Some reduced expressions of the classical Weyl groups and the Weyl groupoids of the Lie superalgebras \( \text{osp}(2m|2n) \)

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**Abstract**

We give some reduced expressions of the classical Weyl groups \( W(A_{N-1}), W(B_N) = W(C_N), W(D_N) \) and the Weyl groupoid of the Lie superalgebra \( \text{osp}(2m|2(N - m)) \).

### 1 Some reduced expressions of the classical Weyl groups

For \( m, n \in \mathbb{Z} \), let \( J_{n,m} := \{k \in \mathbb{Z} | m \leq k \leq n\} \).

Let \( N \in \mathbb{N} \). Let \( M_N(\mathbb{R}) \) be the \( \mathbb{R} \)-algebra of \( N \times N \)-matrices. For \( k, r \in J_{1,N} \), let \( E_{k,r} := [\delta_{k,k'}\delta_{r,r'}]_{k',r' \in J_{1,N}} \in M_N(\mathbb{R}) \), that is \( E_{k,r} \) is the matrix unite such that its \((k, r)\)-component is 1 and the other components is 0. Then \( M_N(\mathbb{R}) = \bigoplus_{k,r \in J_{1,N}} \mathbb{R}E_{k,r} \). Let \( \mathbb{R}^N \) denote the \( \mathbb{R} \)-linear space of \( N \times 1 \)-matrices. For \( k \in J_{1,N} \), let \( e_k \) is the element of \( \mathbb{R}^N \) such that its \((k, 1)\)-component is 1 and the other components is 0. That is \( \{e_k | k \in J_{1,N}\} \) is the standard basis of \( \mathbb{R}^N \). The \( \mathbb{R} \)-algebra \( M_N(\mathbb{R}) \) acts on \( \mathbb{R}^N \) in the ordinal way, that is \( E_{k,r}e_p = \delta_{r,p}e_r \). Let \( \text{GL}_N(\mathbb{R}) \) be the group of invertible \( N \times N \)-matrices, that is \( \text{GL}_N(\mathbb{R}) = \{X \in M_N(\mathbb{R}) | \det X \neq 0\} \). Let \( (,): \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R} \) be the \( \mathbb{R} \)-bilinear map defined by \( (e_k, e_r) := \delta_{kr} \).

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Definition 1.1. For $v \in \mathbb{R}^N \setminus \{0\}$, define $s_v \in \text{GL}_N(\mathbb{R})$ by $s_v(u) := u - \frac{2(u,v)}{(v,v)}v$ ($u \in \mathbb{R}^N$), that is, $s_v$ is the reflection with respect to $v$.

Note that

\begin{equation} \tag{1.1} \label{eq:refl_def}
    s_v^2 = 1.
\end{equation}

We say that a subset $R$ of $\mathbb{R}^N \setminus \{0\}$ is a root system (in $\mathbb{R}^N$) if $|R| < \infty$, $s_v(R) = R$ and $\mathbb{R}v \cap R = \{v, -v\}$ for all $v \in R$, see [Hum, 1.1].

Let $R$ be a root system in $\mathbb{R}^N$. We say that a subset $\Pi$ of $R$ is a root basis of $\Pi$ if $\Pi$ is a (set) basis of $\text{Span}_R(\Pi)$ as an $\mathbb{R}$-linear space and $R \subset \text{Span}_{\mathbb{R}_\geq 0}(\Pi) \cup -\text{Span}_{\mathbb{R}_\geq 0}(\Pi)$ (this is called a simple system in [Hum, 1.3]).

Let $R$ be a root system in $\mathbb{R}^N$. Let $\Pi$ be a root basis of $R$. Let $R^+(\Pi) := R \cap \text{Span}_{\mathbb{R}_\geq 0}(\Pi)$. We call $R^+(\Pi)$ a positive root system of $R$ associated with $\Pi$ (this is called a positive system in [Hum, 1.3]).

Definition 1.2. (See [Hum, 2.10].) Let $R$ be a root system in $\mathbb{R}^N$. Let $\Pi$ be a root basis of $R$.

(1) Assume $N \geq 2$. We call $R$ the $A_{N-1}$-type root system if

$$R = \{ e_x - e_y \mid x, y \in J_{1,N}, x \neq y \}.$$

We call $\Pi$ the $A_{N-1}$-type standard root basis if

$$\Pi = \{ e_x - e_{x+1} \mid x \in J_{1,N-1} \}.$$

(2) Assume $N \geq 2$. We call $R$ the $B_N$-type standard root system if

$$R = \{ ce_x + c'e_y \mid x, y \in J_{1,N}, x < y, c, c' \in \{1, -1\} \} \cup \{ c''e_x \mid c'' \in \{1, -1\} \}.$$

We call $\Pi$ the $B_N$-type standard root basis if

$$\Pi = \{ e_x - e_{x+1} \mid x \in J_{1,N-1} \} \cup \{ e_N \}.$$

(3) Assume $N \geq 2$. We call $R$ the $C_N$-type root system if

$$R = \{ ce_x + c'e_y \mid x, y \in J_{1,N}, x < y, c, c' \in \{1, -1\} \} \cup \{ 2c''e_x \mid c'' \in \{1, -1\} \}.$$

