数理解析研究所講究録
第 1842 巻 2013 年 40-60

REGULARITY FOR A DOUBLY NONLINEAR PARABOLIC EQUATION

JUHA KINNUNEN

ABSTRACT. This survey focuses on regularity results for certain degenerate doubly nonlinear parabolic equations in the case when the Lebesgue measure is replaced with a doubling Borel measure which supports a Poincaré inequality. Possible extensions and connections to analysis on metric measure spaces are also discussed.

1. INTRODUCTION

This note focuses on the regularity of nonnegative weak solutions to the doubly nonlinear parabolic equation

$$\frac{\partial(u^{p-1})}{\partial t}-\operatorname{div}(|Du|^{p-2}Du)=0, \qquad 1<p<\infty. \tag{1.1}$$

When $p=2$ we have the standard heat equation. The equation is degenerate in the sense that the modulus of ellipticity vanishes when the spatial gradient Du vanishes. The main challenge of the equation is the double nonlinearity. Indeed, both the term containing the time derivative and also the term containig the spatial derivatives are nonlinear. Observe that the solutions to (1.1) can be scaled by nonnegative factors, but due to the nonlinearity of the term containing the time derivative, constants cannot be added to a solution.

Parabolic equations of the p-Laplacian type have been studied extensively in the literature. Studies for the p-parabolic equation

$$\frac{\partial u}{\partial t}-\operatorname{div}(|Du|^{p-2}Du)=0, \qquad 1<p<\infty, \tag{1.2}$$

or more general equations of the form

$$\frac{\partial u}{\partial t}-\operatorname{div}(u^{m-1}|Du|^{p-2}Du)=0, \qquad 1<p<\infty, \quad m\in\mathbb{R}, \tag{1.3}$$

seem to be easier to find than for (1.1). These equations are linear with respect to the term containing the time derivative and the function spaces for weak solutions are different compared to (1.1). Formally we obtain the porous medium equation by choosing $p=2$ and $m>1$ and

2000 *Mathematics Subject Classification.* 35K55.

the p-parabolic equation by choosing $m = 1$ in (1.3). In addition, the substitution $v = u^{p-1}$ in (1.1) gives (1.3) with $m = 3 - p$. With this formal change of variable the obtained equations seem to be equivalent. However, since the function spaces are different, it is not a priori clear that the weak solutions for these equations are same. These and more general equations have been studied in [FS], [GV], [Iv], [PV], [V1] and [V2] for certain values of the parameter m. In this note we only consider the doubly nonlinear equation of the form (1.1). We can also consider more general equations

$$\frac{\partial(u^{p-1})}{\partial t} - \operatorname{div} A(x, t, u, Du) = 0,$$

of the p-Laplacian type, but for expository purposes we shall only focus on the prototype equation. We would like to point out that there are certain unexpected difficulties in dealing with the doubly nonlinear equation. We would also like to oppose the general belief that the doubly nonlinear equation is easier and less interesting than the p-parabolic equation. Indeed, it seems that the theory for the p-parabolic equation is needed in the regularity theory for the doubly nonlinar equation, the doubly nonlinear equation seems to be relevant in connections with analysis on metric measure spaces and there are still many interesting open problems.

Harnack type estimates play a fundamental role in the regularity theory for parabolic equations of the p-Laplacian type. A scale and location invariant parabolic Harnack inequality for nonnegative weak solutions of (1.1) has been obtained in [T]. This reflects the scaling property of the doubly nonlinear equation. The proof is based on Moser's celebrated work [M1] and it uses a rather delicate parabolic John-Nirenberg lemma. For this, see also [FG]. For another approach based on a De Giorgi type argument, see [GV]. A relatively transparent proof for Harnack's inequality using the approach of Moser in [M2] can also be found in [KK]. In particular, the parabolic John-Nirenberg lemma is replaced with a very elegant real analysis lemma to Bombieri in [BG] and [B].

In contrast with the case $p = 2$, Harnack estimates do not immediately imply the local Hölder continuity of weak solutions of the doubly nonlinear equation. The main problem is that we cannot add constants to solutions. Recent investigations [KSU] and [KLSU] show that nonnegative weak solutions are, indeed, locally Hölder continuous. See also the recent PhD thesis of Juhana Siljander [Si2]. There seems to be a dichotomic behaviour related to the doubly nonlinear equation. In large scales the scale and location invariant Harnack estimates dominate and the equation behaves, roughly speaking, as the classical heat equation. On the other hand, in small scales the equation looks like a p-parabolic equation. Consequently, relatively heavy regularity theories both for the doubly nonlinear equation and the p-parabolic equation are invoked in the argument. Moreover, quite recently also the spatial gradient of a positive weak solution is shown to be locally bounded, see [Si2]. This is the first step to show that the gradient is locally Hölder continuous. Similar regularity results for certain equations of type (1.3) have been obtained in [Iv], [PV] and [V2].

