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Existence and Approximation of Attractive Points for Nonlinear Mappings in Banach Spaces

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Abstract. Let H be a real Hilbert space norm $\|\cdot\|$. Let C be a nonempty subset of H and let T be a mapping of C into H. We denote by $F(T)$ the set of fixed points of T and by $A(T)$ the set of attractive points of T, i.e.,

(i) $F(T)=\{z\in C: Tz=z\}$;
(ii) $A(T)=\{z\in H: \|Tx-z\|\leq\|x-z\|,\ \ \forall x\in C\}$.

In this article, we extend the concept of attractive points in a Hilbert space to that in a Banach space and then prove attractive point theorems and mean convergence theorems without convexity for nonlinear mappings in a Banach space.

1 Introduction

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Let C be a nonempty subset of H. A mapping $T: C\rightarrow H$ is said to be *nonexpansive* if $\|Tx-Ty\|\leq\|x-y\|$ for all $x, y\in C$. We know that if C is a bounded, closed and convex subset of H and $T: C\rightarrow C$ is nonexpansive, then $F(T)$ is nonempty. Furthermore, from Baillon [4] we know the first nonlinear mean convergence theorem for nonexpansive mappings in a Hilbert space. An important example of nonexpansive mappings in a Hilbert space is a firmly nonexpansive mapping. A mapping F is said to be *firmly nonexpansive* if

$$\|Fx-Fy\|^{2}\leq\langle x-y, Fx-Fy\rangle$$

for all $x, y\in C$. Kohsaka and Takahashi [16], and Takahashi [24] introduced the following nonlinear mappings which are deduced from a firmly nonexpansive mapping in a Hilbert space. A mapping $T: C\rightarrow H$ is called *nonspreading* [16] if

$$2\|Tx-Ty\|^{2}\leq\|Tx-y\|^{2}+\|Ty-x\|^{2}$$

for all $x, y\in C$. A mapping $T: C\rightarrow H$ is called *hybrid* [24] if

$$3\|Tx-Ty\|^{2}\leq\|x-y\|^{2}+\|Tx-y\|^{2}+\|Ty-x\|^{2}$$

for all $x, y\in C$. The class of nonspreading mappings was first defined in a smooth, strictly convex and reflexive Banach space. The resolvents of a maximal monotone operator are nonspreading mappings; see [16] for more details. These three classes of nonlinear mappings are important in the study of the geometry of infinite dimensional spaces. Indeed, by using the fact that the resolvents of a maximal monotone operator are nonspreading mappings, Takahashi, Yao and Kohsaka [27] solved an open problem which is related to Ray's theorem [19] in the geometry of Banach spaces. Kocourek, Takahashi and Yao [12] defined a broad class of nonlinear mappings containing nonexpansive mappings, nonspreading mappings and hybrid mappings in a Hilbert space. A mapping $T : C \to H$ is called *generalized hybrid* [12] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2$$

for all $x, y \in C$, where $\mathbb{R}$ is the set of real numbers. We call such T an (α, β)-*generalized hybrid* mapping; see also [2]. Kocourek, Takahashi and Yao [12] proved a fixed point theorem for such mappings in a Hilbert space.

Theorem 1.1 ([12]). *Let C be a nonempty, closed and convex subset of a Hilbert space H and let $T : C \to C$ be a generalized hybrid mapping. Then T has a fixed point in C if and only if $\{T^n z\}$ is bounded for some $z \in C$.*

They also proved a mean convergence theorem which generalizes Baillon's nonlinear ergodic theorem [4] in a Hilbert space.

Theorem 1.2 ([12]). *Let H be a real Hilbert space, let C be a nonempty, closed and convex subset of H, let T be a generalized hybrid mapping from C into itself with $F(T) \neq \emptyset$ and let P be the metric projection of H onto $F(T)$. Then for any $x \in C$,*

$$S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$$

converges weakly to $p \in F(T)$, where $p = \lim_{n\to\infty} PT^n x$.

Recently, Takahashi and Takeuchi [25] introduced the concept of attractive points of nonlinear mappings in a Hilbert space and then they proved attractive point and mean convergence theorems without convexity for generalized hybrid mappings.