We call $\Pi$ the $C_N$-type standard root basis if

$$\Pi = \{ e_x - e_{x+1} \mid x \in J_{1,N-1} \} \cup \{ 2e_N \}.$$
(4) Assume $N \geq 4$. We call $R$ the $D_N$-type root system if
\[ R = \{ c e_x + c' e_y \mid x, y \in J_{1,N}, x < y, c, c' \in \{1, -1\} \}. \]
We call $\Pi$ the $D_N$-type standard root basis if
\[ \Pi = \{ e_x - e_{x+1} \mid x \in J_{1,N-1} \} \cup \{ e_{N-1} + e_N \}. \]
Let $R$ be a root system in $\mathbb{R}^N$. Let $\Pi$ be a root basis of $R$. We call $W(\Pi)$ the Coxeter group associated with $(R, \Pi)$. Let $S(\Pi) := \{ s_v \in W(\Pi) \mid v \in \Pi \}$. We call $(W(\Pi), S(\Pi))$ the Coxeter system associated with $(R, \Pi)$, see [Hum, 1.6]. Define the map $\ell : W(\Pi) \to \mathbb{Z}_{\geq 0}$ in the following way, see [Hum, 1.6]. Let $\ell(1) := 0$, where 1 is a unit of $W(\Pi)$. Note that an arbitrary $w \in W(\Pi)$ can be written as a product of finite $s_v$'s with some $v \in \Pi$, say $w = s_{v_1} \cdots s_{v_r}$ for some $r \in \mathbb{N}$ and some $v_x \in \Pi (x \in J_{1,r})$. If $w \neq 1$, let $\ell(w)$ be the smallest $r$ for which such an expression exists, and call the expression reduced. For $w \in W(\Pi)$, we call $\ell(w)$ the length of $w$. Let
\[ \mathfrak{L}(w) := \{ v \in R^+(\Pi) \mid w(v) \in -R^+(\Pi) \}. \]
It is well-known that
\[ \ell(w) = |\mathfrak{L}(w)| \quad (\text{see [Hum, Corollary 1.7]}) \]
(see [Hum, Propsoition 1.4]), and
\[ \ell(w s_v) = \begin{cases} 
\ell(w) + 1 & \text{if } w(v) \in R^+(\Pi), \\
\ell(w) - 1 & \text{if } w(v) \in -R^+(\Pi)
\end{cases} \]
(see [Hum, Lemma 1.6 and Corollary 1.7]). Assume that $|R| < \infty$. By the above properties, we can see that there exists a unique $w_o \in W(\Pi)$ such that $w_o(\Pi) = -\Pi$, see [Hum, 1.8]. It is well-known that
\[ \ell(w_o) = |R^+(\Pi)|, \]
which can easily be proved by (1.2), (1.3) and (1.4). Note that $w_o$ is the only element $W(\Pi)$ that $\ell(w) \leq \ell(w_o)$ for all $w \in W(\Pi)$, and $\ell(w) = \ell(w_o) - \ell(w_o w^{-1})$ for all $w \in W(\Pi)$. We call $w_o$ the longest element of the Coxeter system of $(W(\Pi), S(\Pi))$.

Let $k, r \in J_{1,N}$ be such that $k \leq r$. For $z_p \in J_{k,r} \cup (-J_{k,r}) \ (p \in J_{k,r})$ with $|u_p| \neq |u_t| \ (p \neq t)$, let
\[
\left\{ \begin{array}{lll}
k & k+1 & \ldots & r \\
z_k & z_{k+1} & \ldots & z_r \end{array} \right\} := \sum_{p \in J_{k,r}} \frac{z_p}{|z_p|} E_{|z_p|,p} + \sum_{t \in J_{1,N \setminus J_{k,r}}} E_{t,t} \in \text{GL}_N(\mathbb{R}).
\]

We have
\[
s_{ek} = \left\{ \begin{array}{ll}
k \\
-k \end{array} \right\} \quad (k \in J_{1,N}),
\]
\[
s_{ek-e_{k+1}} = \left\{ \begin{array}{ll}
k & k+1 \\
k+1 & k \end{array} \right\} \quad (k \in J_{1,N-1}),
\]
and
\[
s_{ek+e_{k+1}} = \left\{ \begin{array}{ll}
k & k+1 \\
-(k+1) & -k \end{array} \right\} \quad (k \in J_{1,N-1}).
\]

Let $k, p, r \in J_{k,r}$ with $k < r$ and $k \leq p \leq r$, let
\[
\left\{ \begin{array}{lll}
k & \ldots & p \\
z_k & \ldots & z_p \\
p+1 & \ldots & r \end{array} \right\} := \left\{ \begin{array}{lll}
k & \ldots & p \\
z_k & \ldots & z_p \\
p+1 & \ldots & z_r \end{array} \right\}.
\]

Let $k, r \in J_{1,N-1}$ with $k \leq r$. Define $s_{(k,r)}$ inductively by
\[
s_{(k,r)} := \left\{ \begin{array}{ll}
1 & \text{if } k = r \\
s_{(k,r-1)} s_{e_{r-1}-e_r} & \text{if } k < r.
\end{array} \right.
\]

Then, if $r > k$, we have
\[
s_{(k,r)} = \left\{ \begin{array}{lll}
k & \ldots & p \\
k+1 & \ldots & p+1 \\
r-1 & \ldots & r \end{array} \right\},
\]
since (if $r \geq k+2$)
\[
s_{(k,r)} = s_{(k,r-1)} s_{e_{r-1}-e_r}
\]
\[
= \left\{ \begin{array}{lll}
k & \ldots & p \\
k+1 & \ldots & p+1 \\
r-2 & \ldots & r-1 \end{array} \right\} \left\{ \begin{array}{lll}
r-1 & \ldots & r \\
r & \ldots & r \end{array} \right\}
\]
(by (1.7) and an induction)
\[
= \left\{ \begin{array}{lll}
k & \ldots & p \\
k+1 & \ldots & p+1 \\
r-1 & \ldots & r \end{array} \right\}.
\]
Define \( s_{(r,k)} \) inductively by \( s_{(r,k)} := s_{e_{r-1}-e_{r}}s_{(r-1,k)} \) if \( r \geq k+1 \). Clearly (if \( r > k \)) we have

\[
(1.12) \quad s_{(r,k)} = s_{(k,r)}^{-1} = \begin{cases} k & k+1 \ldots \ldots \ldots p \ldots r \\ r & k \ldots p-1 \ldots r-1 \end{cases}.
\]

**Lemma 1.3.** Let \( \Pi \) be the \( A_{N-1} \)-type standard root basis. Let \( w_o \) be the longest element of \((W(\Pi), S(\Pi))\). Let \( s_k := s_{e_k-e_{k+1}} \in S(\Pi) \) for \( k \in J_{1,N-1} \).