The previous regularity results are studied in the case when the Lebesgue measure is replaced with a more general Borel measure, which is assumed to satisfy the doubling condition and supporting a Poincaré inequality. The precise definitions will be given below. These are rather standard assumptions in analysis on Riemannian manifolds and more general metric spaces, see, for example, [BB], [H], [HK] and [SC1]. It is well known that regularity theory for partial differential equations is essentially based on a combination of a Sobolev and a Caccioppoli type energy estimates. The corresponding result in the elliptic case for measures induced by Muckenhoupt's weights has been studied in [FKS]. See also [CF]. The weighted theory in the parabolic case has been studied in [CS], [GW1] and [GW2]. However, in their approach the role of the measure is somewhat different compared to ours. See also [Su] for weighted results for the p-parabolic equation.

Let us briefly explain our motivation to study the regularity theory with more general measure than the Lebesgue measure. For the heat equation Grigor'yan and Saloff-Coste observed that the doubling condition and the Poincaré inequality are not only sufficient but also necessary conditions for a scale invariant parabolic Harnack principle on Riemannian manifolds, see [SC1], [SC2] and [G]. The main result of [KK] shows that the doubling condition and the Poincaré inequality are sufficient conditions for a scale and location invariant Harnack inequality for the doubly nonlinear equation also when $p \neq 2$. It is a very interesting question whether this would also give a characterization for the doubling condition and the Poincaré inequality. Another motivation comes from the boundary Harnack estimates for equations of the p-Laplacian type. In the elliptic case this has been studied in [LN] and it would be very interesting to obtain the corresponding results for the doubly nonlinear equation. Already in the elliptic case, regularity theory in the weighted case plays a central role in the argument. It it likely that the parabolic version of the theory is needed in the future extension of the boundary Harnack estimates to the parabolic case.

Using the methods discussed in this note regularity results can be obtained in many different contexts and ultimately even in more general metric measure spaces. For an approach based on the Dirichlet forms, we refer to [BBK], [BM], [D], [St1] and [St2]. In these references several characterizations of parabolic Harnack inequalities are given in various contexts. It is known that doubling and Poincaré are sufficient for lot of analysis on metric measure spaces, but few necessary conditions are available. Some of the few results concerning sufficient conditions are by Semmes, see [Se]. It would be very interesting to obtain characterizations of the doubling condition and the Poincaré inequality through scale and location invariant parabolic Harnack estimates related to parabolic quasiminimizers introduced in [W]. See also [Z]. The regularity theory for parabolic quasiminimizers on metric measure spaces is currently developed in [KMPP], [MM] and [MS], but many interesting questions remain open.

2. Preliminariess

2.1. **Doubling condition.** The doubling condition gives a uniform bound for the growth of the measure of a ball if the radius is doubled. A Borel measure μ in $\mathbb{R}^N$ is doubling, if there exists a constant $D_0 \geq 1$, called the doubling constant of μ, such that

$$\mu(B(x,2r)) \leq D_0\mu(B(x,r))$$

for every $x \in \mathbb{R}^N$ and $r > 0$. Here $B(x,r) = \{y \in \mathbb{R}^N : d(x,y) < r\}$ is an open ball with center x and radius r. More generally, quasimetric spaces, in which the triangle inequality holds only up to a multiplicative constant, with a doubling measure are sometimes called spaces of homogeneous type.

Roughly speaking, the doubling condition gives an upper bound for the dimension related to the measure. Indeed, if μ is doubling and $r < R$, then

$$\frac{\mu(B(x,R))}{\mu(B(x,r))} \leq C\Big(\frac{R}{r}\Big)^{d_\mu},$$

where

$$d_\mu = \log_2 D_0$$

and C is a constant that depends only on the doubling constant. The exponent d_μ is not necessarily optimal.

2.2. **Poincaré inequality.** The Poincaré inequality gives a link between the metric, measure and the gradient and it provides a passage from the infinitesimal notion of a gradient to larger scale behaviour of a function. Roughly speaking this means that if the gradient is small in average, then also the mean oscillation of a function is small.

