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The Hodge realization of mixed Tate motives

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1 Introduction

This is a progress report on a recent joint work of the author with Masaki Hanamura and Tomohide Tersoma on a reformulation of the Hodge realization of the mixed Tate motives. We have been trying to understand the Hodge realization of the mixed Tate motives which is constructed by Bloch and Kriz. So far it has become clear that the existence of a certain complex of topological chains, which will be denoted by TC, is sufficient to define the Hodge realization. We conjecture that such a complex can be constructed from semi-algebraic sets, but we are still working on the proofs of the neccesary properties of TC. In section 2 the definition of mixed Tate motives due to Bloch and Kriz is reviewed. In section 3 we explain how to define the Hodge realization from the complex TC. In section 4 the Hodge realization of Polylog motives constructed by Bloch is computed. The proofs are mostly omitted.

2 Mixed Tate motives of Bloch and Kriz

A general reference for this section is [1]. Bloch and Kriz construct a certain Hopf algebra by the bar construction. The bar construction is a procedure to construct a commutative Hopf algebra from a DGA. By a DGA we mean a graded commutative ($a \cdot b = (-1)^{\deg(a)\deg(b)} b \cdot a$), associative differential graded algebra A over $\mathbb{Q}$ with unit. A should be given an augmentation

$$\epsilon : \ A \to \mathbb{Q}$$

which is a map of differential graded algebras. First we briefly recall the bar construction. The differential of A is denoted by ∂. It is of degree 1.

Let M and N be differential graded left A-modules. We can view M and N as right A-modules by defining

$$n \cdot a = (-1)^{\deg(a)\deg(n)} a \cdot n$$

Write

$$T(N, A, M) = N \otimes T(A) \otimes M = \underset{r \geq 0}{\oplus} N \otimes T^r(A) \otimes M$$

where $\otimes$ denotes $\otimes_{\mathbb{Q}}$ and $T(A) = \mathbb{Q} \oplus A \oplus A \otimes A \oplus \cdots$ is the tensor algebra. $T(N, A, M)$ is generated by elements of the form

$$n \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_r \otimes m = n[a_1|a_2|\cdots|a_r]m$$

We give $T(N, A, M)$ a complex structure. The differential d is given as a sum of two differetials $d_\otimes$ and δ. The inner differential $d_\otimes$ is the differential of the total complex of the tensor product $T(N, A, M)$. The outer differential δ is defined as follows. On $N \otimes T^r(A) \otimes M$ let

$$\begin{aligned}
\delta_0(n[a_1|\cdots|a_r]m) &= n \cdot a_1[a_2|\cdots|a_r]m \\
\delta_i(n[a_1|\cdots|a_r]m) &= n[a_1|\cdots|a_i \cdot a_{i+1}|\cdots|a_r]m \ (1 \leq i \leq r-1) \\
\delta_r(n[a_1|\cdots|a_r]m) &= a[a_1|\cdots|a_{r-1}]a_r \cdot m
\end{aligned}$$

The differential $\delta = \sum(-1)^i \delta_i$. The total degree of the element

$$n[a_1|\cdots|a_r]m = \deg(n) + \deg(m) + \sum_i \deg(a_i) - r$$

and the total differential is defined by

$$d(n[a_1|\cdots|a_r]m) = d_\otimes(n[a_1|\cdots|a_r]m) + (-1)^{\deg(n)+\deg(m)+\sum_i \deg(a_i)} \delta(n[a_1|\cdots|a_r]m)$$

