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# Nondifferentiable higher order symmetric duality in multiobjective programming involving cones

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## 1 Introduction and Preliminaries

Higher order duality in nonlinear programming has been studied by many researchers. By introducing two differentiable functions $h : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ and $k : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$, Mangasarian [11] formulated the higher order dual problems for nonlinear programming problem. Later, in [19], Mond and Weir gave the conditions for duality and considered other higher order duals. Mond and Zhang [20] obtained results for various higher order dual programming problems under higher order invexity assumptions. Also, under invexity type conditions, such as higher order type I, higher order pseudo type I and higher order quasi type I conditions, Mishra and Rueda [15] gave various duality results, which included Mangasarian [11] higher order duality and Mond Weir [18] higher order duality.

Recently, Mishra and Rueda [16] considered higher order duality for the nondifferentiable mathematical programming. They formulated a number of higher order duals to a nondifferentiable programming problem and established duality under the higher order generalized invexity conditions introduced in [15]. In [21], Yang et al. extended the results in [16] to a class of nondifferentiable multiobjective programming programs. A unified higher order dual model for nondifferentiable multiobjective programs was presented, where every component of the objective function contains a term involving the support function of a compact convex set. Later, Chen [4] studied higher order symmetric duality for scalar and multiobjective nondifferentiable programming problems by introducing higher order $F$-convexity. Mond Weir type duality has been discussed in both these papers. In [12], Mishra established a pair of nondifferentiable higher order symmetric dual model in mathematical programming. Then, Gulati and Gupta [5] formulated Wolfe type higher order nondifferentiable symmetric dual programs and discussed duality relations between them. Very recently, Agarwal et al. [1] established a strong duality theorem for Mond-Weir type multiobjective higher-order nondifferentiable symmetric dual programs.

In this paper, we focus on symmetric duality with cone constraints. Bazaraa and Goode [3] established symmetric duality results for convex function with arbitrary cones. Very recently, Gulati and Gupta [6] studied higher order symmetric duality over arbitrary cones for Wolfe and Mond Weir type models under higher order invexity/pseudoinvexity assumptions, respectively. Gupta and Jayswal [7] formulated Mond-Weir type higher-order multiobjective symmetric dual programs over arbitrary cones and appropriate duality theorems were established under higher-order cone-invexity/cone-pseudoinvexity assumptions. In [2], Agarwal et al. extended the results of Chen [4] over arbitrary cones and proved Mond-Weir type duality theorems under higher-order $K-F$-convexity assumptions. Mond-Weir type duality has been discussed in both the paper. Later, Gupta et al. [8] formulated a pair of higher-order Wolfe type and Mond-Weir type differentiable multiobjective symmetric dual programs over arbitrary cones.

In this paper, we introduce two pairs of nondifferentiable multiobjective higher order symmetric dual problems with cone constraints over arbitrary closed convex cones, where every component of the objective function contains a term involving the support function of a compact convex set. For these problems, Wolfe and Mond-Weir type duals are proposed. Under suitable higher order convexity conditions, we establish weak, strong and converse duality theorems for $K$-weakly efficient solutions. Moreover, we give some special cases of our symmetric duality results. Our symmetric duality results extended and improved the symmetric duality results in Gulati and Gupta [6] to the nondifferentiable multiobjective symmetric dual problems. First we consider the following multiobjective programming problem.

$$
\begin{array}{lll}
(\mathbf{P}) & \text{Minimize} & f(x) \\
 & \text{subject to} & -g(x) \in Q, \; x \in C,
\end{array}
$$

where $f : \mathbb{R}^n \to \mathbb{R}^l$, $g : \mathbb{R}^n \to \mathbb{R}^m$, $C \subset \mathbb{R}^n$ and $Q$ is a closed convex cone with nonempty interior in $\mathbb{R}^m$. We shall denote the feasible set of $(\mathbf{P})$ by

$$X = \{x | -g(x) \in Q, x \in C\}.$$

**Definition 1.1** *Let $S \subseteq \mathbb{R}^n$ be open and $f : S \to \mathbb{R}$ be a differentiable function. The function $f : S \to \mathbb{R}$ is said to be higher order convex at $u \in S$ with respect to $\eta : S \times S \to \mathbb{R}^n$ and $h : S \times \mathbb{R}^n \to \mathbb{R}$, if for all $(x, p) \in S \times \mathbb{R}^n$,*

$$f(x) - f(u) \geqq (x-u)^T[\nabla_x f(u) + \nabla_p h(u,p)] + h(u,p) - p^T \nabla_p h(u,p).$$