Let $1 < p < \infty$. The measure is said to support a $(1,p)$-Poincaré inequality, if there exists a constant $P_0 > 0$ such that

$$\fint_{B(x,r)} |u - u_{B(x,r)}| \, d\mu \le P_0 r \Big(\fint_{B(x,r)} |Du|^p \, d\mu \Big)^{1/p},$$

for every $u \in C^\infty(\mathbb{R}^N)$, $x \in \mathbb{R}^N$ and $r > 0$. Here

$$u_{B(x,r)} = \fint_{B(x,r)} u \, d\mu = \frac{1}{\mu(B(x,r))} \int_{B(x,r)} u \, d\mu$$

denotes the integral average. The crucial property is that the $(1,p)$-Poincaré inequality is assumed to hold uniformly in all scales and locations.

By Hölder's inequality, it is clear that $(1,p)$-Poincaré inequality implies $(1,q)$-Poincaré inequality for every $q > p$. Both sides of the Poincaré inequality also enjoy somewhat unexpected self-improving property. Indeed, the exponent on the left hand side can be increased. If the measure is doubling, then the $(1,p)$-Poincaré inequality implies the following Sobolev-Poincaré inequality. There is a constant $C = C(D_0, p) > 0$ such that

$$\Big(\fint_{B(x,r)} |u - u_{B(x,r)}|^{\kappa p} \, d\mu \Big)^{1/(\kappa p)} \le C r \Big(\fint_{B(x,r)} |Du|^p \, d\mu \Big)^{1/p}$$

for every for every $u \in C^\infty(\mathbb{R}^N)$, $x \in \mathbb{R}^N$ and $r > 0$, where

$$\kappa = \begin{cases} \dfrac{d_\mu}{d_\mu - p}, & 1 < p < d_\mu, \\ 2, & p \ge d_\mu. \end{cases}$$

The factor κ is related to the Sobolev conjugate exponent. When $p = d_\mu$ there is an exponential estimate and for $p > d_\mu$ there is a Hölder estimate, but we do not need these refinements here. For the proof, we refer to [BCLS],[HK], [SC1] and [SC2].

For functions with the zero boundary values we have the following version of Sobolev's inequality. There exists a constant $C = C(D_0, p) > 0$ such that

$$\Big(\fint_{B(x,r)} |u|^{\kappa p} \, d\mu \Big)^{1/(\kappa p)} \le C r \Big(\fint_{B(x,r)} |Du|^p \, d\mu \Big)^{1/p}$$

for every $u \in C_0^\infty(B(x,r))$. For the proof we refer, for example, to [KS]. Observe carefully that the exponent on the left hand side is strictly larger than on the right hand side. This is essential in the regularity theory for partial differential equations. Also the exponent on the right hand side of the Poincaré inequality can be decreased, see [KZ]. This is a very useful fact in maximal function estimates. Sometimes there is a larger ball on the right hand side of the Poincaré inequality, but in the Euclidean case this is an equivalent with the standard Poincaré inequality. The doubling condition for the measure and the Poincaré inequality are available also in the context of more general metric spaces than the Euclidean space and he mentioned self improving phenomena are extremely useful results in analysis on metric measure spaces, see [BB] and [H].

2.3. **Standing assumptions.** From now on we assume that the measure μ is doubling and supports the $(1,p)$-Poincaré inequality for some $1 < p < \infty$. Moreover, we assume that the measure is nontrivial in the sense that the measure of every nonempty open set is strictly positive and measure of every bounded set is finite. As an example, we mention that Muckenhoupt's weights satisfy these assumptions, see [FKS] and [CF].

2.4. **Sobolev spaces.** Let Ω be an open subset of $\mathbb{R}^N$. The elliptic Sobolev space $H^{1,p}(\Omega,\mu)$ is defined to be the completion of $C^\infty(\Omega)$ with respect to the Sobolev norm

$$\|u\|_{1,p,\Omega} = \Big(\int_\Omega |u|^p \,d\mu\Big)^{1/p} + \Big(\int_\Omega |Du|^p \,d\mu\Big)^{1/p}.$$

A function belongs to the local Sobolev space $H^{1,p}_{\mathrm{loc}}(\Omega,\mu)$ if it belongs to $H^{1,p}(\Omega',\mu)$ for every $\Omega' \Subset \Omega$. Here Ω' is an open subset of Ω, whose closure is a compact subset of Ω. The Sobolev space with zero boundary values $H^{1,p}_0(\Omega,\mu)$ is the completion of $C^\infty_0(\Omega)$ with respect to the Sobolev norm. For the basic properties of weighted Sobolev spaces we refer to [FKS] and [HKM]. Observe, that Sobolev inequalities hold for Sobolev functions by a density argument under our assumptions.