In the case where the modules N and M equal to $\mathbb{Q}$ and the module structure is given by the augmentation, the complex

$$B(A) = B(\mathbb{Q}, A, \mathbb{Q})$$

is a graded Hopf algebra. The product is given by the shuffle product

$$[a_1|\cdots|a_r] \otimes [a_{r+1}|\cdots|a_{r+s}] \mapsto \sum_\mu (-1)^{\sigma(\mu)} [a_{\mu(1)}|\cdots|a_{\mu(r+s)}]$$

where the sum is over the set of (r, s) shuffles in the symmetric group on $r+s$ letters, and $(-1)^{\sigma(\mu)}$ is the sign of the graded permutation. For example when $\mu = (1,2)$ acting on $[a_1|a_2]$ then the sign $(-1)^{\sigma(\mu)} = (-1)^{1+\deg(a_1)\deg(a_2)}$.
The coproduct $\psi: \ B(A) \to B(A) \otimes B(A)$ is given by

$$\psi[a_1|a_2|\cdots|a_r] = \sum_{p=0}^{r} [a_1|\cdots|a_p] \otimes [a_{p+1}|\cdots|a_r]$$

The shuffle product is a map of complexes and the copoduct is a map of complexes and also is a map of algebras under the shuffle product. So that the complex $B(A)$ is a graded commutative differential Hopf algebra, and $H^0(B(A))$ is a commutative Hopf algebra.
In the construction of Bloch and Kriz the DGA $\mathcal{N}$ is defined as follows. Let k be the base field and let $\square^n = (\mathbb{P}^1_k - \{1\})^n$. Then the wreath product

$$G_n = S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$$

acts on $\square^n$. Let $\mathrm{Cycle}^r\langle n\rangle$ be the $\mathbb{Q}$-vector space freely generated by codimension r subvarieties of $\square^n$ meeting all faces (defined by setting several coordinates of $\square^n$ to 0 or ∞) properly. For $i \geq 0$ and $r \geq 0$ let $\mathcal{N}(r)^i$ be $\mathrm{AltCycle}^r\langle 2r - i\rangle$ where Alt means the alternating part under the action of the group G_{2r-i}. The DGA $\mathcal{N}$ is defined by

$$\mathcal{N} = \oplus_{r\geq 0}\mathcal{N}(r)^{\cdot}$$

The product on $\mathcal{N}$ is given by the exterior product and the differential

$$\partial = \sum_{p=1}^{n} (-1)^{p-1}(\partial^p_\infty - \partial^p_0)$$

where the map

$$\partial^p_* : \ \mathcal{N}(r)^i \to \mathcal{N}(r)^{i+1}$$

is the pullback by the map

$$i_{p,*} : \ \square^{n-1} \hookrightarrow \square^n$$

which is the inclusion given by setting the p-th coordinate to be $*$. By Lemma 4.3 in [1] the complex $\mathcal{N}$ is a DGA in our sense. $\mathcal{N}$ has an Adams grading given by r and so does the bar complex $B(\mathcal{N})$. As a consequense the commutative Hopf algebra

$$\chi_{\mathrm{mot}} = H^0(B(\mathcal{N})) = \oplus_{r\geq 0} H^0(B(\mathcal{N}))(r)$$

is also Adams graded. The category of mixed Tate motives over k is defined to be the category of finite dimensional graded comodules over χ_{mot}.

3 Hodge realization

In the following the base field is $\mathbb{C}$. First we recall the definition of a mixed Hodge structure. A Hodge structure of weight n is a finite dimensional $\mathbb{Q}$-vector space H (the betti lattice) with a finite decreasing filtration $F^\bullet$ on $H_\mathbb{C} = H \otimes \mathbb{C}$ such that

$$H_\mathbb{C} = \bigoplus_{p+q=n} H^{p,q}$$

where $H^{p,q}$ is defined to be $F^p H_\mathbb{C} \cap \overline{F^q H_\mathbb{C}}$. The filtration $F^\bullet$ is called the Hodge filtration. For $r \in \mathbb{Z}$ let $\mathbb{Q}(r)$ be the Hodge structure with the betti lattice $H = \mathbb{Q}(2\pi i)^r$ with the Hodge filtration

$$F^j H_\mathbb{C} = \begin{cases} \{0\} & j < -r \\ \mathbb{C} & j \geq -r \end{cases}$$

