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MORE ON OPERATOR MONOTONE FUNCTIONS

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ABSTRACT. We investigate some properties of operator monotone functions. In particular, we show that if f is a non-constant operator monotone function on an interval J and A, B are self-adjoint operators with spectra in J such that $A > B$, then $f(A) > f(B)$. As an application we extend the celebrated Löwner–Heinz inequality.

1. INTRODUCTION

Let $(\mathscr{H}, \langle\cdot,\cdot\rangle)$ be a complex Hilbert space and $\mathbb{B}(\mathscr{H})$ denote the algebra of all bounded linear operators on $\mathscr{H}$ equipped with the operator norm $\|\cdot\|$. An operator $A \in \mathbb{B}(\mathscr{H})$ is called *positive* if $\langle Ax, x\rangle \geq 0$ holds for every $x \in \mathscr{H}$ and then we write $A \geq 0$. For self-adjoint operators $A, B \in \mathbb{B}(\mathscr{H})$, we say $A \leq B$ if $B - A \geq 0$. Also for self-adjoint operators $A, B \in \mathbb{B}(\mathscr{H})$, we say $A \succ B$ if $\langle Ax, x\rangle > \langle Bx, x\rangle$ holds for all non-zero elements $x \in \mathscr{H}$. Also $A > B$ if $A \geq B$ and $A - B$ is invertible.

A continuous real valued function f defined on an interval J is called operator monotone if $A \geq B$ implies $f(A) \geq f(B)$ for all self adjoint operators A, B acting on a Hilbert space with spectra in J.

The Löwner theorem says that a function f is operator monotone on an interval J if and only if f has an analytic continuation to the upper half plan Π_+ such that f maps Π_+ into itself. If $f(t)$ is an operator monotone function on (a, b), then clearly $f\left(\frac{2t-a-b}{b-a}\right)$ is operator monotone on $(-1, 1)$, so in this paper we study the family of operator monotone functions on $(-1, 1)$.

Let $\mathcal{K}$ denote the family of all operator monotone functions on $(-1, 1)$ such that $f(0) = 0$ and $f'(0) = 1$. Hansen and Pedersen [8] showed that $\mathcal{K}$ is a compact convex subset of the space of all bounded functions on $(-1, 1)$ with pointwise convergence topology and that the extreme points of $\mathcal{K}$ are of the form $f_\lambda(t) = \frac{t}{1-\lambda t}$ with $|\lambda| < 1$. They [8] also proved that every $f \in \mathcal{K}$ can be represented as

$$f(t) = \int_{-1}^{1} \frac{t}{1-\lambda t} d\mu(\lambda),$$

where μ is a positive measure on $(-1, 1)$, see also [3].

The Löwner–Heinz inequality says that, $f(x) = x^r$ $(0 < r \leq 1)$ is operator monotone on $[0, \infty)$. Löwner proved the inequality for matrices. Heinz proved it for positive operators acting on a Hilbert space of arbitrary dimension. Based on the C^*-algebra theory, Pedersen [14] gave a shorter proof of the inequality.

There exist several operator norm inequalities each of which is equivalent to the Löwner-Heinz inequality. One of them is $\|A^rB^r\| \leq \|AB\|^r$, called the Cördes inequality in the literature, in which A and B are positive operators and $0 < r \leq 1$. A generalization of the Cördes inequality for operator monotone functions is given in [5]. It is shown in [2] that this norm inequality is related to the Finsler structure of the space of positive invertible elements.

Kwong [10] sowed that if $A > B$ ($A \succ B$, resp.), then $A^r > B^r$ ($A^r \succ B^r$, ressp.) for $0 < r \leq 1$. Uchiyama [15] showed that for every non-constant operator monotone function f on an interval J, $A \succ B$ implies $f(A) \succ f(B)$ for all self-adjoint operators A, B with spectra in J.