We denote by $L^p(0,T;H^{1,p}(\Omega))$, $T > 0$, the space of functions $u = u(x,t)$ such that for almost every t with $0 < t < T$ the function $x \mapsto u(x,t)$ belongs to $H^{1,p}(\Omega,\mu)$ and

$$\int_0^T \int_\Omega \big(|u|^p + |Du|^p\big) \,d\mu \,dt < \infty.$$

Notice that the time derivative u_t is deliberately avoided. Roughly speaking the functions in $L^p(0,T;H^{1,p}(\Omega))$ are elliptic Sobolev functions in the spatial variable for a fixed moment of time and L^p-functions in the time variable at a fixed point in Ω. The definitions for spaces $L^p_{\mathrm{loc}}(0,T;H^{1,p}_{\mathrm{loc}}(\Omega,\mu))$ and $L^p(0,T;H^{1,p}_0(\Omega))$ are clear.

2.5. **Parabolic Sobolev estimate.** Next we show how a parabolic Sobolev inequality follows from the elliptic one. The argument is very simple and it can be easily modified to give various versions of the parabolic Sobolev estimate. The most important fact for us is that the exponent on the left hand side is strictly greater than on the right hand side.

Lemma 2.1. *There is a constant $C = C(D_0, p)$ such that*

$$\int_0^T \int_{B(x,r)} |u|^{(2-1/\kappa)p} \, d\mu \, dt$$

$$\leq Cr^p \Big(\operatorname*{ess\,sup}_{0<t<T} \fint_{B(x,r)} |u|^p \, d\mu \Big)^{1-1/\kappa} \int_0^T \int_{B(x,r)} |Du|^p \, d\mu \, dt,$$

for every $u \in L^p(0,T; H_0^{1,p}(B(x,r)))$. Here $\kappa > 1$ is the factor in the Sobolev inequality.

Proof. By Hölder's and Sobolev's inequalities, we have

$$\fint_{B(x,r)} |u|^{(2-1/\kappa)p} \, d\mu$$

$$\leq \Big(\fint_{B(x,r)} |u|^p \, d\mu\Big)^{1-1/\kappa} \Big(\fint_{B(x,r)} |u|^{\kappa p} \, d\mu\Big)^{1/\kappa}$$

$$\leq Cr^p \Big(\fint_{B(x,r)} |u|^p \, d\mu\Big)^{1-1/\kappa} \fint_{B(x,r)} |Du|^p \, d\mu$$

and an integration over the time variable gives

$$\int_0^T \fint_{B(x,r)} |u|^{(2-1/\kappa)p} \, d\mu \, dt$$

$$\leq Cr^p \int_0^T \Big[\Big(\fint_{B(x,r)} |u|^p \, d\mu\Big)^{1-1/\kappa} \fint_{B(x,r)} |Du|^p \Big] \, d\mu \, dt$$

$$\leq Cr^p \Big(\operatorname*{ess\,sup}_{0<t<T} \fint_{B(x,r)} |u|^p \, d\mu \Big)^{1-1/\kappa} \int_0^T \fint_{B(x,r)} |Du|^p \, d\mu \, dt.$$

This proves the claim. □

3. Properties of the doubly nonlinear equation

To be on the safe side, we recall the definition of a weak solution with test functions under the integrals. Formally this is obtained by multiplying the equation (1.1) with a test function and then integrating by parts.

3.1. **Weak solutions.** Let $1 < p < \infty$. A nonnegative function u which belongs to $L^p_{\mathrm{loc}}(0,T; H^{1,p}_{\mathrm{loc}}(\Omega, \mu))$ is a weak solution to (1.1) in $\Omega \times (0,T)$ if

$$\int_0^T \int_\Omega \Big(|Du|^{p-2} Du \cdot D\varphi - u^{p-1} \frac{\partial \varphi}{\partial t} \Big) \, d\mu \, dt = 0 \tag{3.1}$$

for all $\varphi \in C_0^\infty(\Omega \times (0,T))$. Further, we say that u is a supersolution to (1.1), if the integral (3.1) is nonnegative for all $\varphi \in C_0^\infty(\Omega \times (0,T))$ with

$\varphi \geq 0$. If this integral is nonpositive, we say that u is a subsolution. Observe that the time derivative u_t is avoided in the definition and, a priori, the weak solution is not assumed to have the weak derivative in the time direction. The assumption that the function belongs to $L^p_{\text{loc}}(0,T; H^{1,p}_{\text{loc}}(\Omega,\mu))$ guarantees that the integral (3.1) is well defined.