**Definition 1.2** *[17] Let $C$ be a compact convex set in $\mathbb{R}^n$. The support function $s(x|C)$ of $C$ is defined by*

$$s(x|C) := max\{x^T y : y \in C\}.$$

*It is readily verified that for a compact convex set $C$, $y$ is in $N_C(x)$ if and only if $s(y|C) = x^T y$, or equivalently, $x$ is in the subdifferential of $s$ at $y$.*

# 2 Higher Order Symmetric Duality

In this section, we propose the following a pair of higher order Mond-Weir type nondifferentiable multiobjective programming problem:

**(MHNP)** Minimize

$$\begin{aligned} & P_M(x, y, \lambda, z, p) \\ & = \Big(f_1(x, y) + s(x|C_1) - y^T z_1 + h_1(x, y, p_1) - p_1^T \nabla_{p_1} h_1(x, y, p_1), \cdots, \\ & \qquad f_l(x, y) + s(x|C_l) - y^T z_l + h_l(x, y, p_l) - p_l^T \nabla_{p_1} h_l(x, y, p_l)\Big) \end{aligned}$$

$$\text{subject to} - \Big(\sum_{i=1}^{l} \lambda_i [\nabla_y f_i(x, y) - z_i + \nabla_{p_i} h_i(x, y, p_i)]\Big) \in Q_2^*, \tag{1}$$

$$y^T \sum_{i=1}^{l} \lambda_i [\nabla_y f_i(x, y) - z_i + \nabla_{p_i} h_i(x, y, p_i)] \geqq 0, \tag{2}$$

$$x \in Q_1, \ z_i \in D_i, \ \lambda \in K^*, \ \lambda^T e = 1,$$

**(MHND)**

Maximize

$$\begin{aligned} & D_M(u, v, \lambda, w, r) \\ & = \Big(f_1(u, v) - s(v|D_1) + u^T w_1 + g_1(u, v, r_1) - r_1^T \nabla_{r_1} g_1(u, v, r_1), \cdots, \\ & \qquad f_l(u, v) - s(v|D_l) + u^T w_l + g_l(u, v, r_l) - r_l^T \nabla_{r_l} g_l(u, v, r_l)\Big) \end{aligned}$$

$$\text{subject to} \quad \sum_{i=1}^{l} \lambda_i [\nabla_x f_i(u, v) + w_i + \nabla_{r_i} g_i(u, v, r_i)] \in Q_1^*, \tag{3}$$

$$u^T \sum_{i=1}^{l} \lambda_i [\nabla_x f_i(u, v) + w_i + \nabla_{r_i} g_i(u, v, r_i)] \leqq 0, \tag{4}$$

$$v \in Q_2, \ w_i \in C_i, \ \lambda \in K^*, \ \lambda^T e = 1,$$

where
(i) $f_i : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}, h_i : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ and $g_i : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$ are differentiable functions,
(ii) $K$ is a closed convex cone in $\mathbb{R}^l$ with $intK \neq \emptyset$ and $\mathbb{R}^l_+ \subset K$,
(iii) $Q_1$ and $Q_2$ are closed convex cones in $\mathbb{R}^n$ and $\mathbb{R}^m$ with nonempty interiors, respectively,
(iv) $K^*, Q_1^*$ and $Q_2^*$ are positive polar cones of $K, Q_1$ and $Q_2$, respectively,
(v) $r_i, w_i (i = 1, \cdots, l)$ and $p_i, z_i (i = 1, \cdots, l)$ are vectors in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively,
(vi) $C_i$ and $D_i (i = 1, \cdots, l)$ are compact convex sets in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively,
(vii) $e = (1, \cdots, 1)^T$ is a vector in $\mathbb{R}^l$.

