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Elementary proofs of Petz-Hasegawa theorem and Hansen results by only using Löwner-Heinz inequality

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To the memory of Professor Tsuneo Kanno passed away by tsunami disaster at Tohoku district on 2011.3.11 with deep sorrow

2011.3.11 東日本大津波で亡くなられた菅野恒雄東京工業大学名誉教授の霊に捧げます

We show elementary proofs of useful and important operator monotone functions by Petz-Hasegawa and Frank Hansen. Also we consider some extensions of results by Hansen.

A capital letter means a bounded linear operator on a Hilbert space H. An operator T is said to be *positive* (denoted by $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$ and also an operator T is said to be *strictly positive* (denoted by $T > 0$) if T is positive and invertible. A real valued continuous function $f(t)$ on $(0, \infty)$ is said to be *operator monotone* if $f(A) \geq f(B)$ holds for $A \geq B$.

§1 An elementary proof of Theorem B by Petz-Hasegawa

Theorem B is the essential factor of the operator monotone function giving the famous and important *Wigner-Yanase-Dyson skew information* and it turns out that *Wigner-Yanase-Dyson skew information* is closely ralated to the special case of *quantum Fisher information.*

We cite the following almost obvious Proposition A which is an immediate consequence of the well known celebrated result as (LH) (abbreviation of Löwner-Heinz) that t^{α} is an operator monotone for any $\alpha \in [0,1]$ to give a proof of Theorem B.

Proposition A ([2]).
(LH) t^{α} *is an operator monotone for any* $\alpha \in [0,1]$.
(LH-1) *Let* $\alpha_j, \beta_j, \gamma_j ... \in [0,1]$ *for* $j = 1, 2, ..., n$. *Then*
$(\frac{1}{t^{\alpha_1}+t^{\alpha_2}+...+t^{\alpha_n}} + \frac{1}{t^{\beta_1}+t^{\beta_2}+...+t^{\beta_n}} + \frac{1}{t^{\gamma_1}+t^{\gamma_2}+...+t^{\gamma_n}} + ...)^{-1}$ is operator monotone,
in particular: $(t^{-\alpha_1} + t^{-\alpha_2} + ... + t^{-\alpha_n})^{-1}$ *is operator monotone.*

First of all, recall the following obvious Lemma 1.

Lemma 1. *For any natural number n and any positive real number t, the following inequality holds:*

$$(t^{n-1}+t^{n-2}+......t^2+t+1)^2$$
$$=t^{2n-2}+t^{2n-3}+...+t^{n+1}+t^n+t^{n-1}+...+t^2+t+1\ +t(t^{n-2}+t^{n-3}......t^2+t+1)^2. \quad (1.0)$$

Proof. $(t^{n-1}+(t^{n-2}+......t^2+t+1))^2$
$=t^{2(n-1)}+t^{n-1}(t^{n-2}+......t^2+t+1)+t^{n-1}(t^{n-2}+......t^2+t+1)+(t^{n-2}+......t^2+t+1)^2$
$=t^{2n-2}+t^{2n-3}+...+t^{n-1}+(t^{n-2}+...+t^2+t+1)$
$+(t^{n-2}+......t^2+t+1)\{(t^{n-1}+(t^{n-2}+......t^2+t+1)-1)\}$
$=t^{2n-2}+t^{2n-3}+..+t^{n+1}+t^n+t^{n-1}+...+t^2+t+1\ +t(t^{n-2}+t^{n-3}......t^2+t+1)^2$. □

Lemma 2. *Let natural number n and k such that $n-1\geq k\geq 1$. Then the following inequality holds for any positive real number t:*

$$(t^{n-1}+t^{n-2}+......t^2+t+1)^2$$
$$=(t^{n-k-1}+t^{n-k-2}+.....+t^2+t+1)(t^{n+k-1}+t^{n+k-2}+.....+t^2+t+1)$$
$$+t^{n-k}(t^{k-1}+t^{k-2}+.....+t^2+t+1)^2. \quad (1.1)$$

Proof. We show (1.1) by mathematical inductioin on k such that $n-1\geq k\geq 1$.