In this talk, we extend the concept of attractive points in a Hilbert space to that in a Banach space and then prove attractive point theorems and mean convergence theorems without convexity for nonlinear mappings in a Banach space.

2 Preliminaries

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be the topological dual space of E. We denote the value of $y^* \in E^*$ at $x \in E$ by $\langle x, y^*\rangle$. The modulus δ of convexity of E is defined by

$$\delta(\epsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \epsilon\right\}$$

for all ϵ with $0 \leq \epsilon \leq 2$. A Banach space E is said to be uniformly convex if $\delta(\epsilon) > 0$ for all $\epsilon > 0$. A uniformly convex Banach space is strictly convex and reflexive. Let E be a Banach space. The duality mapping J from E into 2^{E^*} is defined by

$$Jx = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\}$$

for all $x \in E$. Let $U = \{x \in E : \|x\| = 1\}$. The norm of E is said to be *Gâteaux differentiable* if for each $x, y \in U$, the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists. In the case, E is called *smooth.* We know that E is smooth if and only if J is a single-valued mapping of E into E^*. We also know that E is reflexive if and only if J is surjective, and E is strictly convex if and only if J is one-to-one. Therefore, if E is a smooth, strictly convex and reflexive Banach space, then J is a single-valued bijection. The norm of E is said to be *uniformly Gâteaux differentiable* if for each $y \in U$, the limit (2.1) is attained uniformly for $x \in U$. It is also said to be *Fréchet differentiable* if for each $x \in U$, the limit (2.1) is attained uniformly for $y \in U$. A Banach space E is called *uniformly smooth* if the limit (2.1) is attained uniformly for $x, y \in U$. It is known that if the norm of E is uniformly Gâteaux differentiable, then J is uniformly norm-to-weak* continuous on each bounded subset of E, and if the norm of E is Fréchet differentiable, then J is norm-to-norm continuous. If E is uniformly smooth, J is uniformly norm-to-norm continuous on each bounded subset of E. For more details, see [22, 23]. The following result is well known; see [22].

Lemma 2.1 ([22]). *Let E be a smooth Banach space and let J be the duality mapping on E. Then, $\langle x - y, Jx - Jy \rangle \geq 0$ for all $x, y \in E$. Furthermore, if E is strictly convex and $\langle x - y, Jx - Jy \rangle = 0$, then $x = y$.*

Let E be a smooth Banach space. The function $\phi\colon E \times E \to \mathbb{R}$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2$$

for all $x, y \in E$; see [1] and [11]. We have from the definition of ϕ that

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy \rangle \tag{2.2}$$

for all $x, y, z \in E$. From $(\|x\| - \|y\|)^2 \leq \phi(x, y)$ for all $x, y \in E$, we can see that $\phi(x, y) \geq 0$. Furthermore, we can obtain the following equality:

$$2\langle x - y, Jz - Jw \rangle = \phi(x, w) + \phi(y, z) - \phi(x, z) - \phi(y, w) \tag{2.3}$$

for all $x, y, z, w \in E$. Let $\phi_*\colon E^* \times E^* \to \mathbb{R}$ be the function defined by

$$\phi_*(x^*, y^*) = \|x^*\|^2 - 2\langle J^{-1}y^*, x^* \rangle + \|y^*\|^2$$

for all $x^*, y^* \in E^*$, where J is the duality mapping of E. It is easy to see that

$$\phi(x, y) = \phi_*(Jy, Jx) \tag{2.4}$$

for all $x, y \in E$. If E is additionally assumed to be strictly convex, then

$$\phi(x, y) = 0 \Longleftrightarrow x = y. \tag{2.5}$$

The following results are in Xu [28] and Kamimura and Takahashi [11].

Lemma 2.2 ([28]). *Let E be a uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, \infty) \to [0, \infty)$ such that $g(0) = 0$ and*

$$\|\lambda x + (1 - \lambda)y\|^2 \leq \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)g(\|x - y\|)$$

for all $x, y \in B_r$ and λ with $0 \leq \lambda \leq 1$, where $B_r = \{z \in E : \|z\| \leq r\}$.

