数理解析研究所講究録
第 1841 巻 2013 年 50-57

Fixed point and nonlinear ergodic theorems for new nonlinear mappings in Hilbert spaces

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Abstract

In this paper we introduce a broad class of nonlinear mappings which contains the class of constractive mappings and the class of generalized hybrid mappings in a Hilbert space. Then we prove a fixed point theorem for such mappings in a Hilbert space. Furthermore, we prove a nonlinear ergodic theorem of Baillon's type in a Hilbert space. Their results generalize the fixed point theorem and the nonlinear ergodic theorem proved by Kocourek, Takahashi and Yao [10].

1 Introduction

Let H be a real Hilbert space. A mapping T from H into H is said to be contractive if there exists a real number α with $0 < \alpha < 1$ such that

$$\|Tx - Ty\| \leq \alpha\|x - y\|$$

for any $x, y \in H$. By Banach [2] it is known that any constraction mapping has a unique fixed point. Let C be a non-empty subset of H. A mapping T from C into H is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|$$

for any $x, y \in C$. By Baillon [1] we know the following nonlinear ergodic theorem in a Hilbert space.

Theorem 1.1. *Let H be a real Hilbert space, let C be a non-empty closed convex subset of H and let T be a nonexpansive mapping from C into C with a fixed point. Then for any $x \in C$,*

$$S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$$

is weakly convergent to a fixed point of T.

An important example of nonexpansive mappings in a Hilbert space is a firmly nonexpansive mapping. A mapping T from C into H is said to be firmly nonexpansive if

$$\|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty \rangle$$

for any $x, y \in C$; see Browder [4] and Goebel and Kirk [6]. It is known that a firmly nonexpansive mapping can be deduced from an equilibrium problem in a Hilbert space; see Blum and Oettli [3] and Combettes and Hirstoaga [5]. Recently Kohsaka and Takahashi [12], and Takahashi [16] introduced the following nonlinear mappings which are deduced from a firmly nonexpansive mapping in a Hilbert space. A mapping T from C into H is said to be nonspreading if

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2$$

for any $x, y \in C$. A mapping T from C into H is said to be hybrid if

$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2$$

for any $x, y \in C$. Motivated by these mappings, Kocourek, Takahashi and Yao [10] defined a class of nonlinear mappings in a Hilbert space. A mapping T from C into H is said to be generalized hybrid if there exist real numbers α and β such that

$$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2$$

for any $x, y \in C$. We call such a mapping an (α, β)-generalized hybrid mapping. We observe that the class of the mappings above covers the classes of well-known mappings. For example, an (α, β)-generalized hybrid mapping is nonexpansive for $\alpha = 1$ and $\beta = 0$, nonspreading for $\alpha = 2$ and $\beta = 1$, and hybrid for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$. They proved fixed point theorems for such mappings; see also Kohsaka and Takahashi [11] and Iemoto and Takahashi [7]. Moreover Kocourek, Takahashi and Yao [10] proved the following nonlinear ergodic theorem.

Theorem 1.2. *Let H be a real Hilbert space, let C be a non-empty closed convex subset of H, let T be a generalized hybrid mapping from C into C which has a fixed point, and let P be the metric projection of H onto the set of fixed points of T. Then for any $x \in C$,*

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

is weakly convergent to a fixed point p of T, where $p = \lim_{n\to\infty} PT^n x$.

In this paper we introduce a broad class of nonlinear mappings T from C into H which contains the class of constractive mappings and the class of generalized hybrid mappings. Then we prove a fixed point theorem for such mappings in a Hilbert space. Furthermore, we prove a nonlinear ergodic theorem of Baillon's type in a Hilbert space. There results generalize the fixed point theorem and the nonlinear ergodic theorem proved by Kocourek, Takahashi and Yao [10].