Example 3.2. The function

$$u(x,t) = t^{-n/(p(p-1))} \exp\Big(-\frac{p-1}{p}\Big(\frac{|x|^p}{pt}\Big)^{1/(p-1)}\Big),$$

where $x \in \mathbb{R}^n$ and $t > 0$, is so-called Barenblatt-Zel'dovich-Kompaneets solution of the doubly nonlinear equation with the Lebesgue measure in the upper half space. Observe that this function is strictly positive for every $x \in \mathbb{R}^N$ and $t > 0$. This indicates an infinite speed of propagation for disturbancies.

Let $0 \leq t_1 < t_2 \leq T$. If the test function φ vanishes only on the lateral boundary $\partial\Omega \times (t_1,t_2)$, then the boundary terms

$$\int_\Omega u(x,t_1)^{p-1}\varphi(x,t_1)\,d\mu = \lim_{\tau\to 0}\frac{1}{\tau}\int_{t_1}^{t_1+\tau}\int_\Omega u(x,t)^{p-1}\varphi(x,t)\,d\mu\,dt$$

and

$$\int_\Omega u(x,t_2)^{p-1}\varphi(x,t_2)\,d\mu = \lim_{\tau\to 0}\frac{1}{\tau}\int_{t_2-\tau}^{t_2}\int_\Omega u(x,t)^{p-1}\varphi(x,t)\,d\mu\,dt$$

appear. In this case (3.1) reads

$$\begin{aligned}\int_{t_1}^{t_2}\int_\Omega &|Du|^{p-2}Du\cdot D\varphi\,d\mu\,dt \\ &-\int_{t_1}^{t_2}\int_\Omega u^{p-1}\frac{\partial\varphi}{\partial t}\,d\mu\,dt + \Big[\int_\Omega u^{p-1}\varphi\,d\mu\Big]_{t=t_1}^{t_2} = 0\end{aligned} \tag{3.3}$$

for almost every t_1, t_2 with $0 \leq t_1 < t_2 \leq T$. This is a useful version of the definition in derivation of energy estimates.

There is a well-recognized difficulty with the test functions. Indeed, in proving estimates we usually need a test function which depends on the solution itself. Then we cannot avoid that the forbidden quantity u_t shows up in the calculation. In most cases one can easily overcome this difficulty by using an equivalent definition in terms of Steklov averages, as on pages 18 and 25 in [DB] and in Chapter 2 of [WZYL]. Alternatively, we can proceed using convolutions with smooth mollifiers as on pages 199–121 in [AS]. Let f_ε denote the mollification of the function f with respect to the time variable. For every $\varphi \in C_0^\infty(\Omega \times (0,T))$, the definition of a weak solution reads

$$\int_0^T\int_\Omega \Big((|Du|^{p-2}Du)_\varepsilon \cdot D\varphi + \varphi\frac{\partial(u^{p-1})_\varepsilon}{\partial t}\Big)\,d\mu\,dt = 0$$

for small enough $\varepsilon > 0$. Observe that the forbidden quantity has disappeared.

For expository purposes, we do not discuss the mollification procedure in our arguments. Instead, we make formal computations and the final estimates will be free of forbidden quantities. Everything can be made precise with the mollification procedure described above, but we leave this to the interested reader.

3.2. **Caccioppoli estimates.** Energy estimates are of fundamental importance in the regularity theory. Here we recall a prototype of such an estimate. Caccioppoli estimates can be obtained by choosing a correct test function in the definition of a weak solution.

Lemma 3.4. *Suppose that u is a nonnegative weak subsolution in $\Omega \times (0,T)$. Then there exists a constant $C = C(p)$ such that*

$$\begin{aligned}&\int_0^T \int_\Omega |Du|^p \varphi^p \, d\mu \, dt + \operatorname*{ess\,sup}_{0<t<T} \int_\Omega u^p \varphi^p \, d\mu \\ &\le C \int_0^T \int_\Omega u^p |D\varphi|^p \, d\mu \, dt + C \int_0^T \int_\Omega u^p \varphi^{p-1} \left| \frac{\partial \varphi}{\partial t} \right| d\mu \, dt\end{aligned}$$

for every $\varphi \in C_0^\infty(\Omega \times (0,T))$ with $\varphi \ge 0$.

Proof. Formally we choose the test function $\eta = u\varphi^p$ so that

$$D\eta = \varphi^p Du + p\varphi^{p-1} D\varphi u$$

and

$$\frac{\partial \eta}{\partial t} = \varphi^p \frac{\partial u}{\partial t} + p\varphi^{p-1} \frac{\partial \varphi}{\partial t} u,$$

where $\varphi \in C_0^\infty(\Omega \times (0,T))$ with $\varphi \ge 0$. Let $0 \le t_1 < t_2 \le T$. The test function vanishes only on the lateral boundary.