(1) We have

\[
(1.13) \quad w_o = \begin{cases} 1 \ldots p \ldots N \\ N \ldots N-p+1 \ldots 1 \end{cases}.
\]

Moreover

\[
(1.14) \quad w_o = (s_1s_2 \cdots s_{N-1})(s_1s_2 \cdots s_{N-2}) \cdots (s_1s_2) \frac{s_1}{2} \quad 1.
\]

Furthermore RHS of \( (1.14) \) is the reduced expression of \( w_o \).

(2) Let \( m \in J_{2,N-1} \). Then

\[
(1.15) \quad w_o = \underbrace{(s_1s_2 \cdots s_{m-1})}_{m-1} \underbrace{(s_1s_2 \cdots s_{m-2})}_{m-2} \cdots \underbrace{(s_1s_2)}_{2} \frac{s_1}{1} \quad 1
\]

\[
\quad \cdot \underbrace{(s_{m+1}s_{m+2} \cdots s_{N-1})}_{N-m-1} \underbrace{(s_{m+1}s_{m+2} \cdots s_{N-2})}_{N-m-2} \cdots \underbrace{(s_{m+1}s_{m+2})}_{2} \frac{s_{m+1}}{1}
\]

\[
\quad \cdot \underbrace{(s_{m}s_{m+1} \cdots s_{N-1})}_{N-m} \underbrace{(s_{m-1}s_{m} \cdots s_{N-2})}_{N-m} \cdots \underbrace{(s_{1}s_{2} \cdots s_{N-m})}_{N-m},
\]

and RHS of \( (1.15) \) is a reduced expression of \( w_o \).

**Proof.** By (1.5), we have

\[
(1.16) \quad \ell(w) = \frac{N(N-1)}{2}.
\]

Let \( k, r \in J_{1,n} \) with \( k < r \). Let

\[
x_{(k,r)} := \begin{cases} k \ldots p \ldots r \\ r \ldots r-p+k \ldots k \end{cases}.
\]
Then
\[(1.17) \quad s_{(k,r)}s_{(k,r-1)}\cdots s_{(k,k+1)} = x_{(k,r)},\]
since, if \(r \geq k + 2\), we have
\[
s_{(k,r)}(s_{(k,r-1)}\cdots s_{(k,k+1)}) = x_{(k,r-1)}.
\]

We have
\[(1.18) \quad x_{(k,r)} \in W(\Pi) \quad \text{and} \quad \ell(x_{(k,r)}) = \frac{(k-r+1)(k-r)}{2},\]
where the first claim follows from (1.17) and the second claim follows from
by (1.2), since \(\mathfrak{L}(x_{(k,r)}) = \{e_x - e_y | k \leq x < y \leq r\}\).

We obtain the claim (1) from (1.16), (1.17) and (1.18) for \(k = 1\) and
\(r = N\).

For \(k, r, t \in J_{1,N-1}\) with \(k < r \leq t\), let
\[(1.19) \quad y_{(k,r-1;r,t)} = \{k + t - r + 1 \ldots z \ldots r - 1 \ldots r \ldots t \ldots t + k - r\}.\]

We have
\[(1.20) \quad s_{(k+t-r,t)}s_{(k+t-r-1,t-1)}\cdots s_{(k+1,r+1)}s_{(k,r)} = y_{(k,r-1;r,t)},\]
since, if \(t > r\),
\[
(s_{(k+t-r,t)}s_{(k+t-r-1,t-1)}\cdots s_{(k+1,r+1)})s_{(k,r)}
= y_{(k+1,r;r+1,t)} \cdot \{k + 1 \ldots p + 1 \ldots r \ldots k\}
= y_{(k,r-1;r,t)}.
\]
We have

\begin{equation}
(1.21) \quad y_{(k,r-1;r,t)} \in W(\Pi) \quad \text{and} \quad \ell(y_{(k,r-1;r,t)}) = (t - r + 1)(r - k),
\end{equation}

where the first claim follows from (1.20) and the second claim follows from (1.2), since $\mathfrak{L}(x_{(k,r)}) = \{e_x - e_y | x \in J_{k,r-1}, x \in J_{r,t}\}$.

Let $m \in J_{2,N-1}$. By (1.13), we have

\begin{equation}
(1.22) \quad w_o = x_{(1,m)}x_{(m+1,N)}y_{(1,N-m;N-m+1,N)}.
\end{equation}

Then we obtain the claim (2) from (1.16), (1.18), (1.21) and (1.22), since $\frac{m(m-1)}{2} + \frac{(N-m)(N-m-1)}{2} + (N - m)m = \frac{N(N-1)}{2}.$ \hfill \Box

Let $k, r \in J_{1,N}$ with $k \leq r$. Let

\begin{equation}
(1.23) \quad b_{(k,r)} := s_{e_k} \cdots s_{e_r} = \begin{cases} k \cdots p \cdots r \cr -k \cdots -p \cdots -r \end{cases},
\end{equation}

see also (1.6). By (1.10), we have

\begin{equation}
(1.24) \quad (s_{(k,r)})^{r-k+1} = 1.
\end{equation}

By (1.6) and (1.10), we have

\begin{equation}
(1.25) \quad s_{e_t}s_{(k,r)} = s_{(k,r)}s_{e_{t-1}}
\end{equation}