(i) In case $k=n-1$. In fact (1.1) holds for $k=n-1$ because (1.1) putting $k=n-1$, which just coincides with (1.0) of Lemma 1.

(ii) Assume (1.1) for some k such that $n-1\geq k\geq 1$. We show that (1.1) holds for $k-1$.

$(t^{n-1}+t^{n-2}+......t^2+t+1)^2$
$=(t^{n-k-1}+t^{n-k-2}+......+t+1)\ \{(t^{n+k-1}+t^{n+k-2}+...t^{2k-1})+(t^{2k-2}+t^{2k-3}...+t+1)\}$
$+t^{n-k}\{(t^{2k-2}+t^{2k-3}+......+t+1)+t(t^{k-2}+t^{k-3}+......+t+1)^2\}$ by (1.0)
$=(t^{n-k-1}+t^{n-k-2}+......+t+1)(t^{n-k}+t^{n-k-1}+......+t+1)t^{2k-1}$
$+(t^{n-k}+t^{n-k-1}+......t+1)(t^{2k-2}+t^{2k-3}+......+t+1)+t^{n-k+1}(t^{k-2}+t^{k-3}...+t+1)^2$
$=(t^{n-k}+t^{n-k-1}+......t+1)\{(t^{n+k-2}+t^{n+k-3}+...+t^{2k-1})+(t^{2k-2}+t^{2k-3}...+t+1)\}$
$+t^{n-k+1}(t^{k-2}+t^{k-3}+......t^2+t+1)^2 \quad (1.2)$

and (1.2) shows that (1.1) holds for *any* k such that $n-1\geq k\geq 1$ by mathematical induction on k by (i) and (ii). □

Next we state an elementary proof of the following Theorem B (see Remark 1.1).

Theorem B ([6], [1]). $f_p(t)=p(1-p)\dfrac{(t-1)^2}{(t^p-1)(t^{1-p}-1)}$ *is an operator monotone function for $-1\leq p\leq 2$.*

Proof. (i) In case $0 < p < 1$. Since $p(1-p) > 0$ holds, we show that $g_p(t) = \dfrac{(t-1)^2}{(t^p-1)(t^{1-p}-1)}$ is operator monotone. We have only to prove the result for $p = \dfrac{k}{n} \in (0,1)$ for natural number n and k such that $n-1 \geq k \geq 1$ by continuity of an operator.

$$\begin{aligned}\frac{t-1}{t^{\frac{k}{n}}-1} &= \frac{(t^{\frac{1}{n}}-1)(t^{\frac{n-1}{n}}+t^{\frac{n-2}{n}}+..+t^{\frac{k}{n}}+t^{\frac{k-1}{n}}...t^{\frac{2}{n}}+t^{\frac{1}{n}}+1)}{(t^{\frac{1}{n}}-1)(t^{\frac{k-1}{n}}+t^{\frac{k-2}{n}}+.....t^{\frac{2}{n}}+t^{\frac{1}{n}}+1)} \\ &= \frac{t^{\frac{n-1}{n}}+t^{\frac{n-2}{n}}+..+t^{\frac{k}{n}}}{t^{\frac{k-1}{n}}+t^{\frac{k-2}{n}}+.....t^{\frac{2}{n}}+t^{\frac{1}{n}}+1}+1 \end{aligned} \tag{1.3}$$

and replacing k by $n-k$ in (1.3),

$$\frac{t-1}{t^{\frac{n-k}{n}}-1} = \frac{t^{\frac{n-1}{n}}+t^{\frac{n-2}{n}}+..+t^{\frac{n-k}{n}}}{t^{\frac{n-k-1}{n}}+t^{\frac{n-k-2}{n}}+.....t^{\frac{2}{n}}+t^{\frac{1}{n}}+1}+1. \tag{1.4}$$