2 Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. We denote the strong convergence and the weak convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. Let A be a nonempty subset of H. We denote by $\overline{co}A$ the closure of the convex hull of A. In a Hilbert space, it is known that

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2 \tag{1}$$

for any $x, y \in H$ and for any $\alpha \in \mathbb{R}$; see [15]. Furthermore, in a Hilbert space, we have that

$$2\langle x-y, z-w\rangle = \|x-w\|^2 + \|y-z\|^2 - \|x-z\|^2 - \|y-w\|^2 \tag{2}$$

for any $x, y, z, w \in H$. Let C be a nonempty subset of H and let T be a mapping from C into H. We denote by $F(T)$ the set of fixed points of T. A mapping T from C into H with $F(T) \neq \emptyset$ is called quasi-nonexpansive if $\|x - Ty\| \leq \|x - y\|$ for any $x \in F(T)$ and for any $y \in C$. It is well-known that the set $F(T)$ of fixed points of a quasi-nonexpansive mapping T is closed and convex; see Ito and Takahashi [8]. It is not difficult to prove such a result in a Hilbert space. In fact, for proving that $F(T)$ is closed, take a sequence $\{z_n\} \subset F(T)$ with $z_n \to z$. Since C is weakly closed, we have $z \in C$. Furthermore, from

$$\|z - Tz\| \leq \|z - z_n\| + \|z_n - Tz\| \leq 2\|z - z_n\| \to 0,$$

z is a fixed point of T and so $F(T)$ is closed. Let us show that $F(T)$ is convex. For $x, y \in F(T)$ and $\alpha \in [0,1]$, put $z = \alpha x + (1-\alpha)y$. Then, we have from (1) that

$$\begin{aligned}
\|z - Tz\|^2 &= \|\alpha x + (1-\alpha)y - Tz\|^2 \\
&= \alpha\|x - Tz\|^2 + (1-\alpha)\|y - Tz\|^2 - \alpha(1-\alpha)\|x-y\|^2 \\
&\leq \alpha\|x - z\|^2 + (1-\alpha)\|y - z\|^2 - \alpha(1-\alpha)\|x-y\|^2 \\
&= \alpha(1-\alpha)^2\|x-y\|^2 + (1-\alpha)\alpha^2\|x-y\|^2 - \alpha(1-\alpha)\|x-y\|^2 \\
&= \alpha(1-\alpha)(1-\alpha+\alpha-1)\|x-y\|^2 \\
&= 0.
\end{aligned}$$

This implies $Tz = z$. So, $F(T)$ is convex. Let C be a nonempty closed convex subset of H and $x \in H$. Then, we know that there exists a unique nearest point $z \in C$ such that $\|x - z\| = \inf_{y \in C}\|x - y\|$. We denote such a correspondence by $z = P_C x$. The mapping P_C is called the metric projection of H onto C. It is known that P_C is nonexpansive and

$$\langle x - P_C x, P_C x - u\rangle \geq 0$$

for any $x \in H$ and for any $u \in C$; see [15] for more details. For proving a nonlinear ergodic theorem in this paper, we also need the following lemma proved by Takahashi and Toyoda [17].

Lemma 2.1. *Let D be a nonempty closed convex subset of H. Let P be the metric projection from H onto D. Let $\{u_n\}$ be a sequence in H. If $\|u_{n+1} - u\| \leq \|u_n - u\|$ for any $u \in D$ and for any $n \in \mathbb{N}$, then $\{Pu_n\}$ converges strongly to some $u_0 \in D$.*

Let l^∞ be the Banach space of bounded sequences with supremum norm. Let μ be an element of $(l^\infty)^*$ (the dual space of l^∞). Then, we denote by $\mu(f)$ the value of μ at $f = (x_1, x_2, x_3, \dots) \in l^\infty$. Sometimes, we denote by $\mu_n(x_n)$ the value $\mu(f)$. A linear functional μ on l^∞ is called a mean if $\mu(e) = \|\mu\| = 1$, where $e = (1, 1, 1, \dots)$. A mean μ is called a Banach limit on l^∞ if $\mu_n(x_{n+1}) = \mu_n(x_n)$. We know that there exists a Banach limit on l^∞. If μ is a Banach limit on l^∞, then for $f = (x_1, x_2, x_3, \dots) \in l^\infty$,

$$\liminf_{n\to\infty} x_n \leq \mu_n(x_n) \leq \limsup_{n\to\infty} x_n.$$

In particular, if $f = (x_1, x_2, x_3, \dots) \in l^\infty$ and $x_n \to a \in \mathbb{R}$, then we have $\mu(f) = \mu_n(x_n) = a$. See [14] for the proof of existence of a Banach limit and its other elementary properties.

Using means and the Riesz theorem, we can obtain the following result; see [13] and [14].