By (1.23), (1.24) and (1.25), for $t \in J_{k+1,r}$, we have

\begin{equation}
(1.26) \quad (s_{(k,r)}s_{e_r})^{r-k+1} = (s_{(k,r)})^{r-k+1}s_{e_k} \cdots s_{e_r} = b_{(k,r)}.
\end{equation}

By (1.6), (1.10) and (1.12), we have

\begin{equation}
(1.27) \quad s_{e_k - e_{k+1}} \cdots s_{e_{r-1} - e_r} s_{e_{r-1} - e_r} \cdots s_{e_k - e_{k+1}} = (s_{(k,r)}s_{e_r} s_{(r,k)}) = s_{e_k}.
\end{equation}

**Lemma 1.4.** Let $\Pi$ be the $B_N$-type standard root basis. Let $w_o$ be the longest element of $(W(\Pi), S(\Pi))$. Let $s_k := s_{e_k - e_{k+1}} \in S(\Pi)$ for $k \in J_{1,N-1}$ and let $s_N := s_{e_N} \in S(\Pi)$.

(1) We have

\begin{equation}
(1.28) \quad w_o = b_{(1,N)} = \left(s_1s_2 \cdots s_N\right)^N.
\end{equation}
Moreover the rightmost hand side of (1.28) is a reduced expression of \( w_o \).

(2) Let \( k, r \in J_{1,N} \) with \( k \leq r \). Then

\[
b_{(k,r)} = \left(\frac{s_k s_{k+1} \cdots s_{N-1} s_N s_{N-1} \cdots s_{r+1} s_r}{2N-k-r+1}\right)^{r-k+1}.
\]

Moreover RHS of (1.29) is a reduced expression of \( b_{(k,r)} \).

(3) Let \( k_1, k_2, \ldots, k_{r-1} \in J_{1,N} \) with \( k_1 < k_2 < \ldots < k_{r-1} \). Let \( b'_y := b_{(k_{y-1}, k_y)} \) \( (y \in J_{1,r}) \), where let \( k_0 := 1 \) and \( k_r := N + 1 \). Then we have \( w_o = b'_1 b'_2 \cdots b'_r \) and \( \ell(w_o) = \sum_{y=1}^{r} \ell(b'_y) \). Moreover \( b'_y b'_z = b'_z b'_y \) for \( y, z \in J_{1,r} \).

(4) Let \( m \in J_{1,N-1} \). Then

\[
w_o = \left(\frac{s_{N-m+1} s_{N-m+2} \cdots s_N}{m}\right)^m \left(\frac{s_1 s_2 \cdots s_{N-1} s_N s_{N-1} \cdots s_{N-m+1} s_{N-m}}{N+m}\right)^{N-m}.
\]

Moreover RHS of (1.30) is a reduced expression of \( w_o \).

**Proof.** We can easily show (1.29) by (1.26) and (1.27).

Let \( k, r \in J_{1,N} \) be such that \( k \leq r \). Note that

\[
\mathcal{L}(b_{(k,r)}) = \{ e_t \mid t \in J_{k,r} \} \cup \{ e_t + c e_{t'} \mid c \in \{-1, 1\}, t \in J_{k,r}, t' \in J_{t',N} \}.
\]

Hence by (1.2), we have

\[
\ell(b_{(k,r)}) = (r - k + 1) + 2 \sum_{t=k}^{r}(N - t)
\]

\[
= (r - k + 1) + 2N(r - k + 1) - 2\left(\frac{r(r+1)}{2} - \frac{k(k-1)}{2}\right)
\]

\[
= (r - k + 1)(1 + 2N - (r + k))
\]

\[
= (2N - k - r + 1)(r - k + 1).
\]

Hence we obtain the second claim of the claim (2). We also obtain the claim (1) since \(|R^+(\Pi)| = N^2\).

Let \( k, t, r \in J_{1,N} \) be such that \( k \leq t < r \). By (1.23), we have

\[
b_{(k,t)} b_{(t+1,r)} = b_{(k,r)}.
\]
By (1.31), we have

\[
\ell(b_{(k,t)}) + \ell(b_{(t+1,r)})
= (2N - k - t + 1)(t - k + 1) + (2N - t - r)(r - t)
= 2N(r - k + 1) - (k + t - 1)(t - k + 1) - (t + r)(r - t)
= 2N(r - k + 1) - (-k^2 + t^2 + 2k - 1) - (r^2 - t^2)
= 2N(r - k + 1) + (k^2 - r^2 - 2k + 1)
= 2N(r - k + 1) + (k - 1 + r)(k - 1 - r)
= (2N - r - k - 1)(r - k + 1)
= \ell(b_{(k,r)}).
\]

(1.33)

By (1.32), (1.32) and the claim (1), we get the claim (3). The claim (4) follows immediately from the claims (1) and (2).