Since $g_p(t)$ is the product of (1.3) and (1.4), we have

$$\begin{aligned} g_p(t) &= \frac{(t-1)^2}{(t^{\frac{k}{n}}-1)(t^{\frac{n-k}{n}}-1)} \\ &= \frac{t^{\frac{n-1}{n}}+t^{\frac{n-2}{n}}+..+t^{\frac{k}{n}}}{t^{\frac{k-1}{n}}+t^{\frac{k-2}{n}}+.....t^{\frac{2}{n}}+t^{\frac{1}{n}}+1} + \frac{t^{\frac{n-1}{n}}+t^{\frac{n-2}{n}}+..+t^{\frac{n-k}{n}}}{t^{\frac{n-k-1}{n}}+t^{\frac{n-k-2}{n}}+.....t^{\frac{2}{n}}+t^{\frac{1}{n}}+1}+t+1 \end{aligned} \tag{1.5}$$

because the product of the first term of (1.3) and the first one of (1.4) equals to t as follows:

$$\left(\frac{t^{\frac{k}{n}}(t^{\frac{n-k-1}{n}}+t^{\frac{n-k-2}{n}}+..+t^{\frac{2}{n}}+t^{\frac{1}{n}}+1)}{t^{\frac{k-1}{n}}+t^{\frac{k-2}{n}}+.....t^{\frac{2}{n}}+t^{\frac{1}{n}}+1}\right)\left(\frac{t^{\frac{n-k}{n}}(t^{\frac{k-1}{n}}+t^{\frac{k-2}{n}}+..t^{\frac{1}{n}}+1)}{t^{\frac{n-k-1}{n}}+t^{\frac{n-k-2}{n}}+.....t^{\frac{2}{n}}+t^{\frac{1}{n}}+1}\right) = t.$$

Thanks to an appropriate modification of (1.5) to apply Proposition A, we have

$$\begin{aligned} &g_p(t) \\ &= \frac{1}{t^{\frac{k-1}{n}}+t^{\frac{k-2}{n}}+...t^{\frac{2}{n}}+t^{\frac{1}{n}}+1}\sum_{j=k}^{n-1}t^{\frac{j}{n}} + \frac{1}{t^{\frac{n-k-1}{n}}+t^{\frac{n-k-2}{n}}+..t^{\frac{2}{n}}+t^{\frac{1}{n}}+1}\sum_{j=n-k}^{n-1}t^{\frac{j}{n}}+t+1 \\ &= \underbrace{\Big(\overbrace{t^{\frac{-(n-k)}{n}}+t^{\frac{-(n-k+1)}{n}}+..+t^{\frac{-(n-1)}{n}}}^{k \quad terms}\Big)^{-1} + \Big(\overbrace{t^{\frac{-(n-1-k)}{n}}+t^{\frac{-(n-k)}{n}}+..+t^{\frac{-(n-2)}{n}}}^{k \quad terms}\Big)^{-1} + ... + \Big(\overbrace{t^{\frac{-1}{n}}+t^{\frac{-2}{n}}+..+t^{\frac{-k}{n}}}^{k \quad terms}\Big)^{-1}}_{n-k \quad terms} \\ &+ \underbrace{\Big(\overbrace{t^{\frac{-k}{n}}+t^{\frac{-(k+1)}{n}}+..+t^{\frac{-(n-1)}{n}}}^{n-k \quad terms}\Big)^{-1} + \Big(\overbrace{t^{\frac{-(k-1)}{n}}+t^{\frac{-k}{n}}+...+t^{\frac{-(n-2)}{n}}}^{n-k \quad terms}\Big)^{-1} + ... + \Big(\overbrace{t^{\frac{-1}{n}}+t^{\frac{-2}{n}}+..+t^{\frac{-(n-k)}{n}}}^{n-k \quad terms}\Big)^{-1}}_{k \quad terms} \\ &+t+1 \end{aligned}$$

so that $g_p(t)$ is operator monotone by (LH-1) of Proposition A ans so is $f_p(t) = p(1-p)g_p(t)$.