The Hodge structure $\mathbb{Q}(r)$ is of the weight $-2r$. For $m \in \mathbb{Z}_{\geq 0}$ the direct sum $\mathbb{Q}(r)^m$ is called a Tate Hodge structure. A mixed Hodge structure is a finite dimensional $\mathbb{Q}$-vector space H with an increasing filtration $W_\bullet H$ (the weight filtration) and a decreasing filtration $F^\bullet H_\mathbb{C}$ (the Hodge filtration) such that for each r the image of $F^\bullet H_\mathbb{C}$ to the graded quotients of the weight filtration $\mathrm{gr}_r^W H$ gives a Hodge structure of weight r. Here the image of $F^j H_\mathbb{C}$ to $\mathrm{gr}_r^W H$ is defined to be the image of

$$F^j H_\mathbb{C} \cap W_r H_\mathbb{C} \to \mathrm{gr}_r^W H_\mathbb{C} = (\mathrm{gr}_r^W H) \otimes \mathbb{C}.$$

A mixed Tate Hodge structure is a mixed Hodge structure

$$(H, W_\bullet, F^\bullet)$$

such that the weight graded quotients $\mathrm{gr}_r^W H$ are Tate Hodge structures of weight r. To define the Hodge realization we need to associate to each graded comodule over $\chi_{mot} = H^0(B(\mathcal{N}))$ a mixed Tate Hodge structure in a natural way.

Proposition 3.1. *Suppose we have a mixed Tate Hodge structure J such that*

1. *J has a comodule structure*

$$\Delta : \; J \to J \otimes H^0(B(\mathcal{N}))$$

which is a morphism of mixed Tate Hodge structures. Here $H^0(B(\mathcal{N})) = \oplus_{r\geq 0} H^0(B(\mathcal{N}))(r)$ is regarded as a direct sum of the pure Tate Hodge structure $H^0(B(\mathcal{N}))(r)$ of weight $2r$.

2. *There is an isomorphism*

$$gr^W_{2r} J \to H^0(B(\mathcal{N}))(r)$$

for $r \geq 0$ *and the coproduct on* $gr^W_{2r} J$ *induced by* Δ *is compatible with the coproduct* ψ *on* $H^0(B(\mathcal{N}))(r)$ *under this isomorphism.*

Then for a graded left $H^0(B(\mathcal{N}))$*-comodule* M *the cotensor product*

$$J \square M = \mathrm{Ker}(J \otimes M \xrightarrow{\Delta\otimes 1 - 1\otimes\Delta} J \otimes H^0(B(\mathcal{N})) \otimes M)$$

is a mixed Tate Hodge structure such that there are isomorphisms

$$gr^W_{2r}(J \otimes M) \simeq M(r)$$

for $r \in \mathbb{Z}$.

So it suffices to give a mixed Tate Hodge structure J with the properties above. The main claim of this note is the following.

Theorem 3.1. *Assume that there exist certain complexes* $C(n)_\bullet$ *in* $\square^n$ *for* $n \geq 1$ *of topological chains with the following properties.*

1. *For each* $i \geq 0$ $C(n)_i$ *is a* $\mathbb{Q}$*-vector space freely generated by certain topological chains of dimension* i *in* $\square^n$. *The boundary maps* $\delta: C(n)_i \to C(n)_{i-1}$ *induces a complex structure on* $C(n)_\bullet$.
2. *For each* $n \geq 1$ *the complex* $C(n)_\bullet$ *is acyclic.*
3. *Intersection with a face* $\{z_j = 0\}$ *resp.* $\{z_j = \infty\}$ *gives a map*
$$\partial_0^j \; (\mathrm{resp.}\, \partial_\infty^j): \; C(n)_i \to C(n-1)_{i-2}$$
which induces a map of complexes
$$C(n)_\bullet \to C(n-1)_{\bullet-2}$$
4. *For each* $r \geq 0$ *and* $i \geq 0$ *there is a natural inclusion*
$$\mathcal{N}(r)^i \hookrightarrow C(2r-i)_{2r-2i}$$
and this is compatible with the intersection with the faces ∂_0^j *and* ∂_∞^j.