Using Lemma 1.4, we have

**Lemma 1.5.** Let \( \Pi \) be the \( D_N \)-type standard root basis. Let \( w_o \) be the longest element of \((W(\Pi), S(\Pi))\). Let \( s_k := s_{e_k - e_{k+1}} \in S(\Pi) \) for \( k \in J_{1,N-1} \) and let \( s_N := s_{e_k + e_{k+1}} \in S(\Pi) \). For \( k \in J_{1,N-1} \), let

\[
d_{(k)} := (s_k \cdots s_{N-2}s_{N-1}s_N)^{N-k}.
\]

(1.34)

Then

\[
\ell(d_{(k)}) = (N - k)(N - k + 1)
\]

(1.35)

and

\[
d_{(k)} = \begin{cases} 
    b_{(k,N)} & \text{if } N - k \text{ is odd,} \\
    b_{(k,N-1)} & \text{if } N - k \text{ is even.}
\end{cases}
\]

(1.36)

In particular,

\[
w_o = d_{(1)}.
\]

(1.37)
Proof. By (1.6), (1.7) and (1.8), we have

\[
(1.38) \quad s_{N-1}s_N = \begin{pmatrix} N-1 & N \\ -(N-1) & -N \end{pmatrix} = s_{e_{N-1}}s_{e_N}.
\]

Then we have

RHS of (1.34)

\[
= (s_{(k,N-1)}s_{e_{N-1}}s_{e_N})^{N-k} \quad \text{(by (1.38))}
\]

\[
(1.39) \quad = (s_{(k,N-1)}s_{e_{N-1}})^{N-k}s_{e_N}^{N-k} \quad \text{(by (1.6) and (1.10))}
\]

\[
= b_{(k,N-1)}s_{e_N}^{N-k} \quad \text{(by (1.26))}
\]

\[
= \text{RHS of (1.36)}
\]

By (1.36), we have

\[
\mathcal{L}(d_{(k)}) = \{e_t + ce_{t'} | c \in \{-1, 1\}, t \in J_{k,r}, t' \in J_{t',N} \}.
\]

Hence by (1.2), we have (1.35) and (1.37). This completes the proof.

\[\square\]

2 Weyl groupoids of super CD-type

Let \(m \in J_{1,N-1}\). Let \(\mathcal{D}_{m|N-m}\) be the set of maps \(a : J_{1,n} \to J_{0,1}\) with \(|a^{-1}(\{0\})| = m\).

Let \(a \in \mathcal{D}_{m|N-m}\). Let \((, )^a : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}\) be the \(\mathbb{R}\)-bilinear map defined by \((e_i, e_j)^a := \delta_{ij} \cdot (-1)^a(i)\). For \(v \in \mathbb{R}^N\) with \((v, v)^a \neq 0\), define \(s_v \in GL_N(\mathbb{R})\) by \(s_v^a(u) := u - \frac{2(u, v)^a}{(v, v)^a}v\) \((u \in \mathbb{R}^N)\).

Let

\[
\dot{\mathcal{D}}_{m|N-m} := \{(a, d) \in \mathcal{D}_{m|N-m} \times J_{0,1} | d \in J_{0,a(N)} \}.
\]
For $i \in J_{1,N}$, define the bijection $\tau_i : \mathcal{D}_{m|N-m} \rightarrow \mathcal{D}_{m|N-m}$ by

$$
\tau_i(a, d) :=
\begin{cases}
(a \circ s_{e_i-e_{i+1}}, d) & \text{if } i \in J_{1,N-2} \text{ and } a(i) \neq a(i + 1), \\
(a \circ s_{e_{N-1}-e_N}, d) & \text{if } i = N - 1, d = 0 \text{ and } a(N - 1) \neq b(N), \\
(a \circ s_{e_{N-1}-e_N}, 1) & \text{if } i = N, a(N - 1) = 1, a(N) = 0, \\
(a \circ s_{e_{N-1}-e_N}, 0) & \text{if } i = N, a(N - 1) = 0, a(N) = 1 \text{ and } d = 1, \\
(a, d) & \text{otherwise}.
\end{cases}
$$

Then $\tau_i^2 = \text{id}_{\mathbb{R}^N}$.

Let $(a, d) \in \mathcal{D}_{m|N-m}$. Let

$$R_+^{(a,d)} := \{e_x + te_y | x, y \in J_{1,N}, x < y, t \in \{1, -1\}\}$$
$$\cup \{2e_z | z \in J_{1,N}, a(z) = 1\},$$

and $R^{(a,d)} := R_+^{(a,d)} \cup -R_+^{(a,d)}$. Then

$$|R_+^{(a,d)}| = N(N - 1) + (N - m) = N^2 - m.$$  \hspace{1cm} (2.1)

For $i \in J_{1,N}$, let

$$\alpha_i^{(a,d)} := \begin{cases}
e_i - e_{i+1} & \text{if } i \in J_{1,N-2}, \\
e_{N-1} - e_N & \text{if } i = N - 1 \text{ and } d = 0, \\
2e_N & \text{if } i = N - 1 \text{ and } d = 1, \\
e_{N-1} + e_N & \text{if } i = N, a(N) = 0 \text{ and } d = 0, \\
2e_N & \text{if } i = N, a(N) = 1 \text{ and } d = 0, \\
e_{N-1} - e_N & \text{if } i = N, d = 1.
\end{cases}$$

Let $\Pi^{(a,d)} := \{\alpha_i^{(a,d)} | i \in J_{1,N}\}$. Then $\Pi^{(a,d)}$ is an $\mathbb{R}$-basis of $\mathbb{R}^N$. Moreover

$$\Pi^{(a,d)} \subset R_+^{(a,d)} \subset (\bigoplus_{i=1}^{N} \mathbb{Z}_{\geq 0} \alpha_i^{(a,d)}) \setminus \{0\}.$$
Note that
\[ \tau_i(a, d) = (a, d) \text{ if and only if } (\alpha_i^{(a,d)}, \alpha_i^{(a,d)})^a \neq 0. \]