(ii) In case $1 < p < 2$. Since $p(1-p) < 0$ we have only to prove that

$$h_p(t) = -\frac{(t-1)^2}{(t^p-1)(t^{1-p}-1)} = \frac{t^{\frac{k}{n}}(t-1)^2}{(t^{\frac{k}{n}}-1)(t^{1+\frac{k}{n}}-1)} \tag{1.6}$$

is an operator monotone function for natural number n and k such that $n-1 \geq k \geq 1$ by continuity of an operator. Then we have by (1.6)

$$h_p(t) = \frac{t^{\frac{k}{n}}(t-1)^2}{(t^{\frac{n+k}{n}}-1)(t^{\frac{k}{n}}-1)}$$
$$= t^{\frac{k}{n}}\left[\frac{(t^{\frac{n-1}{n}}+t^{\frac{n-2}{n}}+..+t^{\frac{k}{n}}+t^{\frac{k-1}{n}}...t^{\frac{2}{n}}+t^{\frac{1}{n}}+1)^2}{\left(t^{\frac{n+k-1}{n}}+t^{\frac{n+k-2}{n}}+...+t^{\frac{n}{n}}+t^{\frac{n-1}{n}}..+t^{\frac{2}{n}}+t^{\frac{1}{n}}+1\right)\left(t^{\frac{k-1}{n}}+t^{\frac{k-2}{n}}+.....t^{\frac{2}{n}}+t^{\frac{1}{n}}+1\right)}\right]. \tag{1.7}$$

By applying modified Lemma 2 replacing t by $t^{\frac{1}{n}}$ to (1.7), so that the part of [ ] of the numerator in (1.7) can be rewritten as follows:

$$(t^{\frac{n-1}{n}} + t^{\frac{n-2}{n}} + .. + t^{\frac{k}{n}} + t^{\frac{k-1}{n}}...t^{\frac{2}{n}} + t^{\frac{1}{n}} + 1)^2$$
$$= \left(t^{\frac{n-k-1}{n}}+t^{\frac{n-k-2}{n}}+..+t^{\frac{1}{n}}+1\right)\left(t^{\frac{n+k-1}{n}}+t^{\frac{n+k-2}{n}}+.....+t^{\frac{1}{n}}+1\right)+t^{\frac{n-k}{n}}\left(t^{\frac{k-1}{n}}+t^{\frac{k-2}{n}}...t^{\frac{2}{n}}+t^{\frac{1}{n}}+1\right)^2$$

and $h_p(t)$ in (1.7) can be rewitten as follows

$$h_p(t) = t^{\frac{k}{n}}\left[\frac{\left(t^{\frac{n-k-1}{n}}+t^{\frac{n-k-2}{n}}+..+t^{\frac{1}{n}}+1\right)\left(t^{\frac{n+k-1}{n}}+t^{\frac{n+k-2}{n}}+.....+t^{\frac{1}{n}}+1\right)+t^{\frac{n-k}{n}}\left(t^{\frac{k-1}{n}}+t^{\frac{k-2}{n}}...t^{\frac{2}{n}}+t^{\frac{1}{n}}+1\right)^2}{\left(t^{\frac{n+k-1}{n}}+t^{\frac{n+k-2}{n}}+...+t^{\frac{n}{n}}+t^{\frac{n-1}{n}}..+t^{\frac{2}{n}}+t^{\frac{1}{n}}+1\right)\left(t^{\frac{k-1}{n}}+t^{\frac{k-2}{n}}+.....t^{\frac{2}{n}}+t^{\frac{1}{n}}+1\right)}\right]. \tag{1.8}$$