5. *Let*

$$\omega_n = \frac{1}{(2\pi i)^n}\frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2} \wedge \cdots \wedge \frac{dz_n}{z_n}$$

be a n-form on $\square^n$*. For each element* $\gamma \in C(n)_n$ *the integral*

$$\int_\gamma \omega_n$$

is well defined. For each element $\gamma \in C(n)_{n+1}$ *there is a Cauchy formula*

$$\int_{\delta(\gamma)} \omega_n + \int_{\partial\gamma} \omega_{n-1} = 0$$

Here the map $\partial = \sum_{i=1}^{n}(-1)^{i-1}(\partial_0^i - \partial_\infty^i)$ *is the cubical differential.*

Then a mixed Tate Hodge structure J *as in Proposition 3.1 can be constructed.*

Remark 1. We conjecture that a complex $C(n)_\bullet$ as above can be constructed from semi-algebraic sets. A subset of $\mathbb{R}^n$ is said to be semi-algebraic if it belongs to the Boolean class of subsets of $\mathbb{R}^n$ which is generated by those of the form

$$\{x \in \mathbb{R}^n | f(x) \geq 0\}$$

where f is any polynomial function on $\mathbb{R}^n$. A Boolean class of subsets is characterized by the following properties.

1. Closedness under taking finite intersection.
2. Closedness under taking finite union.
3. Closedness under taking complementary set.

We explain how to construct a mixed Tate Hodge structure J from the complexes $C(n)_\bullet$. We need to modify the numbering of the complexes $C(n)_\bullet$ to obtain a cohomological complex. Let $C(n)^j := C(n)_{2n-j}$ and let the total complex

$$TC = \oplus_{n\geq 0} C(n)^\bullet$$

with the differential $d = \delta + \partial$. The total degree of elements in $C(n)^j = j - n$. The inclusion

$$\mathcal{N}(r)^i \hookrightarrow C(2r-i)^r$$

induces a map of complexes $\mathcal{N} \to TC$. Also there is a natural right $\mathcal{N}$ module structure on TC by exterior product.

Proposition 3.2. *The map*

$$I: \quad TC \to \mathbb{C}, \quad \gamma \mapsto \sum_{n=0}^{\infty} \int_{\gamma} \omega_n$$

is a map of complexes.

Proof. This follows from the properties of the complexes $C(N)^{\bullet}$. $\square$

Corollary 3.1. *The map*

$$I_{\mathbb{C}} : TC \otimes \mathbb{C} \to \mathbb{C}, \quad \gamma \otimes \alpha \mapsto I(\gamma)\alpha$$

is a quasiisomorphism.

Lemma 3.1. *Consider the bar complex $B(TC, \mathcal{N})$. Then the complex $B(TC, \mathcal{N}) \otimes \mathbb{C}$ is quasi isomorphic to $B(\mathcal{N}) \otimes \mathbb{C}$.*

Proof. By Corollary 3.1 the map $I_{\mathbb{C}} : TC \otimes \mathbb{C} \to \mathbb{C}$ is a quasiisomorphism and it is a map of right $\mathcal{N}$ modules: For $\gamma \in C(n)^j$ and $z \in \mathcal{N}^i(r)$ one has

$$I(\gamma \cdot z) = I(\gamma)\epsilon(z)$$

for reason of type. Here the map ϵ is the augmentation. Hence there exists a map of complexes

$$\begin{aligned} I \otimes 1 : \ & B(TC, \mathcal{N}) \to B(\mathbb{C}, \mathcal{N}) = B(\mathcal{N}) \otimes \mathbb{C} \\ & \gamma[z_1|\cdots|z_k] \mapsto I(\gamma)[z_1|\cdots|z_k] \end{aligned}$$

which is a quasiisomorphism. $\square$

We will define a mixed Tate Hodge structure J such that the weight graded quotient $\mathrm{gr}^W_{2r} J$ is canonically isomorphic to $H^0(B(\mathcal{N}))(r)$.
The betti lattice J is defined to be $H^0(B(TC, \mathcal{N}))$. For $r \geq 0$, let $W_{2r}B(TC, \mathcal{N}) = W_{2r-1}B(TC, \mathcal{N})$ be the subcomplex of $B(TC, \mathcal{N})$ generated by cochains of the form

$$\gamma \otimes z_1 \otimes \cdots \otimes z_k, \quad \sum_i \mathrm{codim} z_i \leq r$$