A substitution of η in the definition of a weak solution gives

$$\begin{aligned}&\int_{t_1}^{t_2} \int_\Omega |Du|^p \varphi^p \, d\mu \, dt + p \int_{t_1}^{t_2} \int_\Omega |Du|^{p-2} Du \cdot D\varphi \, \varphi^{p-1} u \, d\mu \, dt \\ &\quad - \int_{t_1}^{t_2} \int_\Omega u^{p-1} \frac{\partial u}{\partial t} \varphi^p \, d\mu \, dt - p \int_{t_1}^{t_2} \int_\Omega u^p \varphi^{p-1} \frac{\partial \varphi}{\partial t} \, d\mu \, dt \\ &\quad + \left[\int_\Omega u^p \varphi \, d\mu \right]_{t=t_1}^{t_2} \le 0.\end{aligned}$$

Observe that the forbidden time derivative appears. An integration by parts implies

$$\int_{t_1}^{t_2}\int_\Omega u^{p-1}\frac{\partial u}{\partial t}\varphi^p\,d\mu\,dt$$
$$=\frac{1}{p}\left[\int_\Omega u^p\varphi^p\,d\mu\right]_{t=t_1}^{t_2}-\int_{t_1}^{t_2}\int_\Omega u^p\varphi^{p-1}\frac{\partial\varphi}{\partial t}\,d\mu\,dt.$$

Now the forbidden time derivative has disappeared from the right hand side. We arrive at

$$\begin{aligned}\int_{t_1}^{t_2}\int_\Omega |Du|^p\varphi^p\,d\mu\,dt&+\frac{p-1}{p}\left[\int_\Omega u^p\varphi^p\,d\mu\right]_{t=t_1}^{t_2}\\&\le -p\int_{t_1}^{t_2}\int_\Omega |Du|^{p-2}Du\cdot D\varphi\,\varphi^{p-1}u\,d\mu\,dt \qquad (3.5)\\&\quad+(p-1)\int_{t_1}^{t_2}\int_\Omega u^p\varphi^{p-1}\frac{\partial\varphi}{\partial t}\,d\mu\,dt.\end{aligned}$$

In this estimate, the parameters t_1 and t_2 can be chosen as we please.

The final estimate is obtained in two steps. First, by choosing $t_1=0$ and $t_2=\tau$ such that

$$\int_\Omega u^p(x,\tau)\varphi^p(x,\tau)\,d\mu(x)\ge\frac{1}{2}\operatorname*{ess\,sup}_{0<t<T}\int_\Omega u(x,t)^p\varphi(x,t)^p\,d\mu,$$

we obtain

$$\begin{aligned}\left[\int_\Omega u^p\varphi^p\,d\mu\right]_{t=t_1}^{t_2}&=\int_\Omega u(x,\tau)^p\varphi(x,\tau)^p\,d\mu\\&\ge\frac{1}{2}\operatorname*{ess\,sup}_{0<t<T}\int_\Omega u(x,t)^p\varphi(x,t)^p\,d\mu.\end{aligned}$$

By (3.5), this implies that

$$\begin{aligned}\operatorname*{ess\,sup}_{0<t<T}\int_\Omega u^p\varphi^p\,d\mu&\le C(p)\int_0^T\int_\Omega |Du|^{p-1}|D\varphi|\,\varphi^{p-1}u\,d\mu\,dt\\&\quad+C(p)\int_0^T\int_\Omega u^p\varphi^{p-1}\left|\frac{\partial\varphi}{\partial t}\right|\,d\mu\,dt.\end{aligned}$$

On the other hand, by choosing $t_1=0$ and $t_2=T$ in (3.5), we have

$$\begin{aligned}\int_0^T\int_\Omega |Du|^p\varphi^p\,d\mu\,dt&\le p\int_0^T\int_\Omega |Du|^{p-1}|D\varphi|\,\varphi^{p-1}u\,d\mu\,dt\\&\quad+(p-1)\int_0^T\int_\Omega u^p\varphi^{p-1}\left|\frac{\partial\varphi}{\partial t}\right|\,d\mu\,dt.\end{aligned}$$

Consequently, we arrive at

$$\begin{aligned}\int_0^T \int_\Omega |Du|^p \varphi^p \, d\mu \, dt + \operatorname*{ess\,sup}_{0<t<T} \int_\Omega u^p \varphi^p \, d\mu \\ \le C(p) \int_0^T \int_\Omega |Du|^{p-1} |D\varphi| \, \varphi^{p-1} u \, d\mu \, dt \\ + C(p) \int_0^T \int_\Omega u^p \varphi^{p-1} \left| \frac{\partial \varphi}{\partial t} \right| d\mu \, dt.\end{aligned}$$