By (1.8) we have

$$h_p(t)$$
$$= t^{\frac{k}{n}}\left[\frac{t^{\frac{n-k-1}{n}}+t^{\frac{n-k-2}{n}}+..+t^{\frac{1}{n}}+1}{t^{\frac{k-1}{n}}+t^{\frac{k-2}{n}}+.....t^{\frac{2}{n}}+t^{\frac{1}{n}}+1}\right] + \left[\frac{t\left(t^{\frac{k-1}{n}}+t^{\frac{k-2}{n}}...t^{\frac{2}{n}}+t^{\frac{1}{n}}+1\right)}{t(t^{\frac{k-1}{n}}+t^{\frac{k-2}{n}}+...t^{\frac{1}{n}}+1)+(t^{\frac{n-1}{n}}..+t^{\frac{k}{n}})+(t^{\frac{k-1}{n}}+..t^{\frac{2}{n}}+t^{\frac{1}{n}}+1)}\right]$$
$$= (t^{\frac{-(n-k)}{n}} + ... + t^{\frac{-(n-2)}{n}} + t^{\frac{-(n-1)}{n}})^{-1} + (t^{\frac{-(n-k-1)}{n}} + ... + t^{\frac{-(n-2)}{n}})^{-1} + ... + (t^{\frac{-1}{n}} + ... + t^{\frac{-k}{n}})^{-1}$$
$$+\left[1 + \frac{1}{t} + \frac{1}{t^{\frac{k}{n}}+t^{\frac{k-1}{n}}+...t^{\frac{1}{n}}} + \frac{1}{t^{\frac{k+1}{n}}+t^{\frac{k}{n}}+...t^{\frac{2}{n}}} + ... + \frac{1}{t^{\frac{n-1}{n}}+t^{\frac{n-2}{n}}+...t^{\frac{n-k}{n}}}\right]^{-1}$$

is operator monotone by (LH-1).

(iii). In case $-1 < p < 0$. $f_p(t)$ is operator monotone because the result reduces to the case $1 < p < 2$ by symmetry.

(iv) $\lim_{p\to 0} f_p(t) = \lim_{p\to 1} f_p(t) = \dfrac{t-1}{\log t}$ and $f_{-1}(t) = f_2(t) = \dfrac{2t}{t+1}$ and these functions are both operator monotone.

Whence the proof is complete by (i), (ii), (iii) and (iv). □

Remark 1.1. The proof of the case $0 < p < 1$ in Theorem B was obtained in [6], who also conjectured the case $1 < p < 2$. Imcomplete proofs of this statement have appeared in the literature, but it seems that the first correct proof was obtained by Cai And Hansen [1, Theorem 5.2] (see the footnote on page 11 of [1]). The proof of Theorem B in this paper is based on [3].

§2 Elementary proofs of the results by Hansen and related ones

Theorem C ([4]).
(i) $f(t)=\dfrac{t^q-1}{t^p-1}$ *is an operator monotone function for* $1\geq q\geq p>0$ *and* $t\geq 0$.

(ii) $f^*(t)=\dfrac{t}{f(t)}$ *is also an operator monotone, where* $f(t)$ *is in* (i)

(iii) $f^\sharp(t)=tf(t^{-1})$ *is also an operator monotone, where* $f(t)$ *is in* (i).

Proof. (i). We have only to prove the result for $p=\dfrac{k}{n}$ and $q=\dfrac{m}{n}$ for natural numbers n,m,k such that $n\geq m\geq k\geq 1$ by continuity of an operator.

$$
\begin{aligned}
f(t)=\frac{t^{\frac{m}{n}}-1}{t^{\frac{k}{n}}-1}&=\frac{(t^{\frac{1}{n}}-1)(t^{\frac{m-1}{n}}+t^{\frac{m-2}{n}}+...+t^{\frac{k}{n}}+t^{\frac{k-1}{n}}+...t^{\frac{1}{n}}+1)}{(t^{\frac{1}{n}}-1)(t^{\frac{k-1}{n}}+t^{\frac{k-2}{n}}+...+t^{\frac{1}{n}}+1)}\\
&=\frac{t^{\frac{m-1}{n}}+t^{\frac{m-2}{n}}+...+t^{\frac{k}{n}}}{t^{\frac{k-1}{n}}+t^{\frac{k-2}{n}}+...+t^{\frac{1}{n}}+1}+1 \qquad (2.1)\\
&=\left(t^{\frac{-(m-k)}{n}}+t^{\frac{-(m-k+1)}{n}}+...+t^{\frac{-(m-1)}{n}}\right)^{-1}+\left(t^{\frac{-(m-k-1)}{n}}+t^{\frac{-(m-k)}{n}}+...+t^{\frac{-(m-2)}{n}}\right)^{-1}\\
&\quad +.......+\left(t^{\frac{-1}{n}}+t^{\frac{-2}{n}}+...+t^{\frac{-k}{n}}\right)^{-1}+1
\end{aligned}
$$

so that $f(t)$ is an operator monotone function by (LH-1) of Proposition A.