Then the subspace $W_{2r}H^0(B(TC, \mathcal{N}))$ is defined to be the image of $H^0(W_{2r}B(TC, \mathcal{N}))$ in $H^0(B(TC, \mathcal{N}))$.

Proposition 3.3. *Let $r \geq 0$. The weight graded quotient $\mathrm{gr}^W_{2r} J$ is canonically isomorphic to $H^0(B(\mathcal{N}))(r)$.*

Proof. Let $\mathrm{gr}^W_{2r} B(TC, \mathcal{N})$ be the quotient

$$\frac{W_{2r}B(TC,\mathcal{N})}{W_{2(r-1)}B(TC,\mathcal{N})}.$$

One sees that this is the tensor product

$$TC \otimes B(\mathcal{N})(r)$$

as a complex. So the cohomology

$$\begin{aligned} &H^0(\mathrm{gr}^W_{2r}B(TC,\mathcal{N})) = \oplus_{i+j=0}H^i(TC) \otimes H^j(B(\mathcal{N})(r)) \\ &= H^0(TC) \otimes H^0(B(\mathcal{N})(r)) = H^0(B(\mathcal{N})(r)). \end{aligned}$$

The short exact sequence

$$0 \to W_{2(r-1)}B(TC,\mathcal{N}) \to W_{2r}B(TC,\mathcal{N}) \to \mathrm{gr}^W_{2r}B(TC,\mathcal{N}) \to 0$$

induces the long exact sequence of cohomology

$$\cdots \to H^i(W_{2(r-1)}B(TC,\mathcal{N})) \to H^i(W_{2r}B(TC,\mathcal{N})) \to H^i(\mathrm{gr}^W_{2r}B(TC,\mathcal{N})) \to \cdots.$$

The same argument as in Lemma 0.2 shows that the complex $W_{2r}B(TC,\mathcal{N})\otimes \mathbb{C}$ is quasiisomorphic to

$$W_{2r}B(\mathcal{N}) \otimes \mathbb{C} = \oplus_{j\leq r}B(\mathcal{N})(j) \otimes \mathbb{C}$$

and this long exact sequence becomes direct sum of short exact sequences:

$$0 \to H^i(W_{2(r-1)}B(\mathbb{C},\mathcal{N})) \to H^i(W_{2r}B(\mathbb{C},\mathcal{N})) \to H^i(\mathrm{gr}^W_{2r}B(TC,\mathcal{N})\otimes\mathbb{C}) \to 0.$$

So we have a short exact sequence

$$0 \to H^0(W_{2(r-1)}B(TC,\mathcal{N})) \to H^0(W_{2r}B(TC,\mathcal{N})) \to H^0(\mathrm{gr}^W_{2r}B(TC,\mathcal{N})) \to 0.$$

This concludes the proof. $\square$

We define the Hodge filtration. By Lemma 0.2 $J_{\mathbb{C}} = J \otimes \mathbb{C}$ is isomorphic to $H^0(B(\mathcal{N})) \otimes \mathbb{C}$. For $k \geq 0$, the Hodge filtration $F^k J_{\mathbb{C}}$ is defined to be

$$\oplus_{j\geq k}H^0(B(\mathcal{N})(j)) \otimes \mathbb{C}.$$

4 Polylog motives

As a example we compute the Hodge realization of the polylog motives. This is constructed by Bloch ([1]). Note that the chains $\eta_k(i)$ which will appear in the following is defined in [1]. For $a \in \mathbb{C} - \{0,1\}$ consider the locus in $\mathbb{P}^1(\mathbb{C}) - \{1\}$ parametrized in nonhomogeneous coordinates by