Finally, Young's inequality implies that

$$\begin{aligned}C(p) \int_0^T \int_\Omega |Du|^{p-1} |D\varphi| \, \varphi^{p-1} u \, d\mu \, dt \\ \le \frac{1}{2} \int_0^T \int_\Omega |Du|^p \varphi^p \, d\mu \, dt + C \int_0^T \int_\Omega u^p |D\varphi|^p \, d\mu \, dt.\end{aligned}$$

The claim follows by absorbing terms. □

3.3. **Structure properties.** The solutions of the doubly nonlinear equation do not have much general structure. However, solutions can be scaled by nonnegative factors and the minimum of two supersolutions is a supersolution and a maximum of two subsolutions is a subsolution. In particular, the truncation of a weak solution solution is either a supersolution or a subsolution depending on whether the truncation is from above or from below.

The following property is useful in proving the Harnack estimates for weak solutions. It gives us a passage from estimates for supersolutions to estimates for subsolutions and vice versa. In this section we work under the additional technical assumption that the solution is strictly bounded away from zero.

Lemma 3.6. *Suppose that $u \ge \varepsilon > 0$ is a supersolution in $\Omega \times (0,T)$. Then $v = u^{-1}$ is a subsolution.*

Proof. Let $\varphi \in C_0^\infty(\Omega \times (0,T))$ with $\varphi \ge 0$. Formally we choose the test function $\eta = u^{2(1-p)}\varphi$. Then

$$D\eta = -2(p-1)u^{1-2p}\varphi Du + u^{2(1-p)} D\varphi$$

and

$$\frac{\partial \eta}{\partial t} = -2(p-1)u^{1-2p}\varphi \frac{\partial u}{\partial t} + u^{2(1-p)} \frac{\partial \varphi}{\partial t}.$$

A substitution in the definition of a weak solution leads to

$$\begin{aligned}0 \leq &- 2(p-1)\int_0^T\int_\Omega |Du|^p u^{1-2p}\varphi\,d\mu\,dt\\ &+\int_0^T\int_\Omega u^{2(1-p)}|Du|^{p-2}Du\cdot D\varphi\,d\mu\,dt\\ &+2(p-1)\int_0^T\int_\Omega u^{-p}\varphi\frac{\partial u}{\partial t}\,d\mu\,dt-\int_0^T\int_\Omega u^{1-p}\frac{\partial\varphi}{\partial t}\,d\mu\,dt.\end{aligned}$$

An integration by parts gives

$$\begin{aligned}\int_0^T\int_\Omega u^{-p}\varphi\frac{\partial u}{\partial t}\,d\mu\,dt &= -\frac{1}{p-1}\int_0^T\int_\Omega \frac{\partial u^{1-p}}{\partial t}\varphi\,d\mu\,dt\\ &=\frac{1}{p-1}\int_0^T\int_\Omega u^{1-p}\frac{\partial\varphi}{\partial t}\,d\mu\,dt.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}0 &\leq \int_0^T\int_\Omega |Du|^{p-2}Du\cdot D\varphi\, u^{2(1-p)}\,d\mu\,dt+\int_0^T\int_\Omega u^{1-p}\frac{\partial\varphi}{\partial t}\,d\mu\,dt\\ &=-\int_0^t\int_\Omega\Big(|Dv|^{p-2}Dv\cdot D\varphi - v^{p-1}\frac{\partial\varphi}{\partial t}\Big)\,d\mu\,dt.\end{aligned}$$

Here we used the fact that $Du=-v^{-2}Dv$. □

Another property that is sometimes used in the proof of the Harnack estimates is that the logarithm of a positive solution is a subsolution to the same equation and hence locally bounded. This property is used in connection with the parabolic BMO and John-Nirenberg lemma. The situation is more delicate for the doubly nonlinear equation.

Lemma 3.7. *Suppose that $u \geq \varepsilon > 0$ is a weak supersolution in $\Omega\times(0,T)$. Then $v=\log u$ is a weak subsolution of the equation*

$$(p-1)\frac{\partial v}{\partial t}-\operatorname{div}(|Dv|^{p-2}Dv)=0.$$

Observe that the equation above differs from the original equation if $p\neq 2$. In fact, it is an equation of the p-parabolic type and the proof of the local boundedness of weak subsolutions is more involved than for the doubly nonlinear equation. This is one of the reasons why we consider an alternative approach without referring to the parabolic John-Nirenberg lemma.