There are several extensions of the Löwner–Heinz inequality. The Furuta inequality [6], which states that if $A \geq B \geq 0$, then for $r \geq 0$, $(A^{r/2}A^pA^{r/2})^{1/q} \geq (A^{r/2}B^pA^{r/2})^{1/q}$ holds for $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$, is known as an exquisite extension of the Löwner–Heinz inequality; Also Ando [1] extended the Löwner Heinz inequality for a pair of J-selfadjoint matrices.

Let Ω be a open subset of $\mathbb{C}$. A set $\mathcal{F} \subseteq C(\Omega)$ is bounded if for each compact subset $K \subseteq \Omega$, $\sup\{\|f\|_K : f \in \mathcal{F}\} < \infty$. The Montel theorem states that if $\mathcal{F}$ is a bounded subset of the set $A(\Omega)$ of all analytic functions on Ω, then $\mathcal{F}$ is a normal family, i.e, each sequence $\{f_n\}$ in $\mathcal{F}$ has a subsequence $\{f_{n_j}\}$ converging uniformly on each compact subset of Ω.

2. The results

Throughout this note, let $\Omega = \Pi_+ \bigcup \Pi_- \bigcup (-1, 1)$, where Π_- is the lower half plan.

Theorem 2.1. *The family $\mathcal{K}$ is bounded in $A(\Omega)$, so it is a normal family.*

Proof. Let S be the convex hull of $\{f_\lambda : \ |\lambda| < 1\}$ where $f_\lambda(t) = \frac{t}{1-\lambda t}$. By Krein–Millman's theorem, $\mathcal{K}$ is the closed convex hull of it's extreme points, so $\overline{S} = \mathcal{K}$. Fix $K \subseteq \Omega$ as a compact set. Then $h(\lambda, z) = |1 - \lambda z|$ is continuous on $[-1, 1] \times K$ and so takes its minimum value. It should be noticed that the minimum value m of h $[-1, 1] \times K$ is nonzero. Put $M_K := \sup\{|z| : \ z \in K\}$. Then

$$|f_\lambda(z)| = \frac{|z|}{|1 - \lambda z|} \leq \frac{M_K}{m}$$

If $g = \sum_{i=1}^n c_i f_{\lambda_i} \in \mathcal{S}$, then

$$|g(z)| = |\sum_{i=1}^n c_i f_{\lambda_i}(z)| \leq \sum_{i=1}^n c_i |f_{\lambda_i}(z)| \leq \sum_{i=1}^n c_i \frac{M_k}{m} = \frac{M_k}{m},$$

whence $||g||_K \leq M_K$. Now assume that $g \in \mathcal{K}$ is arbitrary. There exists $\{f_n\}$ in $\mathcal{S}$ such that $f_n(t) \to g(t)$ for each $t \in (-1, 1)$. Since S is bounded, the sequence $\{f_n\}$ is bounded. By Montel's theorem there exists a subsequence $\{f_{n_j}\}$ converging to g'

in uniform compact convergence topology on Ω. Since $g = g'$ on $(-1,1)$, we have $g(z) = g'(z)$ for each $z \in \Omega$. Hence

$$|g(z)| = |g'(z)| = \lim_{n_j \to \infty} |f_{n_j}(z)| \leq \frac{M_K}{m}.$$

Therefore $\mathcal{K}$ is a normal family. □

Proposition 2.2. *Let $f \in \mathcal{K}$ and $f(-1,1) \subseteq (-1,1)$. Then $f(t) = t$ for each $t \in (-1,1)$.*

Proof. Since $f(-1,1) \subseteq (-1,1)$, so $f^n = f \circ f \cdots \circ f \in \mathcal{K}$. Hence by Theorem (2.11), f^n has a convergent subsequence that converges to a function $h \in \mathcal{K}$. Assume that $f(t_0) < t_0$ for some $t_0 \in (-1,1)$. Hence $\{f^{(n)}(t_0)\}$ is an increasing sequence converging to $h(t_0)$. Thus

$$h(f(t_0)) = \lim_{n \to \infty} f^n(f(t_0)) = \lim_{n \to \infty} f^{n+1}(t_0) = h(t_0)$$

Since h is one-one, we infer that $f(t_0) = t_0$, which is a contradiction and this completes the proof. □

Remark 2.3. We can prove Proposition 2.2 directly as follows.