Lemma 2.3 ([11]). *Let E be smooth and uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, 2r] \to \mathbb{R}$ such that $g(0) = 0$ and $g(\|x - y\|) \leq \phi(x, y)$ for all $x, y \in B_r$, where $B_r = \{z \in E : \|z\| \leq r\}$.*

Let E be a smooth Banach space and let C be a nonempty subset of E. A mapping $T : C \to E$ is called *generalized nonexpansive* [8] if $F(T) \neq \emptyset$ and $\phi(Tx, y) \leq \phi(x, y)$ for all $x \in C$ and $y \in F(T)$. Let D be a nonempty subset of a Banach space E. A mapping $R : E \to D$ is said to be *sunny* if $R(Rx + t(x - Rx)) = Rx$ for all $x \in E$ and $t \geq 0$. A mapping $R : E \to D$ is said to be a *retraction* or a *projection* if $Rx = x$ for all $x \in D$. A nonempty subset D of a smooth Banach space E is said to be a *generalized nonexpansive retract* (resp. *sunny generalized nonexpansive retract*) of E if there exists a generalized nonexpansive retraction (resp. sunny generalized nonexpansive retraction) R from E onto D; see [8] for more details. The following results are in Ibaraki and Takahashi [8].

Lemma 2.4 ([8]). *Let C be a nonempty closed sunny generalized nonexpansive retract of a smooth and strictly convex Banach space E. Then the sunny generalized nonexpansive retraction from E onto C is uniquely determined.*

Lemma 2.5 ([8]). *Let C be a nonempty closed subset of a smooth and strictly convex Banach space E such that there exists a sunny generalized nonexpansive retraction R from E onto C and let $(x, z) \in E \times C$. Then the following hold:*

(i) $z = Rx$ if and only if $\langle x - z, Jy - Jz \rangle \leq 0$ for all $y \in C$;
(ii) $\phi(Rx, z) + \phi(x, Rx) \leq \phi(x, z)$.

In 2007, Kohsaka and Takahashi [14] proved the following results:

Lemma 2.6 ([14]). *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty closed subset of E. Then the following are equivalent:*

(a) C is a sunny generalized nonexpansive retract of E;
(b) C is a generalized nonexpansive retract of E;
(c) JC is closed and convex.

Lemma 2.7 ([14]). *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty closed sunny generalized nonexpansive retract of E. Let R be the sunny generalized nonexpansive retraction from E onto C and let $(x, z) \in E \times C$. Then the following are equivalent:*

(i) $z = Rx$;
(ii) $\phi(x, z) = min_{y \in C} \phi(x, y)$.

Let l^∞ be the Banach space of bounded sequences with supremum norm. Let μ be an element of $(l^\infty)^*$ (the dual space of l^∞). Then we denote by $\mu(f)$ the value of μ at $f = (x_1, x_2, x_3, \ldots) \in l^\infty$. Sometimes we denote by $\mu_n(x_n)$ the value $\mu(f)$. A linear functional μ on l^∞ is called a *mean* if $\mu(e) = \|\mu\| = 1$, where $e = (1, 1, 1, \ldots)$. A mean μ is called a *Banach limit* on l^∞ if $\mu_n(x_{n+1}) = \mu_n(x_n)$. We know that there exists a Banach limit on l^∞. If μ is a Banach limit on l^∞, then for $f = (x_1, x_2, x_3, \ldots) \in l^\infty$,

$$\liminf_{n \to \infty} x_n \leq \mu_n(x_n) \leq \limsup_{n \to \infty} x_n.$$

In particular, if $f = (x_1, x_2, x_3, \ldots) \in l^\infty$ and $x_n \to a \in \mathbb{R}$, then we have $\mu(f) = \mu_n(x_n) = a$. See [22] for the proof of existence of a Banach limit and its other elementary properties.

3 Existence of Attractive Points in Banach Spaces

In 2011, Takahashi and Takeuchi [25] proved the following attractive point theorem in a Hilbert space.