Lemma 2.2. *Let H be a Hilbert space, let $\{x_n\}$ be a bounded sequence in H and let μ be a mean on l^∞. Then there exists a unique point $z_0 \in \overline{co}\{x_n \mid n \in \mathbb{N}\}$ such that*

$$\mu_n\langle x_n, y\rangle = \langle z_0, y\rangle, \quad \forall y \in H.$$

3 Fixed point theorems

Let H be a real Hilbert space and let C be a nonempty subset of H. A mapping T from C into H is said to be widely generalized hybrid if there exist $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in \mathbb{R}$ such that

$$\begin{aligned} \alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 \\ + \max\{\varepsilon\|x - Tx\|^2, \zeta\|y - Ty\|^2\} \leq 0 \end{aligned} \tag{3}$$

for any $x, y \in C$; see [9]. Such a mapping T is called $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$-widely generalized hybrid. An $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$-widely generalized hybrid mapping is generalized hybrid in the sense of Kocourek, Takahashi and Yao [10] if $\alpha + \beta = -\gamma - \delta = 1$ and $\varepsilon = \zeta = 0$. We first prove a fixed point theorem for widely generalized hybrid mappings in a Hilbert space.

Theorem 3.1. *Let H be a real Hilbert space, let C be a non-empty closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$-widely generalized hybrid mapping from C into itself which satisfies the following conditions* (1) *and* (2)*:*

(1) $\alpha + \beta + \gamma + \delta \geq 0$;

(2) $\varepsilon + \alpha + \gamma > 0$, *or* $\zeta + \alpha + \beta > 0$.

Then T has a fixed point if and only if there exists $z \in C$ such that $\{T^n z \mid n = 0, 1, \ldots\}$ is bounded. In particular, a fixed point of T is unique in the case of $\alpha + \beta + \gamma + \delta > 0$ on the condition (1).

Remark 3.2. We can also prove Theorem 3.1 by using the following condition instead of the condition (2):

(2)′ $\varepsilon - \beta - \delta > 0$, or $\zeta - \gamma - \delta > 0$.

In the case of the condition $\varepsilon - \beta - \delta > 0$, we obtain from (1) that

$$\varepsilon - \beta - \delta \leq \varepsilon + \alpha + \gamma.$$

Thus we obtain the desired result by Theorem 3.1. Similary, for the case of $\zeta - \gamma - \delta > 0$, we can obtain the result by using the case of $\zeta + \alpha + \beta > 0$.

As a direct consequence of Theorem 3.1, we obtain the following.

Theorem 3.3. *Let H be a real Hilbert space, let C be a non-empty bounded closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$-widely generalized hybrid mapping from C into itself which satisfies the following conditions* (1) *and* (2):

(1) $\alpha + \beta + \gamma + \delta \geq 0$;

(2) $\varepsilon + \alpha + \gamma > 0$, *or* $\zeta + \alpha + \beta > 0$.

Then T has a fixed point. In particular, a fixed point of T is unique in the case of $\alpha + \beta + \gamma + \delta > 0$ on the condition (1).

Note that an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$-widely generalized hybrid mapping T above with $\alpha = 1$, $\beta = \gamma = \varepsilon = \zeta = 0$ and $-1 < \delta < 0$ is a contractive mapping. Using Theorem 3.1, we can show the Banach fixed point theorem in a Hilbert space.

Theorem 3.4 (the Banach fixed point theorem). *Let H be a real Hilbert space and let T be a contractive mapping from H into H, that is, there exists a real number α with $0 < \alpha < 1$ such that*

$$\|Tx - Ty\| \leq \alpha \|x - y\|$$

for any $x, y \in H$. Then T has a unique fixed point.

Using Theorem 3.1, we can show the following fixed point theorem for generalized hybrid mappings in a Hilbert space.

Theorem 3.5 (Kocourek, Takahashi and Yao [10]**).** *Let H be a real Hilbert space, let C be a non-empty closed convex subset of H and let T be a generalized hybrid mapping from C into C, that is, there exist real numbers α and β such that*

$$\alpha \|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta \|Tx - y\|^2 + (1 - \beta)\|x - y\|^2$$

for any $x, y \in C$. Then T has a fixed point if and only if there exists $z \in C$ such that $\{T^n z \mid n = 0, 1, \ldots\}$ is bounded.