It follows from

$$f(t) = \int_{-1}^{1} \frac{t}{1 - \lambda t} d\mu(\lambda),$$

that

$$-1 \leqq \int_{-1}^{1} \frac{t}{1 - \lambda t} d\mu(\lambda) \leqq 1 \quad (-1 < t < 1).$$

Since for each λ the integrand $\frac{t}{1-\lambda t}$ is positive and increasing on $0 < t < 1$, by the Lebesgue's monotone convergence theorem

$$\int_{-1}^{1} \frac{1}{1 - \lambda} d\mu(\lambda) = \lim_{t \to 1-} \int_{-1}^{1} \frac{t}{1 - \lambda t} \leqq 1.$$

Similarly we have

$$\int_{-1}^{1} \frac{-1}{1 + \lambda} d\mu(\lambda) = \lim_{t \to -1+} \int_{-1}^{1} \frac{t}{1 - \lambda t} \geqq -1.$$

Thus we have

$$\int_{-1}^{1} \frac{1}{1 - \lambda^2} d\mu(\lambda) = \frac{1}{2} \int_{-1}^{1} (\frac{1}{1 - \lambda} + \frac{1}{1 + \lambda}) d\mu(\lambda)$$
$$\leqq 1 = \int_{-1}^{1} 1 d\mu(\lambda).$$

From this it follows that $\frac{1}{1-\lambda^2} = 1$ almost everywhere with respect to μ, Thus $\mu\{0\} = 1$, which implies $f(t) = t$. □.

Corollary 2.4. *Let f be an odd operator monotone function on $(-1,1)$ and A is a bounded linear operator on a Hilbert space with spectrum in $(-1,1)$. Then $f(|A|) \geq f'(0)|A|$.*

Proof. If $f(t_0) < f'(0)t_0$ for some $t_0 \in (0,1)$, then $f_1(t) = \frac{1}{f'(0)t_0} f(t_0 t) \in \mathcal{K}$ and $f_1(-1,1) \subseteq (-1,1)$, so, by Proposition (2.2), we have $f_1(1) = 1$, which is a contradiction. Hence

$$f(|t|) \geq f'(0)|t|, \qquad t \in (-1,1) \tag{2.1}$$

Therefore $f(|A|) \geq f'(0)|A|$. □

Remark 2.5. A direct proof of (2.1) reads as follows. Notice that $f(0) = 0$. Hence

$$f(t) = f'(0) \int_{-1}^{1} \frac{t}{1-\lambda t} d\mu(\lambda). \tag{2.2}$$

Since $f(t) = -f(-t)$, we obtain

$$\int_{-1}^{1} \frac{1}{1-\lambda t} d\mu(\lambda) = \int_{-1}^{1} \frac{1}{1+\lambda t} d\mu(\lambda).$$

Thus

$$\int_{-1}^{1} \frac{1}{1-\lambda t} d\mu(\lambda) = \frac{1}{2} \int_{-1}^{1} \left(\frac{1}{1-\lambda t} + \frac{1}{1+\lambda t}\right) d\mu(\lambda)$$
$$= \int_{-1}^{1} \frac{1}{1-(\lambda t)^2} d\mu(\lambda) \geqq \int_{-1}^{1} \frac{1}{1-(\lambda t)^2} d\mu(\lambda) = 1.$$

(2.2) yields $|f(t)| \geqq f'(0)|t|$. □

In the sequel we need the following lemma.