Theorem 3.1 ([25]). *Let H be a Hilbert space, let C be a nonempty subset of H and let T be a generalized hybrid mapping of C into itself. Suppose that there exists an element $z \in C$ such that $\{T^n z\}$ is bounded. Then $A(T)$ is nonempty. Additionally, if C is closed and convex, then $F(T)$ is nonempty.*

In this section, we first try to extend Takahashi and Takeuchi's attractive point theorem [25] to Banach spaces. Let E be a smooth Banach space. Let C be a nonempty subset of E and let T be a mapping of C into E. We denote by $A(T)$ the set of *attractive points* [17] of T, i.e.,

$$A(T) = \{z \in E : \phi(z, Tx) \le \phi(z, x), \quad \forall x \in C\}.$$

From Lin and Takahashi [17], $A(T)$ is a closed and convex subset of E. A mapping $T : C \to E$ is called *generalized nonspreading* [13] if there exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$\begin{aligned} \alpha\phi(Tx, Ty) + (1-\alpha)\phi(x, Ty) + \gamma\{\phi(Ty, Tx) - \phi(Ty, x)\} \\ \le \beta\phi(Tx, y) + (1-\beta)\phi(x, y) + \delta\{\phi(y, Tx) - \phi(y, x)\} \end{aligned} \tag{3.1}$$

for all $x, y \in C$, where $\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$ for $x, y \in E$. We call such T an *$(\alpha, \beta, \gamma, \delta)$-generalized nonspreading mapping.* For example, a (1,1,1,0)-generalized nonspreading mapping is a nonspreading mapping in the sense of Kohsaka and Takahashi [16], i.e.,

$$\phi(Tx, Ty) + \phi(Ty, Tx) \le \phi(Tx, y) + \phi(Ty, x), \quad \forall x, y \in C;$$

see also [15] and [3]. Let T be an $(\alpha, \beta, \gamma, \delta)$-generalized nonspreading mapping. Observe that if $F(T) \neq \emptyset$, then $\phi(u, Ty) \le \phi(u, y)$ for all $u \in F(T)$ and $y \in C$. Using the technique developed by [20] and [21], we can prove an attractive point theorem for generalized nonspreading mappings in a Banach space.

Theorem 3.2 (Lin and Takahashi [17]). *Let E be a smooth and reflexive Banach space. Let C be a nonempty subset of E and let T be a generalized nonspreading mapping of C into itselt. Then, the following are equivalent:*

(a) $A(T) \neq \emptyset$;
(b) $\{T^n x\}$ is bounded for some $x \in C$.

Additionally, if E is strictly convex and C is closed and convex, then the following are equivalent:

(a) $F(T) \neq \emptyset$;
(b) $\{T^n x\}$ is bounded for some $x \in C$.

4 Skew-Attractive Point Theorems

Let E be a smooth Banach space and let C be a nonempty subset of E. Let $T : C \to E$ be a generalized nonspreading mapping; see (3.1). This mapping has the property that if $u \in F(T)$ and $x \in C$, then $\phi(u, Tx) \leq \phi(u, x)$. This property can be revealed by putting $x = u \in F(T)$ in (3.1). Similarly, putting $y = u \in F(T)$ in (3.1), we obtain that for any $x \in C$,

$$\begin{aligned} \alpha\phi(Tx, u) &+ (1-\alpha)\phi(x, u) + \gamma\{\phi(u, Tx) - \phi(u, x)\} \\ &\leq \beta\phi(Tx, u) + (1-\beta)\phi(x, u) + \delta\{\phi(u, Tx) - \phi(u, x)\} \end{aligned} \tag{4.1}$$

and hence

$$(\alpha - \beta)\{\phi(Tx, u) - \phi(x, u)\} + (\gamma - \delta)\{\phi(u, Tx) - \phi(u, x)\} \leq 0. \tag{4.2}$$

Therefore, we have that $\alpha > \beta$ together with $\gamma \leq \delta$ implies $\phi(Tx, u) \leq \phi(x, u)$. Motivated by this property of T and $F(T)$, we give the following defintition. Let E be a smooth Banach space. Let C be a nonempty subset of E and let T be a mapping of C into E. We denote by $B(T)$ the set of *skew-attractive points* of T, i.e.,

$$B(T) = \{z \in E : \phi(Tx, z) \leq \phi(x, z), \quad \forall x \in C\}.$$

The following result was proved by Lin and Takahashi [17].