We establish the symmetric duality theorems between (**MHNP**) and (**MHND**).

**Theorem 2.1 (Weak Duality)** *Let $(x, y, \lambda, z, p)$ and $(u, v, \lambda, w, r)$ be feasible solutions of* (**MHNP**) *and* (**MHND**), *respectively. Assume that*
*(i)$f_i(\cdot, v) + (\cdot)^T w_i$ is higher order convex at $u$ with respect to $g_i(u, v, r_i), (i = 1, \cdots, l)$,*
*(ii)$-[f_i(x, \cdot) - (\cdot)^T z_i]$ is higher order convex at $y$ with respect to and $-h_i(x, y, p_i), (i = 1, \cdots, l)$,*
*Then*

$$D_M(u, v, \lambda, w, r) - P_M(x, y, \lambda, z, p) \notin intK.$$

**Lemma 2.1** *[10] If $\overline{x}$ is a $K$-weakly efficient solution of* (**P**), *then there exist $\alpha \in K^*$ and $\beta \in Q^*$ not both zero such that*

$$(\alpha^T \nabla f(\overline{x}) + \beta^T \nabla g(\overline{x}))(x - \overline{x}) \geqq 0, \quad \textit{for all} \quad x \in C,$$
$$\beta^T g(\overline{x}) = 0.$$

*Equivalently, there exist $\alpha \in K^*$, $\beta \in Q^*$, $\beta_1 \in C^*$ and $(\alpha, \beta, \beta_1) \neq 0$ such that*

$$\alpha^T \nabla f(\overline{x}) + \beta^T \nabla g(\overline{x}) - \beta_1^T I = 0,$$
$$\beta^T g(\overline{x}) = 0,$$
$$\beta_1^T \overline{x} = 0.$$

**Theorem 2.2 (Strong Duality)** *Let $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{z}, \overline{p})$ be a $K$-weakly efficient solution of* (**MHNP**). *Fix $\lambda = \overline{\lambda}$ in* (**MHND**). *Assume that*
*(i)$h_i(\overline{x}, \overline{y}, 0) = 0, g_i(\overline{x}, \overline{y}, 0) = 0, \nabla_{p_i} h_i(\overline{x}, \overline{y}, 0) = 0$,*
*$\nabla_y h_i(\overline{x}, \overline{y}, 0) = 0, \nabla_x h_i(\overline{x}, \overline{y}, 0) = \nabla_{r_i} g_i(\overline{x}, \overline{y}, 0), i = 1, 2, \cdots, l$,*
*(ii)for all $i \in \{1, 2, \cdots, l\}$, the Hessian matrix $\nabla_{p_i p_i} h_i(\overline{x}, \overline{y}, \overline{p}_i)$ is nonsingular,*
*(iii)the set of vectors $\{\nabla_y f_i(\overline{x}, \overline{y}) - \overline{z}_i + \nabla_{p_i} h_i(\overline{x}, \overline{y}, \overline{p}_i), i = 1, 2, \cdots, l\}$ are linearly*

*independent,*
*(iv)for some* $\alpha \in K^* \setminus \{0\}$ *and* $\overline{p}_i \in \mathbb{R}^m, \overline{p}_i \neq 0 (i = 1, 2, \cdots, l)$ *implies that*

$$\sum_{i=1}^{l} \alpha_i \overline{p}_i^T [\nabla_y f_i(\overline{x}, \overline{y}) - \overline{z}_i + \nabla_{p_i} h_i(\overline{x}, \overline{y}, \overline{p}_i)] \neq 0.$$

*Then there exists* $\overline{w}_i \in C_i$ *such that* $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{w}, \overline{r} = 0)$ *is feasible for* (**MHND**) *and the corresponding values of* (**MHNP**) *and* (**MHND**) *are equal. If the assumption of Theorem 2.1 are satisfied, then* $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{z}, \overline{p} = 0)$ *and* $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{w}, \overline{r} = 0)$ *are* $K$*-weakly efficient solutions of* (**MHNP**) *and* (**MHND**), *respectively.*

We now state a converse duality theorem whose proof follows on the lines of Theorem 2.2.