Lemma 2.6. [3, Lemma 2.4] *If f is an operator monotone function on an interval (a,b), then $f^{2p+1}(t) \geq 0$ for all $p = 0, 1, 2, \cdots$ and all $a < t < b$.*

Corollary 2.7. *Let f be an odd operator monotone function on $(-1,1)$. Then f is concave on $(-1,0)$ and convex on $(0,1)$.*

Proof. Without loss of generality we may assume that $f \in \mathcal{K}$. We shall show that f is convex on $(0,1)$. The proof of Lemma 4.1 of [8] shows that $f'(t) \geq \frac{f(t)^2}{t^2}$. It follows from Corollary (2.4) that $f'(t) \geq 1$ for each $t \in (0,1)$. Therefore

$$f''(0) = \lim_{t \to 0^+} \frac{f'(t) - f'(0)}{t} = \lim_{t \to 0^+} \frac{f'(t) - 1}{t} \geq 0.$$

By Lemma (2.6), $f^{(3)}(t) \geq 0$ for all $t \in (-1,1)$, so $f''(t) \geq 0$ for all $t \in (0,1)$ since f'' is monotone. Hence f is a convex function on (0,1). Since f is an odd function, f is concave on $(-1,0)$. □

Theorem 2.8. *An odd operator monotone function on* $(-1,1)$ *is of the form*

$$f(t)=f'(0)\int_{-1}^{1}\frac{t}{1-(\lambda t)^2}d\mu(\lambda), \tag{2.3}$$

where μ *is a probability measure on* $(-1,1)$.

Proof. As before, we may assume that $f\in\mathcal{K}$. The function f can be represented as a power series $f(t)=\sum_{n=1}^{\infty}a_nt^n$, which is convergent for $|t|<1$, cf. [3] . Since f is odd, $a_{2n}=0$ for all n. Due to f is operator monotone, there is a probability measure μ on $(-1,1)$ such that

$$f(t)=\int_{-1}^{1}\frac{t}{1-\lambda t}d\mu(\lambda)=\int_{-1}^{1}\sum_{n=1}^{\infty}t(\lambda t)^n\,d\mu(\lambda)=\sum_{n=1}^{\infty}t^{n+1}\int_{-1}^{1}\lambda^n d\mu(\lambda)$$

Therefore $a_{2n}=\int_{-1}^{1}\lambda^{2n-1}=0$ and so

$$f(t)\;=\;\int_{-1}^{1}\sum_{n=1}^{\infty}t(\lambda t)^{2n-1}\,d\mu(t)=\int_{-1}^{1}\frac{t}{1-(\lambda t)^2}d\mu(\lambda).$$

If f is of the form (2.3), then it is trivially odd. In addition,

$$f(t)\;=\;\int_{-1}^{1}\frac{t}{1-(\lambda t)^2}d\mu(\lambda)=\frac{1}{2}\int_{-1}^{1}\frac{t}{1-\lambda t}+\frac{t}{1+\lambda t}d\mu(\lambda)=\frac{1}{2}(g(t)-g(-t)),$$

where $g(t)=\int_{-1}^{1}\frac{t}{1-\lambda t}d\mu(\lambda)$. Hence f is an odd operator monotone function on $(-1,1)$. □

We start main results with the following useful lemma.

Lemma 2.9. *Let* $A,B\in\mathbb{B}(\mathscr{H})$ *be invertible positive operators such that* $A-B\geq m>0$. *Then*

$$B^{-1}-A^{-1}\geq\frac{m}{(||A||-m)\;||A||}. \tag{2.4}$$

Proof. Since $f(t)=\frac{1}{t}$ is a decreasing operator monotone function on $[0,\infty)$ we have $B^{-1}\geq(A-m)^{-1}$. On the other hand

$$\begin{aligned}
&(A-m)^{-1}\geq A^{-1}+\frac{m}{(||A||-m)||A||}\\
\Longleftrightarrow\;&(A^{-1}+\frac{m}{(||A||-m)||A||})(A-m)\leq 1\\
\Longleftrightarrow\;&\frac{A^2}{(||A||-m)||A||}-\frac{mA}{(||A||-m)||A||}\leq 1\\
\Longleftrightarrow\;&A^2-mA\leq(||A||-m)||A||\\
\Longleftrightarrow\;&||A^2-mA||\leq(||A||-m)||A||.
\end{aligned}$$