$$(x_1, \cdots, x_k, 1 - x_1, 1 - x_2/x_1, \cdots, 1 - x_{k-1}/x_{k-2}, 1 - a/x_{k-1})$$

and let $\rho_k(a)$ be the alternating projection of this locus. $\rho_k(a)$ is an element in $\mathcal{N}(k)^1$ and there is an equality

$$\partial\rho_k(a) = \rho_{k-1}(a) \cdot \rho_1(1-a)$$

Let

$$\mathbf{Li}_k(a) \in B(\mathcal{N})$$

be the element

$$\begin{aligned}&[\rho_k(a)] + [\rho_{k-1}(a)|\rho_1(1-a)]\\ &+[\rho_{k-2}|\rho_1(1-a)|\rho_1(1-a)] + \cdots\\ &\cdots + [\rho_1(a)|\rho_1(1-a)|\cdots|\rho_1(1-a)]\end{aligned}$$

The element $\mathbf{Li}_k(a)$ is a cocycle of degree 0. For $0 \le i \le k-1$ let $\eta_k(i)$ be the $(k+i)$-chain in $(\mathbb{P}^1 - \{1\})^{k+i}$ defined to be the alternating projection of the locus

$$\begin{aligned}&(x_1, \cdots, x_i, t_{i+1}, \cdots, t_{k-1}, 1 - x_1, \cdots, 1 - x_i/x_{i-1}, 1 - t_i/x_i)\\ &t_{k-1} \in (0,a),\ t_{k-2} \in (0, t_{k-1}), \cdots, t_i \in (0, t_{i+1}),\\ &x_1, \cdots, x_i \in \mathbb{C}\end{aligned}$$

Then we have the following equalities.

$$\delta\eta_k(k-1) = \rho_k(a), \quad \delta\eta_k(i) = \eta_{k-1}(i)\cdot\rho_1(1-a) + (-1)^{k+i+1}\partial\eta_k(i+1)\ (0 \le i \le k-2).$$

Let $Z_k(a) \in B(TC, \mathcal{N})$ be the element

$$\begin{aligned}&1[\mathbf{Li}_k(a)] + \xi_k(a)[1] + \xi_{k-1}(a)[\rho_1(1-a)] +\\ &\cdots + \xi_1(a)[\rho_1(1-a)|\cdots|\rho_1(1-a)]\end{aligned}$$

where $\xi_k(a) \in TC$ is the element

$$\sum_{i=0}^{k-1} \eta_k(i)$$

Then $Z_k(a)$ is a cocycle of degree 0 and

$$\begin{aligned}&(I \otimes 1)Z_k(a)\\ &= \mathbf{Li}_k(a) + Li_k(a) + Li_{k-1}(a)\rho_1(1-a)\\ &+ \cdots + Li_1(a)[\rho_1(1-a)|\cdots|\rho_1(1-a)]\\ &\in H^0(B(\mathcal{N})) \otimes \mathbb{C}\end{aligned}$$

Here

$$\begin{aligned}&Li_k(a) =\\ &\int_{\eta_k(0)} \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_k}{z_k}\\ &= \int_{t_{k-1}\in(0,a)} \frac{dt_{k-1}}{t_{k-1}} \int_{t_{k-2}\in(0,t_{k-1})} \frac{dt_{k-2}}{t_{k-2}} \cdots\\ &\int_{t_1\in(0,t_2)} \frac{dt_1}{t_1} \int_{t_0\in(0,t_1)} \frac{d(1-t_0)}{1-t_0}\end{aligned}$$

is the polylogarithm function. The integral on other chains $\eta_k(i)$ vanish for reason of type.

References

[1] Bloch, S., Kriz, I., *Mixed Tate Motives*, Ann. Math. **140**, 557-605.