Example 3.6. Let H be the real line and let T be a mapping from H into H defined by $Tx = 2x$ for any $x \in H$. Taking $\alpha = 1$, $\beta = \gamma = -2$, $\delta = 4$ and $\varepsilon = \zeta = 2$, we have that

$$\begin{aligned}
&\alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 \\
&\qquad + \max\{\varepsilon\|x - Tx\|^2, \zeta\|y - Ty\|^2\} \\
&\quad = |2x - 2y|^2 - 2|x - 2y|^2 - 2|2x - y|^2 + 4|x - y|^2 + \max\{2x^2, 2y^2\} \\
&\quad = 8|x - y|^2 - 2|(x - y) - y|^2 - 2|x + (x - y)|^2 + \max\{2x^2, 2y^2\} \\
&\quad = -2x^2 - 2y^2 + \max\{2x^2, 2y^2\} \leq 0
\end{aligned}$$

for any $x, y \in H$. Furthermore, since $\{T^n 0 \mid n = 0, 1, \ldots\} = \{0\}$, (1) $\alpha + \beta + \gamma + \delta = 1 > 0$ and (2) $\varepsilon + \alpha + \gamma = 1 > 0$, we have from Theorem 3.1 that T has a unique fixed point. However T is not a contractive mapping. Moreover, taking $x = 0$ and $y = 1$, we have that for any real numbers α and β,

$$\begin{aligned}
&\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 - \beta\|Tx - y\|^2 - (1 - \beta)\|x - y\|^2 \\
&\quad = 4\alpha + 4(1 - \alpha) - \beta - (1 - \beta) = 3 > 0.
\end{aligned}$$

Thus T is not generalized hybrid.

4 Nonlinear ergodic theorem

In this section, using the technique developed by Takahashi [13], we prove a nonlinear ergodic theorem of Baillon's type in a Hilbert space. Before proving the result, we need the following lemmas.

Lemma 4.1. *Let H be a real Hilbert space, let C be a non-empty closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$-widely generalized hybrid mapping from C into C which has a fixed point and satisfies the condition:*

(2) $\varepsilon + \alpha + \gamma > 0$, *or* $\zeta + \alpha + \beta > 0$.

Then the set of fixed points of T is closed.

Lemma 4.2. *Let H be a real Hilbert space, let C be a non-empty closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$-widely generalized hybrid mapping from C into C which has a fixed point and satisfies the conditions* (1) *and* (2)*:*

(1) $\alpha + \beta + \gamma + \delta \geq 0$;

(2) $\varepsilon + \alpha + \gamma > 0$, *or* $\zeta + \alpha + \beta > 0$.

Then the set of fixed points of T is convex.

Lemma 4.3. *Let H be a real Hilbert space, let C be a non-empty closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$-widely generalized hybrid mapping from C into C which has a fixed point and satisfies the conditions* (1) *and* (3)*:*

(1) $\alpha+\beta+\gamma+\delta \geq 0$;

(3) $\alpha+\gamma>0$, *or* $\alpha+\beta>0$.

Then T *is quasi-nonexpansive.*

Theorem 4.4. *Let* H *be a real Hilbert space, let* C *be a non-empty closed convex subset of* H *and let* T *be an* $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$*-widely generalized hybrid mapping from* C *into* C *which has a fixed point and satisfies the conditions* (1) *and* (2):

(1) $\alpha+\beta+\gamma+\delta \geq 0$;

(2) $\varepsilon+\alpha+\gamma>0$, *or* $\zeta+\alpha+\beta>0$;

(3) $\alpha+\gamma>0$, *or* $\alpha+\beta>0$;

respectively. Then for any $x \in C$,

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

is weakly convergent to a fixed point p *of* T, *where* P *is the metric projection from* H *onto* $F(T)$ *and* $p=\lim_{n\to\infty} PT^n x$.

Using Theorem 4.4, we can show the following nonlinear ergodic theorem for generalized hybrid mappings in a Hilbert space.

Theorem 4.5 (Kocourek, Takahashi and Yao [10]). *Let* H *be a real Hilbert space, let* C *be a non-empty closed convex subset of* H *and let* T *be a generalized hybrid mapping from* C *into* C, *that is, there exist* $\alpha, \beta \in \mathbb{R}$ *such that*

$$\alpha\|Tx-Ty\|^2+(1-\alpha)\|x-Ty\|^2 \leq \beta\|Tx-y\|^2+(1-\beta)\|x-y\|^2$$

for any $x, y \in C$. *Suppose that* $F(T)$ *is nonempty. Then for any* $x \in C$,

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

is weakly convergent to a fixed point p *of* T, *where* $p=\lim_{n\to\infty} PT^n x$ *and* P *is the metric projection from* H *onto* $F(T)$.

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