There exists $\lambda_0 \in \mathrm{sp}(A)$ such that $||A|| = \lambda_0$. Since $A \geq m > 0$, we have

$$\begin{aligned} ||A^2 - mA|| &= \max\{\lambda \ : \ \lambda \in \mathrm{sp}(A^2 - mA)\} \\ &= \max\{\lambda^2 - m\lambda \ : \ \lambda \in \mathrm{sp}(A)\} \\ &= \lambda_0^2 - m\lambda_0 \\ &= (||A|| - m)||A||. \end{aligned}$$

So $B^{-1} \geq (A-m)^{-1} \geq A^{-1} + \frac{m}{(||A||-m)||A||}$. □

Proposition 2.10. *Let f be a non-constant operator monotone function on an interval J and A, B be self-adjoint operators with spectra in J such that $A > B$. Then $f(A) > f(B)$.*

Proof. Without loss of generality we assume that $J = (-1, 1)$. Let $A, B \in \mathbb{B}(\mathscr{H})$ be self-adjoint operators with spectra in $(-1, 1)$ and $A - B$ is positive and invertible. So there exists $m > 0$ such that $A - B \geq m > 0$. Put $f_\lambda(t) = \frac{t}{1-\lambda t}$ for each λ with $|\lambda| < 1$. We shall show that $f_\lambda(A) - f_\lambda(B)$ is bounded blow and so invertible. It is clear that the claim is true for $\lambda = 0$. If $0 < \lambda < 1$, then $(1 - \lambda B) - (1 - \lambda A) = \lambda(A - B) > \lambda m > 0$ as well as $1 - \lambda B$ and $1 - \lambda A$ are positive invertible operators. Since

$$\frac{t}{1-\lambda t} = \frac{-1}{\lambda} + \frac{1}{\lambda}\left(\frac{1}{1-\lambda t}\right),$$

by Lemma 2.9, we have

$$\begin{aligned} f_\lambda(A) - f_\lambda(B) &= \frac{1}{\lambda}\left(\frac{1}{1-\lambda A} - \frac{1}{1-\lambda B}\right) \\ &\geq \frac{1}{\lambda}\left(\frac{\lambda m}{(||1-\lambda B|| - \lambda m)\ ||1-\lambda B||}\right) \qquad \text{(by (2.9))} \\ &= \frac{m}{(||1-\lambda B|| - \lambda m)\ ||1-\lambda B||} > 0 \end{aligned}$$

A similar argument shows that

$$f_\lambda(A) - f_\lambda(B) \geq \frac{m}{(||1-\lambda A|| + \lambda m)\ ||1-\lambda A||} > 0$$

for each $-1 < \lambda < 0$. Since f is operator monotone on $(-1, 1)$, it can be represented as

$$f(t) = f(0) + f'(0)\int_{-1}^{1} f_\lambda(t)d\mu(\lambda),$$

where μ is a nonzero positive measure on $(-1,1)$. Since f is nonconstant, $f'(0) > 0$, [3, Lemma 2.3]. Hence

$$\begin{aligned} f(A) &- f(B) \\ &= f'(0)\int_{-1}^{1}\left(\frac{A}{1-\lambda A} - \frac{B}{1-\lambda B}\right) d\mu(\lambda) \\ &= f'(0)\int_{-1}^{1}(f_\lambda(A) - f_\lambda(B))\, d\mu(\lambda) \\ &\geq f'(0)\int_{-1}^{1} m_\lambda d\mu(\lambda), \end{aligned}$$