(ii). $f^*(t)=\dfrac{t}{f(t)}=\dfrac{t}{1+\dfrac{t^{\frac{m-1}{n}}+t^{\frac{m-2}{n}}+...+t^{\frac{k}{n}}}{t^{\frac{k-1}{n}}+t^{\frac{k-2}{n}}+...+t^{\frac{1}{n}}+1}}$ by (2.1)

$$
=\left\{\frac{1}{t}+\frac{1}{t^{\frac{n-m+k}{n}}+t^{\frac{n-m+k-1}{n}}+..+t^{\frac{n-m+1}{n}}}+\frac{1}{t^{\frac{n-m+k+1}{n}}+t^{\frac{n-m+k}{n}}+..+t^{\frac{n-m+2}{n}}}+..+\frac{1}{t^{\frac{n-1}{n}}+t^{\frac{n-2}{n}}+..+t^{\frac{n-k}{n}}}\right\}^{-1}
$$

is operator monotone by (LH-1) of Proposition A.

(iii). $f^\sharp(t)=tf(t^{-1})=\left(\frac{t^{\frac{m}{n}}-1}{t^{\frac{k}{n}}-1}\right)t^{1-\frac{m}{n}+\frac{k}{n}}$

$$
\begin{aligned}
&=\frac{(t^{\frac{1}{n}}-1)(t^{\frac{m-1}{n}}+t^{\frac{m-2}{n}}+...t^{\frac{m-k}{n}}+t^{\frac{m-k-1}{n}}+...+t^{\frac{1}{n}}+1)t^{1-\frac{m}{n}+\frac{k}{n}}}{(t^{\frac{1}{n}}-1)(t^{\frac{k-1}{n}}+t^{\frac{k-2}{n}}+...+t^{\frac{1}{n}}+1)}\\
&=\frac{(t^{\frac{m-1}{n}}+t^{\frac{m-2}{n}}+...t^{\frac{m-k}{n}}+t^{\frac{m-k-1}{n}}+...+t^{\frac{1}{n}}+1)t^{1-\frac{m}{n}+\frac{k}{n}}}{t^{\frac{k-1}{n}}+t^{\frac{k-2}{n}}+...+t^{\frac{1}{n}}+1}\\
&=\frac{t(t^{\frac{k-1}{n}}+t^{\frac{k-2}{n}}+t^{\frac{1}{n}}+1)+t^{\frac{n-1}{n}}+t^{\frac{n-2}{n}}+...+t^{\frac{n-m+k+1}{n}}+t^{\frac{n-m+k}{n}}}{t^{\frac{k-1}{n}}+t^{\frac{k-2}{n}}+...+t^{\frac{1}{n}}+1}\\
&=t+\frac{t^{\frac{n-1}{n}}+t^{\frac{n-2}{n}}+...+t^{\frac{n-m+k+1}{n}}+t^{\frac{n-m+k}{n}}}{t^{\frac{k-1}{n}}+t^{\frac{k-2}{n}}+...+t^{\frac{1}{n}}+1}
\end{aligned}
$$

$$= t + (t^{\frac{-(n-k)}{n}} + t^{\frac{-(n-k+1)}{n}} + \ldots + t^{\frac{-(n-1)}{n}})^{-1} + (t^{\frac{-(n-k-1)}{n}} + t^{\frac{-(n-k)}{n}} + \ldots + t^{\frac{-(n-2)}{n}})^{-1}$$
$$+ \ldots\ldots + (t^{\frac{-(n-m+1)}{n}} + t^{\frac{-(n-m+2)}{n}} + \ldots + t^{\frac{-(n-m+k)}{n}})^{-1}$$

so that $f^{\natural}(t)$ is operator monotone by (LH-1) of Proposition A. □

Proposition 2.1. *If $f(t)$ and $g(t)$ are both positive operator monotone on $t \geq 0$, then*

(i) $f(t)^{1-\alpha}g(t)^{\alpha}$ *is positive operator monotone for any $\alpha \in [0,1]$ on $t \geq 0$.* (2.2)