For \( i \in J_{1,N} \), define \( s_i^{(a,d)} \in \text{GL}_N(\mathbb{R}) \) by
\[
s_i^{(a,d)}(\alpha_i^{(a,d)}) :=
\begin{cases}
-\alpha_i^{\tau_i(a,d)} & \text{if } i = j, \\
\alpha_j^{\tau_j(a,d)} & \text{if } i \neq j \text{ and } (\alpha_i^{(a,d)}, \alpha_i^{(a,d)})^a \neq 0, \\
\alpha_j^{\tau_j(a,d)} + \alpha_i^{\tau_i(a,d)} & \text{if } i \neq j, (\alpha_i^{(a,d)}, \alpha_i^{(a,d)})^a = 0 \text{ and } (\alpha_i^{(a,d)}, \alpha_j^{(a,d)})^a \neq 0.
\end{cases}
\]

We can directly see

**Lemma 2.1.** Let \((a, d) \in D_{m|N-m} \) and \( i \in J_{1,N} \). Assume that \( d = 0 \).
Assume that \( i \in J_{1,N-1} \) if \( a(N-1) = 1 \) and \( a(N) = 0 \). Then \( s_i^{(a,d)} = s_{\alpha_i^{(a,d)}} \), where \( s_{\alpha_i^{(a,d)}} \) is the one of Definition 1.1.

**Notation.** Let \((a, d) \in D_{m|N-m} \). Let \( \text{Map}_0^N \) be a set with \(|\text{Map}_0^N| = 1 \).
For \( r \in \mathbb{N} \), let \( \text{Map}_r^N \) be the set of all maps from \( J_{1,r} \) to \( J_{1,N} \). Let \( \text{Map}_\infty^N \) be the set of all maps from \( \mathbb{N} \) to \( J_{1,N} \). For \( r \in \mathbb{Z}_{\geq 0} \), \( f \in \text{Map}_r^N \cup \text{Map}_\infty^N \) and \( t \in J_{1,r} \), let
\[
(a, d)_{f,0} := (a, d), \quad 1^{(a,d)}s_{f,0} := \text{id}_{\mathbb{R}^N} \\
(a, d)_{f,t} := \tau_i((a, d)_{f,t-1}), \quad 1^{(a,d)}s_{f,t} := 1^{(a,d)}s_{f,t-1}s_{f(t)}^{(a,d)_{f,t}}.
\]

**Proposition 2.2.** Let \((a, d) \in D_{m|N-m} \) be such that \( d = 0 \), \( b(z) = 1 \) (\( z \in J_{1,N-m} \)) and \( b(z') = 0 \) (\( z' \in J_{N-m+1,N} \)). Let \( n := |P_+^{(a,d)}| \). Define \( f \in \text{Map}_n^N \) by
\[
f(t) :=
\begin{cases}
N - m + t & (\text{if } t \in J_{1,m}), \\
f(t - m) & (\text{if } t \in J_{m+1,m(m-1)}), \\
t - m(m - 1) & (\text{if } t \in J_{m(m-1)+1,m(m-1)+N}), \\
2N + m(m-1) - t & (\text{if } t \in J_{m(m-1)+N+1,m^2+N}), \\
f(t - (N + m)) & (\text{if } t \in J_{m^2+N+1,n}).
\end{cases}
\]
Then

\[(2.3) \quad 1^{(a,d)}_{s_{f,n}} = \begin{cases} b_{(1,N)} & \text{if } m \text{ is odd}, \\ b_{(1,N-1)} & \text{if } m \text{ is even}. \end{cases}\]

**Proof.** For \(y \in J_{1,m}\), define \(a^{(y)} \in D_{m\mid N-m}\) by

\[a^{(y)}(z) := \begin{cases} 1 & \text{if } z \in J_{1,N-m-1} \cup \{N - m + y\}, \\ 0 & \text{if } z \in J_{N-m,N-m+y-1} \cup J_{N-m+y+1,N}. \end{cases}\]

Then we can directly see that for \(t \in J_{1,n}\),

\[ (a, d)_{f,t} = \begin{cases} (a, d) & \text{if } t \in J_{1,m(m-1)+N-m-1}, \\ (a^{(t-(N-m-1))}, 0) & \text{if } t \in J_{m(m-1)+N-m(m-1)+N-1}, \\ (a^{(m-(t-(m(m-1)+N))}), 0) & \text{if } t \in J_{m(m-1)+N,m(m-1)+N+m}, \\ (a, d)_{f,t-(N+m)} & \text{if } t \in J_{m^2+N+1,n}. \end{cases}\]

So we see that for \(t \in J_{1,n}\),

\[(2.4) \quad s_{f(t)}^{(a,d)_{f,t}} = \begin{cases} s_{e_{f(t)}-e_{f(t)+1}} & \text{if } f(t) \in J_{1,N-1}, \\ s_{e_{N-1}+e_{N}} & \text{if } t \in J_{1,m(m-1)} \text{ and } f(t) = N, \\ s_{2e_{N}}(=s_{e_{N}}) & \text{if } t \in J_{m(m-1)+1,n} \text{ and } f(t) = N. \end{cases}\]

Define \(f' \in \text{Map}_{n-m(m-1)}^{N}\) by \(f'(t) := f(t + m(m - 1))\), so

\[(2.5) \quad 1^{(a,d)}_{s_{f,m(m-1)}} = 1^{(a,d)}_{s_{f',n-m(m-1)}}.\]

By (1.29) and (1.36), \(1^{(a,d)}_{s_{f,m(m-1)}} = b_{(N-m+1,N)}\) (resp. \(b_{(N-m+1,N-1)}\)) if \(m\) is odd (resp. even). By (1.29) and (2.4), \(1^{(a,d)}_{s_{f',n-m(m-1)}} = b_{(1,N-m)}\).