where

$$m_\lambda = \frac{m}{(||1-\lambda B|| - \lambda m)\ ||1-\lambda B||}$$

if $0 \leq \lambda < 1$, and

$$m_\lambda = \frac{m}{(||1-\lambda A|| + \lambda m)\ ||1-\lambda A||}$$

if $-1 < \lambda < 0$. Since μ is a nonzero positive measure and $m_\lambda > 0$, we have

$$f(A) - f(B) \geq f'(0)\int_{-1}^{1} m_\lambda d\mu(\lambda) > 0.$$

Therefore $f(A) > f(B)$. □

Theorem 2.11. *Let $A, B \in \mathbb{B}(\mathscr{H})$ be positive operators such that $A - B \geq m > 0$ and $0 < r \leq 1$. Then*

$$A^r - B^r \geq ||A||^r - (||A|| - m)^r.$$

Proof. Let $0 < r < 1$. It is known that

$$t^r = \frac{\sin(r\pi)}{\pi}\int_0^\infty \frac{t}{\lambda + t}\lambda^{r-1} d\lambda, \tag{2.5}$$

in which $0 < r < 1$, see e.g. [4, Chapter V]. First note that,

$$\begin{aligned} \frac{A}{\lambda + A} - \frac{B}{\lambda + B} &= \lambda\left(\frac{1}{\lambda + B} - \frac{1}{\lambda + A}\right) \\ &\geq \frac{\lambda m}{(||A+\lambda|| - m)||A+\lambda||} \quad \text{by (2.4)} \\ &= \frac{\lambda m}{(||A|| + \lambda - m)(||A|| + \lambda)} \end{aligned}$$

for each $\lambda > 0$. By using (2.5) we have

$$\begin{aligned} &A^r - B^r \\ &= \frac{\sin(r\pi)}{\pi}\int_0^\infty \lambda^{r-1}\left(\frac{A}{\lambda + A} - \frac{B}{\lambda + B}\right) d\lambda \\ &\geq \frac{\sin(r\pi)}{\pi}\int_0^\infty \left(\frac{m\lambda^r}{(||A|| + \lambda - m)(||A|| + \lambda)}\right) d\lambda, \end{aligned}$$

We need to compute

$$I=\int_0^\infty \frac{\lambda^r}{(\lambda+||A||)(\lambda+(||A||-m))}d\lambda$$

where $0<r<1$. We will need the branch cut for $z^r=\rho^r e^{ir\theta}$, in which $z=\rho e^{i\theta}$ and $0\leq\theta\leq 2\pi$. Consider

$$\int_C \frac{z^r}{(z+||A||)(z+(||A||-m))}dz\,,$$

where the keyhole contour C consists of a large circle C_R of radius R, a small circle C_ϵ of radius ϵ and two lines just above and below the branch cuts $\theta=0$; see Figure 1. The contribution from C_R is $O(R^{r-2})2\pi R=O(R^{r-1})=0$ as $R\to\infty$. Similarly the contribution from C_ϵ is zero as $\epsilon\to 0$. The contribution from just above the branch cut and from just below the branch cut is I and $-e^{2r\pi i}I$, respectively, as $\epsilon\to 0$ and $R\to\infty$. Hence, taking the limits as $\epsilon\to 0$ and $R\to\infty$,

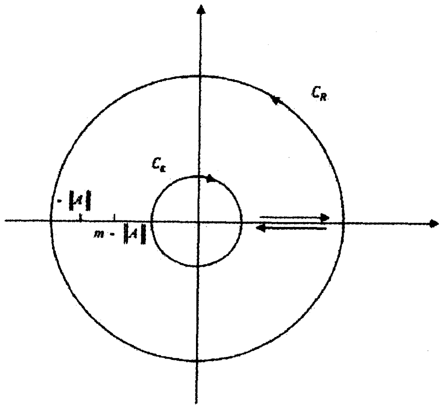

FIGURE 1. Keyhole contour

$$\begin{aligned}(1-e^{2r\pi i})I &= \int_C \frac{z^r}{(z+||A||)(z+(||A||-m))}dz\\ &= -2\pi i e^{r\pi i}\left(\frac{||A||^r-(||A||-m)^r}{||A||-(||A||-m)}\right)\end{aligned}$$

