) \in K^6 \mid x, y, z \in K\}$.
For any arrow $\alpha \in E$, let $f_\alpha : H_{s(\alpha)} \to H_{r(\alpha)}$ be the canonical inclusion map. Then (H, f) is indecomposable.

Theorem 3.4. *Let Γ be a finite, connected quiver. If the underlying undirected graph $|\Gamma|$ contains one of the extended Dynkin diagrams $\tilde{A}_n$ $(n \geq 0)$, $\tilde{D}_n$ $(n \geq 4)$, $\tilde{E}_6$, $\tilde{E}_7$ and $\tilde{E}_8$, then there exists an infinite-dimensional, indecomposable, Hilbert representation of Γ.*

We need to get Hilbert representations of Γ with any orientation. It is a hard task. In order to do this, we need to use Reflection functors, closed conditions and nice mapping property of Hilbert representations.

4. Hilbert representation of the Kronecker quiver

It is known that indecomposable finite dimensional representations of 1-loop are reduced to the Jordan canonical forms. It is realized by Weierstrass using elementary divisors in 1868. Representations of 1-loops are contained in representations of the Kronecker quiver. General forms of indecomposable finite dimensional representations of the Kronecker quiver are obtained by Kronecker in 1890 as follows.

(I) $H_0 = \mathbb{C}^{n+1}$, $H_1 = \mathbb{C}^n$, $f_\alpha = (0 I_n)$, $f_\beta = (I_n 0)$.

(II)$H_0 = H_1 = \mathbb{C}^n$,$f_\alpha = I_n$,$f_\beta = B_n$. (B_n is the backward shift with n size.)

(III)$H_0 = H_1 = \mathbb{C}^n$, $f_\alpha = I_n$,$f_\beta = \lambda + B_n(\lambda \neq 0)$.

(IV) $H_0 = \mathbb{C}^n$,$H_1 = \mathbb{C}^{n+1}$, $f_\alpha = (0I_n)^t$,$f_\beta = (I_n 0)^t$.

In this form, transitive representations are (I) and (IV). (II) and (III) are not transitive except n=1.

5. The Kronecker quiver and 4 subspaces

We shall note the relation between classification of the Kronecker quiver and classification of 4 subspaces.

Gelfand and Ponomarev gave a complete classification of indecomposable systems of four subspaces in a finite-dimensional space.

In order to do this, Gelfand and Ponomarev introduced an integer valued invariant $\rho(\mathcal{S})$, called *defect*, for a system $\mathcal{S} = (H; E_1, E_2, E_3, E_4)$ of four subspaces by

$$\rho(\mathcal{S}) = \sum_{i=1}^{4} \dim E_i - 2 \dim H.$$

The invariant defect characterizes an essential feature of the system.

We put the Kronecker quiver $\Gamma = (V, E, s, r)$ by $V = \{0, 1\}$, $E = \{\alpha, \beta\}$ and $s(\alpha) = 0, s(\beta) = 0$, and $r(\alpha) = 1, r(\beta) = 1$. We put $\widetilde{D_4} = (V, E, s, r)$ by $V = \{v_0, v_1, v_2, v_3, v_4\}$, $E = \{\alpha_i, i = 1, 2, 3, 4\}$ and $s(\alpha_i) = v_i$ and $r(\alpha_i) = v_0$.

For a Hilbert representation (H, f) of the Kronecker quiver, we associate with a Hilbert representation (K, g) of $\widetilde{D_4}$ by $K_{v_1} = H \oplus 0$, $K_{v_2} = 0 \oplus K$, $K_{v_3} = \{(x, Ax); x \in H\}$, $K_{v_4} = \{(x, Bx); x \in H\}$, $K_{v_0} = H \oplus K$. g_{α_i} is the canonical inclusion from K_{v_i} to K_{v_0}. Then $End(H, f)$ is isomorphic to $End(K, g)$.

Let $\mathcal{S}_1(2k+1, -1)$ (resp. $\mathcal{S}_2(2k+1, 1)$) (cf,[EW2006]) be the isomorphism class of 4 subspaces which has the odd whole space dimension and defect -1(resp. 1). $\mathcal{S}_1(2k+1, -1)$ corresponds to Kronecker classification (I) and $\mathcal{S}_2(2k+1, 1)$ corresponds to Kronecker classification (IV). Let $\mathcal{S}_{1,3}(2k, 0)$ be the isomorphism class of 4 subspaces which has $A = B_k$ and $B = 1$. Let $\mathcal{S}(2k, 0; \lambda)$ be the isomorphism class of 4 subspaces which has $A = \lambda + B_k$ and $B = 1$. $\mathcal{S}_{1,3}(2k, 0)$ and $\mathcal{S}(2k, 0; \lambda)$ correspond to Kronecker classification (II) and (III).(cf.[EW2006])

6. Canonical and non-canonical Hilbert representations of the Kronecker quiver

Next we consider indecomposable Hilbert representations of the Kronecker quiver in the infinite dimensional case.

In the infinite dimensional setting, different phenomenon occurs compared with finite dimensional case.

For a Kronecker quiver $\Gamma = (V, E, s, r)$, $V = \{0, 1\}$, $E = \{\alpha, \beta\}$ and $s(\alpha) = 0, s(\beta) = 0$, and $r(\alpha) = 1, r(\beta) = 1$, and an operator $T \in B(H)$,where H is an infinite dimensional Hilbert space, we can associate to a canonical representation (H, f) such that $H_0 = H_1 = H$ and $f_\alpha = I$ and $f_\beta = T$.