**Theorem 2.3 (Converse Duality)** *Let* $(\overline{u}, \overline{v}, \overline{\lambda}, \overline{w}, \overline{r})$ *be a* $K$*-weakly efficient solution of* (**MHND**). *Fix* $\lambda = \overline{\lambda}$ *in* (**MHNP**). *Assume that*
*(i)*$h_i(\overline{u}, \overline{v}, 0) = 0, g_i(\overline{u}, \overline{v}, 0) = 0, \nabla_{r_i} g_i(\overline{u}, \overline{v}, 0) = 0,$
$\nabla_x g_i(\overline{u}, \overline{v}, 0) = 0, \nabla_y g_i(\overline{u}, \overline{v}, 0) = \nabla_{p_i} h_i(\overline{u}, \overline{v}, 0), i = 1, 2, \cdots, l,$
*(ii)for all* $i \in \{1, 2, \cdots, l\}$, *the Hessian matrix* $\nabla_{r_i r_i} g_i(\overline{u}, \overline{v}, \overline{r}_i)$ *is nonsingular,*
*(iii)the set of vectors* $\{\nabla_x f_i(\overline{u}, \overline{v}) - \overline{w}_i + \nabla_{r_i} g_i(\overline{u}, \overline{v}, \overline{r}_i), i = 1, 2, \cdots, l\}$ *are linearly independent,*
*(iv)for some* $\alpha \in K^* \setminus \{0\}$ *and* $\overline{r}_i \in \mathbb{R}^n, \overline{r}_i \neq 0 (i = 1, 2, \cdots, l)$ *implies that*

$$\sum_{i=1}^{l} \alpha_i \overline{r}_i^T [\nabla_x f_i(\overline{u}, \overline{v}) + \overline{w}_i, + \nabla_{r_i} g_i(\overline{u}, \overline{v}, \overline{r}_i)] \neq 0.$$

*Then there exists* $\overline{z}_i \in D_i$ *such that* $(\overline{u}, \overline{v}, \overline{\lambda}, \overline{z}, \overline{p} = 0)$ *is feasible for* (**MHNP**) *and the corresponding values of* (**MHNP**) *and* (**MHND**) *are equal. If the assumption of Theorem 2.1 are satisfied, then* $(\overline{u}, \overline{v}, \overline{\lambda}, \overline{z}, \overline{p} = 0)$ *and* $(\overline{u}, \overline{v}, \overline{\lambda}, \overline{w}, \overline{r} = 0)$ *are* $K$*-weakly efficient solutions of* (**MHNP**) *and* (**MHND**), *respectively.*

Also, we propose the following a pair of higher order Wolfe type nondifferentiable multiobjective programming problem:

**(WHNP)**
Minimize

$$
\begin{aligned}
&P_W(x,y,\lambda,z,p)\\
&= \Big(f_1(x,y)+s(x|C_1)-y^Tz_1+h_1(x,y,p_1)-p_1^T\nabla_{p_1}h_1(x,y,p_1)\\
&\qquad -y^T\sum_{i=1}^{l}\lambda_i[\nabla_y f_i(x,y)-z_i+\nabla_{p_i}h_i(x,y,p_i)],\cdots,\\
&\quad f_l(x,y)+s(x|C_l)-y^Tz_l+h_l(x,y,p_l)-p_l^T\nabla_{p_1}h_l(x,y,p_l)\\
&\qquad -y^T\sum_{i=1}^{l}\lambda_i[\nabla_y f_i(x,y)-z_i+\nabla_{p_i}h_i(x,y,p_i)]\Big)\\
&\text{subject to} -\Big(\sum_{i=1}^{l}\lambda_i[\nabla_y f_i(x,y)-z_i+\nabla_{p_i}h_i(x,y,p_i)]\Big)\in Q_2^*, \qquad (5)\\
&\qquad x\in Q_1,\ z_i\in D_i,\ \lambda\in K^*,\ \lambda^Te=1,
\end{aligned}
$$