by the Cauchy residue theorem. So

$$I=\frac{\pi}{m\sin(r\pi)}\left(||A||^r-(||A||-m)^r\right).$$

Therefore

$$\begin{aligned}&A^r-B^r\\ &\geq \frac{\sin(r\pi)}{\pi}\int_0^\infty\left(\frac{m\lambda^r}{(||A||+\lambda-m)(||A||+\lambda)}\right)d\lambda\\ &= ||A||^r-(||A||-m)^r.\end{aligned}$$

□

Corollary 2.12. *Let $A, B \in \mathbb{B}(\mathscr{H})$ be positive operators such that $A - B \geq m > 0$. Then*

$$\log A - \log B \geq \log ||A|| - \log(||A|| - m).$$

Proof. Put $f_n(t) = n(t^{\frac{1}{n}} - 1)$ on $[0, \infty)$. Then the sequence $\{f_n\}$ uniformly converges to $\log t$ on any compact subset of $(0, \infty)$. Hence

$$\begin{aligned} \log A - \log B &= \lim_{n \to \infty} f_n(A) - f_n(B) \\ &\geq \lim_{n \to \infty} n(||A||^{\frac{1}{n}} - (||A|| - m)^{\frac{1}{n}}) \\ &= \log ||A|| - \log(||A|| - m). \end{aligned}$$

□

References

1. T. Ando, *Löwner inequality of indefinite type*, Linear Algebra Appl., **385** (2004), 73–80.
2. E. Andruchow, G. Corach and D. Stojanoff, *Geometrical significance of Löwner–Heinz inequality*, Proc. Amer. Math. Soc. **128** (2000), no. 4, 1031-1037.
3. J. Bendat and S. Sherman, *Monotone and convex operator functions*, Trans. Amer. Math. Soc. **79** (1955), 58–71.
4. R. Bhatia, *Matrix Analysis*, Springer, New York, 1997.
5. J. Fujii and M. Fujii, *A norm inequality for operator monotone functions*, Math. Japon. **35** (1990), no. 2, 249-252.
6. T. Furuta, *$A \geqslant B \geqslant 0$ assures $(B^rA^pB^r)^{1/q} \geqslant B^{(p+2r)/q}$ for $r \geqslant 0$, $p \geqslant 0$, $q \geqslant 1$ with $(1+r)q \geqslant p+2r$*, Proc. Amer. Math. Soc., **101** (1987), 85-88.
7. T. Furuta, *Norm inequalities equivalent to Lowner-Heinz theorem*, Rev. Math. Phys. **1** (1989), 135–137.
8. F. Hansen and G. Pedersen, *Jensen's inequality for operators and Lowner's theorem*, Math. Ann. **258** (1981/82), no. 3, 229-241.
9. E. Heinz, *Beiträge zur Störungstheorie der Spektralzerlegung*, Math. Ann. *123* (1951), 415–438.
10. M.K. Kwong, *Inequalities for the powers of nonnegative Hermitian operators*, Proc. Amer. Math. Soc. **51** (1975), 401-406.
11. C. Löwner, *Über monotone Matrixfunktionen*, Math. Z. *38* (1934), 177–216.
12. M.S. Moslehian and H. Najafi, *An extension of the Lowner–Heinz inequality*, Linear Algebra Appl. 437 (2012), no. 9, 2359–2365.
13. M.S. Moslehian, H. Najafi and M. Uchiyama, *A normal family of operator monotone functions*, Hokkaido Math. J., to appear.
14. G.K. Pedersen, *Some operator monotone functions*, Proc. Amer. Math. Soc. **36** (1972), 309–310.
15. M. Uchiyama, *Strong monotonicity of operator functions*, Integral Equations Operator Theory **37** (2000), no. 1, 95–105.

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