Lemma 4.1 ([17]). *Let E be a smooth Banach space and let C be a nonempty subset of E. Let T be a mapping from C into E. Then $B(T)$ is closed.*

Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty subset of E. Let T be a mapping of C into E. Define a mapping T^* as follows:

$$T^*x^* = JTJ^{-1}x^*, \quad \forall x^* \in JC,$$

where J is the duality mapping on E and J^{-1} is the duality mapping on E^*. A mapping T^* is called the *adjoint* mapping of T; see also [26] and [6]. It is easy to show that if T is a mapping of C into itselt, then T^* is a mapping of JC into itself. In fact, for $x^* \in JC$, we have $J^{-1}x^* \in C$ and hence $TJ^{-1}x^* \in C$. So, we have $T^*x^* = JTJ^{-1}x^* \in JC$. Then, T^* is a mapping of JC into itself. We can prove the following result in a Banach space which was proved by Lin and Takahashi [17].

Lemma 4.2 ([17]). *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty subset of E. Let T be a mapping of C into E and let T^* be the duality mapping of T. Then, the following hold:*

(1) $JB(T) = A(T^*)$;
(2) $JA(T) = B(T^*)$.

In particular, $JB(T)$ is closed and convex.

Using these results, we can discuss skew-attractive point theorems in Banach spaces. Let E be a smooth Banach space and let C be a nonempty subset of E. A mapping $T : C \to E$ is called *skew-generalized nonspreading* [7] if there exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$\begin{aligned} \alpha\phi(Ty, Tx) &+ (1-\alpha)\phi(Ty, x) + \gamma\{\phi(Tx, Ty) - \phi(x, Ty)\} \\ &\leq \beta\phi(y, Tx) + (1-\beta)\phi(y, x) + \delta\{\phi(Tx, y) - \phi(x, y)\} \end{aligned} \tag{4.3}$$

for all $x, y \in C$. We call such T an $(\alpha, \beta, \gamma, \delta)$-*skew-generalized nonspreading* mapping. For example, a (1,1,1,0)-skew-generalized nonspreading mapping is a skew-nonspreading mapping in the sense of Ibaraki and Takahashi [9], i.e.,

$$\phi(Tx, Ty) + \phi(Ty, Tx) \leq \phi(x, Ty) + \phi(y, Tx), \quad \forall x, y \in C.$$

The following theorem was proved by Lin and Takahashi [17].

Theorem 4.3 ([17]). *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty subset of E. Let T be a skew-generalized nonspreading mapping of C into itselt. Then, the following are equivalent:*

(a) $B(T) \neq \emptyset$;
(b) $\{T^n x\}$ is bounded for some $x \in C$.

Additionally, if C is closed and JC is closed and convex, then the following are equivalent:

(a) $F(T) \neq \emptyset$;
(b) $\{T^n x\}$ is bounded for some $x \in C$.

5 Mean Convergence Theorems in Banach Spaces

In this section, we can prove a mean convergence theorem without convexity for generalized nonspreading mappings in a Banach space. Before proving it, we state the following lemmas.

Lemma 5.1 ([20, 5]). *Let E be a reflexive Banach space, let $\{x_n\}$ be a bounded sequence in E and let μ be a mean on l^∞. Then there exists a unique point $z_0 \in \overline{co}\{x_n : n \in \mathbb{N}\}$ such that*

$$\mu_n \langle x_n, y^* \rangle = \langle z_0, y^* \rangle, \quad \forall y^* \in E^*. \tag{5.1}$$

A unique point $z_0 \in \overline{co}\{x_n : n \in \mathbb{N}\}$ satisfying (5.1) is called the *mean vector* of $\{x_n\}$ for μ.