In particular,

(ii) $f(t)^{1-2\alpha}t^{\alpha}$ *is positive operator monotone for any $\alpha \in [0,1]$ on $t \geq 0$.* (2.3)

(iii) $f(t)^{1-\alpha}f(t^{-1})^{\alpha}t^{\alpha}$ *is positive operator monotone for any $\alpha \in [0,1]$ on $t \geq 0$.* (2.4)

Proof. Let $A \geq B > 0$. Since $f(A)\natural_{\alpha}g(A) \geq f(B)\natural_{\alpha}g(B)$ for $\alpha \in [0,1]$, so we have (2.2). Since $g(t) = f^{*}(t) = \dfrac{t}{f(t)}$ is operator monotone, (2.3) follows by (2.2).
Also since $g(t) = f^{\natural}(t) = tf(t^{-1})$ is operator monotone, (2.4) follows by (2.2). □

Proposition 2.2. *Let $1 \geq q \geq p > 0$ and $\alpha \in [0,1]$. Then*

(i) $\quad F_{p,q,\alpha} = \dfrac{t^q - 1}{t^p - 1}t^{(1-q+p)\alpha}$ *is operator monotone on $t \geq 0$*

and

(ii) $\quad G_{p,q,\alpha} = \left(\dfrac{t^q - 1}{t^p - 1}\right)^{1-2\alpha}t^{\alpha}$ *is operator monotone on $t \geq 0$.*

Proof. (i). Put $f(t) = \frac{t^q-1}{t^p-1}$ for $1 \geq q \geq p > 0$ on $t \geq 0$ and $g(t) = f^{\natural}(T) = tf(t^{-1}) = \frac{t^q-1}{t^p-1}t^{1-q+p}$ on $t \geq 0$. Since $f(t)$ and $g(t)$ are operator monotone shown in §1, $f(t)^{1-\alpha}g(t)^{\alpha} = \dfrac{t^q - 1}{t^p - 1}t^{(1-q+p)\alpha}$ is operator monotone by (2.2). And (ii) follows by (2.3). □

Theorem D ([5]).*Let the exponent $\gamma \in [0,1]$. The functions*

$$f_{\gamma}(t) = \frac{1}{2}(1+t)\left(\frac{4t}{(t+1)^2}\right)^{\gamma} = t^{\gamma}\left(\frac{1+t}{2}\right)^{1-2\gamma}$$

are operator monotone, normalized in the sense $f_{\gamma}(1) = 1$ and $f_{\gamma}(t) = tf_{\gamma}(t^{-1})$ for $t \geq 0$.

Proof. Since $f(t) = \frac{1+t}{2}$ is operator monotone and $f_{\gamma}(t) = f(t)^{1-2\gamma}t^{\gamma} = t^{\gamma}(\frac{1+t}{2})^{1-2\gamma}$ is operator monoyone by (2.3) of Proposition 2.1. Others are immediate consequences of calculations. □

Remark 2.1. $F_{p,q,\alpha}(t) = \dfrac{t^q - 1}{t^p - 1} t^{(1-q+p)\alpha}$ is operator monotone on $t \geq 0$ in Proposition 2.2 contains the following useful operator monotone functions as follows;

$$F_{p,p,\alpha}(t) = t^\alpha \text{ for } \alpha \in [0,1] \quad \text{and} \quad F_{p,q,0}(t) = \frac{t^q - 1}{t^p - 1} \text{ for } 1 \geq q \geq p > 0.$$

Although it is well known that $f^*(t) = \dfrac{t}{f(t)}$ and $f^\sharp(t) = tf(t^{-1})$ in Theorem C in §2 are both operator monotone if $f(t)$ is operator monotone, we give elementary direct proofs of (ii) and (iii) in Theorem C without use of the operator monotonicity of $f(t)$ in (i).

Both Proposition 2.1 and Proposition 2.2 for $\alpha = \frac{1}{2}$ in §2 are shown in [4] and Theorem D is shown in [5] by using a canonical representation and these results are considered in [4][5] closely associated with useful *Morozova-Chentsov function.*

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