$T \in B(H)$ is strongly irreducible if and only if there does not exist a non-trivial idempotent P such that $TP = PT$. If we take a strongly irreducible operator $T \in B(H)$, then we get an indecomposable Hilbert representation of the Kronecker quiver.

Theorem 6.1. *Let S be a shift and $\lambda \in \mathbb{C}$. Put $A_\lambda = S + \lambda$. Take a canonical representation (H^λ, f^λ) such that $H_0^\lambda = H_1^\lambda = H$ and $f_\alpha^\lambda = I$ and $f_\beta^\lambda = A_\lambda$. Then $\{(H^\lambda, f^\lambda)\}_\lambda$ is an uncountable family of canonical indecomposable Hilbert representations of Γ.*

Theorem 6.2. *Let A, B be strongly irreducible operators and $\lambda(\neq 0) \in \sigma(A)$. Put (H, f) by $H_0 = H_1 = H$ and $f_\alpha = \lambda - A, f_\beta = A$. Put (K, g) by $K_0 = K_1 = H$ and $g_\alpha = I, g_\beta = B$. Then (H, f) and (K, g) are indecomposable representations and they are not isomorphic.*

In the following we can construct continuously many non-canonical indecomposable representations of the Kronecker quiver.

Theorem 6.3. *Let S be a unilateral shift on an infinite dimensional Hilbert space. Let $T_\lambda = S + \lambda$, $T_\mu = S + \mu$, $(\lambda, \mu \in \mathbb{C}, \mid \lambda - 1 \mid \leq 1, \mid \mu - 1 \mid \leq 1, \mid \lambda \mid \leq 1. \mid \mu \mid \leq 1)$.*

Put (H^λ, f^λ) by $H_0^\lambda = H_1^\lambda = H$ and $f_\alpha^\lambda = I - T_\lambda, f_\beta^\lambda = T_\lambda$.

Then $\lambda = \mu$ if and only if (H^λ, f^λ) and (H^μ, f^μ) are isomorphic.

7. Construction of transitive representations of the Kronecker quiver

In this section we present examples of transitive representations of the Kronecker quiver on infinite dimensional Hilbert spaces by two methods.

For the Kronecker quiver $\Gamma = (V, E, s, r)$, $V = \{0, 1\}$, $E = \{\alpha, \beta\}$ and $s(\alpha) = 0, s(\beta) = 0$, and $r(\alpha) = 1, r(\beta) = 1$, and an operator $T \in B(H)$,where H is an infinite dimensional Hilbert space, we can associate to a canonical representation (H, f) such that $H_0 = H_1 = H$ and $f_\alpha = I$ and $f_\beta = T$. These canonical representations are not transitive. Our non-canonical representations which we constructed above are not transitive.

Next we construct transitive representations of the Kronecker quiver by weight sequences. Let $H = \ell^2(\mathbb{Z})$ and $\lambda > 1$. We define two weight sequences $a(n), b(n) (n \in \mathbb{Z})$ by

$a(n) = e^{-\lambda^n}(n \geq 1, even)$, $1(n$ is otherwise$)$.

$b(n) = e^{-\lambda^n}(n \geq 1, odd)$, $1(n$ is otherwise$)$. We put $A = D_a(D_a$ is a diagonal operator with $a(n)$ diagonal coefficients) and $B = UD_b($

the product of bilateral forward shift U and diagonal operator D_b with $b(n)$ diagonal coefficients). A is a positive operator and B is a weighted shift operator with weight $b(n)$.

We put $H_0^\lambda = H, H_1^\lambda = H$. We put $f_\alpha^\lambda = A$ and $f_\beta^\lambda = B$. Then we get a Hilbert representation (H^λ, f^λ) of the Kronecker quiver on an infinite dimensional Hilbert space.

Theorem 7.1. *This Hilbert representation (H^λ, f^λ) of the Kronecker quiver is transitive. That is, we have End$(H^\lambda, f^\lambda) = \mathbb{C}$.*

Theorem 7.2. *For $\lambda, \mu > 1$,$\lambda \neq \mu$, $(H^\lambda, f^\lambda)$and (H^μ, f^μ) are not isomorphic.*

Next we construct transitive representations of the Kronecker quiver by perturbation method. Let S be a unilateral shift on $K = \ell^2(\mathbb{N})$. Let $e_1, e_2, ..$ be a basis of K. Take $\lambda = (\lambda_i)_i \in \ell^\infty(\mathbb{N})$ such that $\lambda_i \neq \lambda_j (i \neq j)$ and $\overline{w} = (\overline{w_n})_n \in \ell^2(\mathbb{N})$ such that $w_n \neq 0$ for any $n \in \mathbb{N}$. Let $\theta_{e_1,\overline{w}}$ be a rank one operator. Let Γ be the Kronecker quiver. Put (K, g) be a Hilbert representation of Γ as follows.

$K_0 = K_1 = K, g_\alpha = S, g_\beta = SD_\lambda + \theta_{e_1,\overline{w}}$.

Theorem 7.3. *This Hilbert representation (K, g) of Γ is transitive and this representation is not isomorphic to the above transitive representations.*

References

[EW2006] M. Enomoto and Y. Watatani, *Relative position of four subspaces in a Hilbert space,* Adv. Math. 201 (2006), 263-317.

[EW2009] M.Enomoto and Y.Watatani:Indecomposable representations of quivers on infinite-dimensional Hilbert spaces,Journal of Functional Analysis **256**(2009),959-991.

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