Lemma 5.2 ([18]). *Let E be a smooth, strictly convex and reflexive Banach space with the duality mapping J and let D be a nonempty, closed and convex subset of E. Let $\{x_n\}$ be a bounded sequence in D and let μ be a mean on l^∞. If $g : D \to \mathbb{R}$ is defined by*

$$g(z) = \mu_n \phi(x_n, z), \quad \forall z \in D,$$

then the mean vector z_0 of $\{x_n\}$ for μ is a unique minimizer in D such that

$$g(z_0) = \min\{g(z) : z \in D\}.$$

Lemma 5.3 ([18]). *Let E be a smooth and reflexive Banach space and let C be a nonempty subset of E. Let T be a generalized nonspreading mapping of C into itself. Suppose that $\{T^n x\}$ is bounded for some $x \in C$. Define*

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x, \quad \forall n \in \mathbb{N}.$$

If a subsequence $\{S_{n_i} x\}$ of $\{S_n x\}$ converges weakly to a point u, then $u \in A(T)$. Additionally, if E is strictly convex and C is closed and convex, then $u \in F(T)$.

Lemma 5.4 ([18]). *Let E be a uniformly convex and smooth Banach space. Let C be a nonempty subset of E and let $T : C \to C$ be a mapping such that $B(T) \neq \emptyset$. Then, there exists a unique sunny generalized nonexpansive retraction R of E onto $B(T)$. Furthermore, for any $x \in C$, $\lim_{n\to\infty} RT^n x$ exists in $B(T)$.*

Using these lemmas, we prove the following mean convergence theorem for generalized non-spreading mappings in a Banach space.

Theorem 5.5 (Lin and Takahashi [17]). *Let E be a uniformly convex Banach space with a Fréchet differentiable norm and let C be a nonempty subset of E. Let $T : C \to C$ be a generalized nonspreading mapping such that $A(T) = B(T) \neq \emptyset$. Let R be the sunny generalized nonexpansive retraction of E onto $B(T)$. Then, for any $x \in C$, the sequence $\{S_n x\}$ of Cesàro means*

$$S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$$

converges weakly to an element q of $A(T)$, where $q = \lim_{n\to\infty} RT^n x$.

Using Theorem 5.5, we obtain the following theorems.

Theorem 5.6 (Kocourek, Takahashi and Yao [13]). *Let E be a uniformly convex Banach space with a Fréchet differentiable norm. Let $T : E \to E$ be an $(\alpha, \beta, \gamma, \delta)$-generalized nonspreading mapping such that $\alpha > \beta$ and $\gamma \leq \delta$. Assume that $F(T) \neq \emptyset$ and let R be the sunny generalized nonexpansive retraction of E onto $F(T)$. Then, for any $x \in E$, the sequence $\{S_n x\}$ of Cesàro means*

$$S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$$

converges weakly to an element q of $F(T)$, where $q = \lim_{n\to\infty} RT^n x$.

Proof. We also know that $\alpha > \beta$ together with $\gamma \leq \delta$ implies that $\phi(Tx, u) \leq \phi(x, u)$ for all $x \in E$ and $u \in F(T)$. We also note that $A(T) = F(T)$ and $B(T) = F(T)$. So, we have the desired result from Theorem 5.5. □

References

[1] Y. I. Alber, *Metric and generalized projections in Banach spaces: Properties and applications*, in Theory and Applications of Nonlinear Operators of Accretive and Monotone Type (A. G. Kartsatos Ed.), Marcel Dekker, New York, 1996, pp. 15–50.

[2] K. Aoyama, S. Iemoto, F. Kohsaka and W. Takahashi, *Fixed point and ergodic theorems for λ-hybrid mappings in Hilbert spaces*, J. Nonlinear Convex Anal. **11** (2010), 335-343.

[3] K. Aoyama, F. Kohsaka and W. Takahashi, *Three generalizations of firmly nonexpansive mappings: Their relations and continuity properties*, J. Nonlinear Convex Anal. **10** (2009), 131-147.

[4] J.-B. Baillon, *Un theoreme de type ergodique pour les contractions non lineaires dans un espace de Hilbert*, C. R. Acad. Sci. Paris Ser. A-B **280** (1975), 1511–1514.

[5] N. Hirano, K. Kido and W. Takahashi, *Nonexpansive retractions and nonlinear ergodic theorems in Banach spaces*, Nonlinear Anal. **12** (1988), 1269–1281.

[6] T. Honda, T. Ibaraki and W. Takahashi, *Duality theorems and convergence theorems for nonlineaqr mappings in Banach spaces*, Int. J. Math. Statis. **6** (2010), 46–64.