Hence by (1.22) and (2.5), we have (2.3), as desired. \(\square\)

For \((a, d) \in \dot{D}_{m\mid N-m}\) and \(i, j \in J_{1,N}\), define \(C^{(a,d)} = [c^{(a,d)}_{ij}]_{i,j \in J_{1,N}} \in M_{N}(\mathbb{Z})\) by

\[s_{i}^{(a,d)}(\alpha_{j}^{(a,d)}) = \alpha_{j}^{(a,d)} - c^{(a,d)}_{ij} \alpha_{i}^{(a,d)}.\]
Then $C^{(a,d)}$ is a generalized Cartan matrix, i.e., (M1) and (M2) below hold.

(M1) $c_{ii}^{(a,d)} = 2$ $(i \in J_{1,N})$.
(M2) $c_{jk}^{(a,d)} \leq 0$, $\delta_{c_{jk}^{(a,d)},0} = \delta_{c_{kj}^{(a,d)},0}$ $(j, k \in J_{1,N}; j \neq k)$.

Then the data

$$\hat{C}_{m|N-m} := C(J_{1,N}, \hat{D}_{m|N-m}, (\tau_i)_{i \in J_{1,N}}; (C^{(a,d)})^{(a,d) \in \hat{D}_{m|N-m}})$$

is a (rank-$N$) Cartan scheme, i.e., (C1) and (C2) below hold.

(C1) $\tau_i^2 = \text{id}_{\hat{D}_{m|N-m}}$ $(i \in J_{1,N})$.
(C2) $c_{ij}^{\tau_i((a,d))} = c_{ij}^{(a,d)}$ $(i \in J_{1,N})$.

Note that

$$-c_{ij}^{(a,d)} = |R_+^{(a,d)} \cap (\mathbb{Z}\alpha_i^{(a,d)} \oplus \mathbb{Z}\alpha_j^{(a,d)})| (i, j \in J_{1,N}, i \neq j).$$

The data

$$\hat{R}_{m|N-m} := R(\hat{C}_{m|N-m}, (R_+^{(a,d)})^{(a,d) \in \hat{D}_{m|N-m}}).$$

is a generalized root system of type $C$, i.e., (R1)-(R4) below hold.

(R1) $R^{(a,d)} = R_+^{(a,d)} \cup -R_+^{(a,d)}$ $((a, d) \in \hat{D}_{m|N-m})$.
(R2) $R^{(a,d)} \cap \mathbb{Z}\alpha_i = \{ \alpha_i, -\alpha_i \}$ $((a, d) \in \hat{D}_{m|N-m}, i \in J_{1,N})$.
(R3) $s_i^{(a,d)}(R^{(a,d)}) = R_+^{\tau_i(a,d)}$ $((a, d) \in \hat{D}_{m|N-m}, i \in J_{1,N})$.
(R4) $(\tau_i \tau_j)^{-c_{ij}^{(a,d)}}(a, d) = (a, d)$ $((a, d) \in \hat{D}_{m|N-m}, i, j \in J_{1,N})$.

For $(a, d) \in \hat{D}_{m|N-m}$, let

$$W^{(a,d)} := \{ 1^{(a,d)} s_{f,r} \in \text{GL}_N(\mathbb{R}) \mid r \in \mathbb{Z}_{\geq 0}, f \in \text{Map}_r^N \},$$

and define the map $\ell^{(a,d)} : W^{(a,d)} \to \mathbb{Z}_{\geq 0}$ by

$$\ell^{(a,d)}(w) := \min \{ r \in \mathbb{Z}_{\geq 0} \mid \exists f \in \text{Map}_r^N, w = 1^{(a,d)} s_{f,r} \}.$$

By [HY08, Lemma 8 (iii)], we see that

(2.6) $1^{(a,d)} s_{f,r} = 1^{(a,d)} s_{f',r'}$ implies $(a, d)_{f,r} = (a, d)_{f',r'}$.  

and that
\begin{equation}
\ell^{(a,d)}(w) = |w^{-1}(R_{+}^{(a,d)}) \cap -\oplus_{i=1}^{N}\mathbb{Z}_{\geq 0}\alpha_{i}|.
\end{equation}

For \((a, d) \in \mathcal{D}_{m|N-m}, w \in W^{(a,d)}\) and \(f \in \text{Map}_{\ell^{(a,d)}}(w)\), if \(w = 1^{(a,d)}s_{f,\ell^{(a,d)}(w)}\), we call \(f\) a reduced word map of \(w\).

By (2.6) and (2.7), we have formulas for \(W^{(a,d)}\) similar to (1.3) and (1.4). In particular, for each \((a, d) \in \mathcal{D}_{m|N-m}\), there exists a unique \(w_{0}^{(a,d)} \in W^{(a,d)}\) such that
\[ \ell^{(a,d)}(w_{0}^{(a,d)}) = |R_{+}^{(a,d)}|, \]
and we call \(w_{0}^{(a,d)}\) the longest element of \(W^{(a,d)}\).