**(WHND)**
Maximize

$$
\begin{aligned}
&D_W(u,v,\lambda,w,r)\\
&= \Big(f_1(u,v)-s(v|D_1)+u^Tw_1+g_1(u,v,r_1)-r_1^T\nabla_{r_1}g_1(u,v,r_1)\\
&\qquad -u^T\sum_{i=1}^{l}\lambda_i[\nabla_x f_i(u,v)+w_i+\nabla_{r_i}g_i(u,v,r_i)],\cdots,\\
&\quad f_l(u,v)-s(v|D_l)+u^Tw_l+g_l(u,v,r_l)-r_l^T\nabla_{r_l}g_l(u,v,r_l)\\
&\qquad -u^T\sum_{i=1}^{l}\lambda_i[\nabla_x f_i(u,v)+w_i+\nabla_{r_i}g_i(u,v,r_i)]\Big)\\
&\text{subject to} \quad \sum_{i=1}^{l}\lambda_i[\nabla_x f_i(u,v)+w_i+\nabla_{r_i}g_i(u,v,r_i)]\in Q_1^*, \qquad (6)\\
&\qquad v\in Q_2,\ w_i\in C_i,\ \lambda\in K^*,\ \lambda^Te=1,
\end{aligned}
$$

Similarly, we can establish weak, strong and converse duality theorems between (**WHNP**) and (**WHND**).

# 3 Special Cases

We give some special cases of our duality.

(i) If $C_i = \{0\}$ and $D_i = \{0\}$, $i = 1, \cdots, l$, then **(MHNP)** and **(MHNP)** reduced to the symmetric dual programs in Gupta and Jayswal [7].
(ii) If $C_i = \{0\}$, $D_i = \{0\}$, $i = 1, \cdots, l$, and $l = 1$, then our programs reduced to the symmetric dual programs in Gulati and Gupta [6].
(iii) If $Q_1 = \mathbb{R}^n_+$ and $Q_2 = \mathbb{R}^m_+$, then **(MHNP)** and **(MHNP)** become dual programs considered in Chen [4] and Agarwal et al.[1].
(iv) If $Q_1 = \mathbb{R}^n_+, Q_2 = \mathbb{R}^m_+$ and $l = 1$, then **(MHNP)** and **(MHNP)** become dual programs in Mishra [12].
(v) If $Q_1 = \mathbb{R}^n_+, Q_2 = \mathbb{R}^m_+$ and $l = 1$, then **(WHNP)** and **(WHNP)** become dual programs in Gulati and Gupta [5].

If $h_i(x, y, p_i) = (\frac{1}{2})p_i^T \nabla_{yy} f_i(x, y)p_i$, $g_i(u, v, r_i) = (\frac{1}{2})r_i^T \nabla_{xx} f_i(u, v)r_i$,
(vi) $C_i = \{0\}$ and $D_i = \{0\}$, $i = 1, \cdots, l$, then **(MHNP)** and **(MHNP)** reduce to the problems considered in Mishra and Lai [14].
(vii) $Q_1 = \mathbb{R}^n_+$, $Q_2 = \mathbb{R}^m_+$ and $l = 1$, then **(MHNP)** and **(MHND)** become the problems considered by Hou and Yang [9].
(viii) $Q_1 = \mathbb{R}^n_+$, $Q_2 = \mathbb{R}^m_+$, $C_i = \{0\}$, $D_i = \{0\}$, $i = 1, \cdots, l$, and $l = 1$, then our programs reduce to the problems considered in Mishra [13].

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