[7] M.-H. Hsu, W. Takahashi and J.-C. Yao, *Generalized hybrid mappings in Hilbert spaces and Banach spaces*, Taiwanese J. Math. 16–1 (2012), 129–149.

[8] T. Ibaraki and W. Takahashi, *A new projection and convergence theorems for the projections in Banach spaces*, J. Approx. Theory **149** (2007), 1–14.

[9] T. Ibaraki and W. Takahashi, *Fixed point theorems for new nonlinear mappings of nonexpansive type in Banach spaces*, J. Nonlinear Convex Anal. **10** (2009), 21–32.

[10] T. Ibaraki and W. Takahashi, *Generalized nonexpansive mappings and a proximal-type algorithm in Banach spaces*, Contemp. Math., **513**, Amer. Math. Soc., Providence, RI, 2010, pp. 169–180.

[11] S. Kamimura and W. Takahashi, *Strong convergence of a proximal-type algorithm in a Banach space*, SIAM J. Optim. **13** (2002), 938–945.

[12] P. Kocourek, W. Takahashi and J. -C. Yao, *Fixed point theorems and weak convergence theorems for generalized hybrid mappings in Hilbert spaces*, Taiwanese J. Math. **14** (2010), 2497–2511.

[13] P. Kocourek, W. Takahashi and J. -C. Yao, *Fixed point theorems and ergodic theorems for nonlinear mappings in Banach spaces*, Adv. Math. Econ. **15** (2011), 67–88.

[14] F. Kohsaka and W. Takahashi, *Generalized nonexpansive retractions and a proximal-type algorithm in Banach spaces*, J. Nonlinear Convex Anal. **8** (2007), 197-209.

[15] F. Kohsaka and W. Takahashi, *Existence and approximation of fixed points of firmly nonexpansive type mappings in Banach spaces*, SIAM J. Optim. **19** (2008), 824-835.

[16] F. Kohsaka and W. Takahashi, *Fixed point theorems for a class of nonlinear mappings related to maximal monotone operators in Banach spaces*, Arch. Math. **91** (2008), 166-177.

[17] L.-J. Lin and W. Takahashi, *Attractive point theorem for nonspreading mappings in Banach space*, J. Convex Anal., to appear.

[18] L.-J. Lin, W. Takahashi and Z.-T. Yu, *Attractive point theorems and ergodic theorems for 2-generalized nonspreading mappings in Banach spaces*, J. Nonlinear Convex Anal., to appear.

[19] W. O. Ray, *The fixed point property and unbounded sets in Hilbert space*, Trans. Amer. Math. Soc. **258** (1980), 531–537.

[20] W. Takahashi, *A nonlinear ergodic theorem for an amenable semigroup of nonexpansive mappings in a Hilbert space*, Proc. Amer. Math. Soc. **81** (1981), 253–256.

[21] W. Takahashi, *A nonlinear ergodic theorem for a reversible semigroup of nonexpansive mappings in a Hilbert space*, Proc. Amer. Math. Soc. **97** (1986), 55–58.

[22] W. Takahashi, *Nonlinear Functional Analysis, Fixed Point Theory and its Applications*, Yokohama Publishers, Yokohama 2000.

[23] W. Takahashi, *Convex Analysis and Approximation of Fixed Points (Japanese)*, Yokohama Publishers, Yokohama, 2000.

[24] W. Takahashi, *Fixed point theorems for new nonlinear mappings in a Hilbert space*, J. Nonlinear Convex Anal. **11** (2010), 79–88.

[25] W. Takahashi and Y. Takeuchi, *Nonlinear ergodic theorem without convexity for generalized hybrid mappings in a Hilbert space*, J. Nonlinear Convex Anal. **12** (2011), 399–406.

[26] W. Takahashi and J.-C. Yao, *Nonlinear operators of monotone type and convergence theorems with equilibrium problems in Banach spaces*, Taiwanese J. Math. **15** (2011), 787–818.

[27] W. Takahashi, J.-C. Yao and F. Kohsaka, *The fixed point property and unbounded sets in Banach spaces*, Taiwanese J. Math. **14** (2010), 733–742.

[28] H. K. Xu, *Inequalities in Banach spaces with applications*, Nonlinear Anal. **16** (1981), 1127–1138.