By Proposition 2.2, we have

**Theorem 2.3.** Let \((a, d) \in \mathcal{D}_{m|N-m}\) be such that \(d = 0\), \(a(z) = 1\) \((z \in J_{1,N-m})\) and \(a(z') = 0\) \((z' \in J_{N-m+1,N})\). Then a reduced word map of \(w_{0}^{(a,d)}\) is given by (2.2). Moreover,
\begin{equation}
(\mathcal{W}_{m|N-m})' := \bigcup_{(a,d),(a',d') \in \mathcal{D}_{m|N-m}} \mathcal{H}_{(a,d)}^{(a,d)},
\end{equation}
and \(\mathcal{W}_{m|N-m} := (\mathcal{W}_{m|N-m})' \cup \{o\}\), where \(o\) is an element such that \(o \notin (\mathcal{W}_{m|N-m})'\). We regard \(\mathcal{W}_{m|N-m}\) as the semigroup by \(o\omega := \omega o := o\) \((\omega \in \mathcal{W}_{m|N-m})\) and
\[
((a_{1}, d_{1}), w_{1}, (a_{2}, d_{2}))(a_{3}, d_{3}), w_{2}, (a_{4}, d_{4})) := \begin{cases} ((a_{1}, d_{1}), w_{1}w_{2}, (a_{4}, d_{4})) & \text{if } (a_{2}, d_{2}) = (a_{3}, d_{3}), \\ o & \text{if } (a_{2}, d_{2}) \neq (a_{3}, d_{3}). \end{cases}
\]
We call \(\mathcal{W}_{m|N-m}\) the Weyl groupoid of the Lie superalgebra \(\text{osp}(2m|2(N-m))\).
For \((a, d) \in \dot{D}_{m|N-m}\), let 
\[
\varepsilon^{(a,d)} := ((a, d), id_{\mathbb{R}^{N}}, (a, d)) \in \mathcal{H}_{(a,d)}^{(a,d)}.
\]

For \((a, d) \in \dot{D}_{m|N-m}\) and \(i \in J_{1,N}\), let 
\[
\sigma_{i}^{(a,d)} := (\tau_{i}(a, d), s_{i}^{(a,d)}, (a, d)) \in \mathcal{H}_{\tau_{i}(a,d)}^{(a,d)}.
\]

For \(r \in \mathbb{Z}_{\geq 0}\), \(t \in J_{0,r}\) and \(f \in \text{Map}_{r}^{N}\), let 
\[
1^{(a,d)}\sigma_{f,r} := ((a, d), 1^{(a,d)}s_{f,r}, (a, d)) \in \mathcal{H}_{(a,d)_{f,r}}(t \in \mathbb{N})
\]

For \(i, j \in J_{1,N}\), define \(f_{ij} \in \text{Map}_{\infty}^{N}\) by 
\[
f_{ij}(2t-1) := i, \quad f_{ij}(2t) := j (t \in \mathbb{N}).
\]

By [HY08, Theorem 1], we have

**Theorem 2.5.** The semigroup \(\dot{W}_{m|N-m}\) can also be defined by the generators

\[
o, \varepsilon^{(a,d)}, \sigma_{i}^{(a,d)} ((a, d) \in \dot{D}_{m|N-m}, i \in J_{1,N}),
\]

and relations

\[
\omega \omega = \omega o = o \quad (\omega \in \dot{W}_{m|N-m}),
\]

\[
\varepsilon^{(a,d)}\varepsilon^{(a,d)} = \varepsilon^{(a,d)}, \quad \varepsilon^{(a,d)}\varepsilon^{(a',d')} = o \quad ((a, d) \neq (a', d')),
\]

\[
\varepsilon^{\tau_{i}(a,d)}\sigma_{i}^{(a,d)} = \sigma_{i}^{(a,d)}\varepsilon^{(a,d)} = \sigma_{i}^{(a,d)}, \quad \sigma_{i}^{\tau_{i}(a,d)}\sigma_{i}^{(a,d)} = \varepsilon^{(a,d)},
\]

\[
1^{(a,d)}\sigma_{f_{ij},-2c_{ij}^{(a,d)}} = \varepsilon^{(a,d)} (i \neq j).
\]

Let \((a, d) \in \dot{D}_{m|N-m}, r \in \mathbb{Z}_{\geq 0}\) and \(f, f' \in \text{Map}_{r}^{N}\). We write \(f \sim_{r}^{(a,d)} f'\) if there exist \(i, j \in J_{1,N}\) such that \(i \neq j, \quad t - c_{ij}^{(a,d)_{f,k}} \leq r, \quad f(k_{1}) = f'(k_{1}) (k_{1} \in J_{1,t} \cup J_{t-c_{ij}^{(a,d)_{f,k}}+1,r}), \quad f(k_{2}) = i, \quad f'(k_{2}) = j (k_{2} \in J_{t+1,t-c_{ij}^{(a,d)_{f,k}} \cap 2N}).\)

We write \(f \sim_{r}^{(a,d)} f'\) if \(f = f'\) or there exists \(t \in \mathbb{N}\) and \(f_{k} \in \text{Map}_{r}^{N}\) \((k \in J_{1,t})\) such that \(f \sim_{r}^{(a,d)_{f_{k}}} f_{k} \sim_{r}^{(a,d)_{f_{k+1}}} f_{k+1} (k \in J_{t-1,t})\) and \(f_{t} \sim_{r}^{(a,d)_{f_{k}} f'}\).

By [HY08, Theorem 5, Corollary 6], we have

**Theorem 2.6.** Let \((a, d) \in \dot{D}_{m|N-m}\) and \(w \in W^{(a,d)}\).

1. Let \(f, f' \in \text{Map}_{\ell^{(a,d)}(w)}^{N}\) be such that 
\[
1^{(a,d)}s_{f,\ell^{(a,d)}(w)} = 1^{(a,d)}s_{f',\ell^{(a,d)}(w)} = w.
\]

Then \(f \sim_{\ell^{(a,d)}(w)}^{(a,d)} f'\).

2. Let \(r \in \mathbb{N}\) and \(f \in \text{Map}_{r}^{N}\) be such that \(r > \ell^{(a,d)}(w)\) and 
\[
1^{(a,d)}s_{f,r} = w.
\]

Then there exist \(f' \in \text{Map}_{r}^{N}\) and \(t \in J_{1,r-1}\) such that \(f \sim_{r}^{(a,d)} f'\) and \(f'(t) = f'(t + 1)\).
References