Proof. Let $\varphi\in C_0^\infty(\Omega\times(0,T))$ with $\varphi\geq 0$. Formally we choose the test function $\eta=u^{1-p}\varphi$. Then

$$D\eta=(1-p)u^{-p}\varphi Du+u^{1-p}D\varphi$$

and
$$\frac{\partial \eta}{\partial t} = (1-p)u^{-p}\varphi\frac{\partial u}{\partial t} + u^{1-p}\frac{\partial \varphi}{\partial t}.$$
A substitution in the definition of a weak solution gives
$$\begin{aligned} 0 \leq &(1-p)\int_0^T\int_\Omega |Du|^p u^{-p}\varphi\, d\mu\, dt \\ &+ \int_0^T\int_\Omega u^{1-p}|Du|^{p-2}Du\cdot D\varphi\; d\mu\, dt \\ &-(1-p)\int_0^T\int_\Omega u^{-1}\varphi\frac{\partial u}{\partial t}\, d\mu\, dt - \int_0^T\int_\Omega \frac{\partial \varphi}{\partial t}\, d\mu\, dt. \end{aligned}$$
By throwing away the first nonpositive term and observing that the last term is zero, we have
$$\int_0^T\int_\Omega u^{1-p}|Du|^{p-2}Du\cdot D\varphi\; d\mu\, dt + (p-1)\int_0^T\int_\Omega u^{-1}\varphi\frac{\partial u}{\partial t}\, d\mu\, dt \geq 0.$$
An integration by parts gives
$$\begin{aligned} \int_0^T\int_\Omega u^{-1}\varphi\frac{\partial u}{\partial t}\, d\mu\, dt &= \int_0^T\int_\Omega \Big(\frac{\partial}{\partial t}\log u\Big)\varphi\, d\mu\, dt \\ &= -\int_0^T\int_\Omega \log u\frac{\partial \varphi}{\partial t}\, d\mu\, dt. \end{aligned}$$
On the other hand, we have
$$\int_0^T\int_\Omega u^{1-p}|Du|^{p-2}Du\cdot D\varphi\, d\mu\, dt = \int_0^T\int_\Omega |Dv|^{p-2}Dv\cdot D\varphi\, d\mu\, dt.$$
Therefore, we obtain
$$\int_0^T\int_\Omega |Dv|^{p-2}Dv\cdot D\varphi\, d\mu\, dt - (p-1)\int_0^T\int_\Omega v\frac{\partial \varphi}{\partial t}\, d\mu\, dt \geq 0.$$
This completes the proof. □

3.4. **Quasiminimizers.** There is also a variational approach to the doubly nonlinear equation. Let $K \geq 1$. A nonnegative function u which belongs to $L^p_{\mathrm{loc}}(0,T,;H^{1,p}_{\mathrm{loc}}(\Omega,\mu))$ is a parabolic K-quasiminimizer in $\Omega\times(0,T)$ if for every $\Omega' \Subset \Omega$ and $0<t_1<t_2<T$ we have
$$\begin{aligned} &\int_{t_1}^{t_2}\int_{\Omega'} u^{p-1}\frac{\partial \varphi}{\partial t}\, d\mu\, dt + \frac{1}{p}\int_{t_1}^{t_2}\int_{\Omega'} |Du|^p\, d\mu\, dt \\ &\leq \frac{K}{p}\int_{t_1}^{t_2}\int_{\Omega'} |Du + D\varphi|^p\, d\mu\, dt \end{aligned}$$
for all $\varphi \in C_0^\infty(\Omega'\times(t_1,t_2))$. Parabolic quasiminimizers have been studied in [W] and [Z].

By the following result the class of quasiminimizers is precisely the same class as the weak solutions when $K = 1$.

Theorem 3.8. *Every weak solution of the doubly nonlinear equation is a K-quasiminimizer with $K = 1$ and, conversely, every K-quasiminimizer with $K = 1$ is a weak solution of the doubly nonlinear equation.*

Proof. First assume that u is a weak solution of the doubly nonlinear equation, let $\Omega' \Subset \Omega$, $0 < t_1 < t_2 < T$ and $\varphi \in C_0^\infty(\Omega' \times (t_1, t_2))$. Then

$$\int_{t_1}^{t_2} \int_{\Omega'} |Du|^p \, d\mu \, dt = \int_{t_1}^{t_2} \int_{\Omega'} |Du|^{p-2} Du \cdot Du \, d\mu \, dt$$
$$= \int_{t_1}^{t_2} \int_{\Omega'} |Du|^{p-2} Du \cdot (Du + D\varphi) \, d\mu \, dt - \int_{t_1}^{t_2} \int_{\Omega'} u^{p-1} \frac{\partial \varphi}{\partial t} \, d\mu \, dt,$$

from which it follows that

$$\int_{t_1}^{t_2} \int_{\Omega'} u^{p-1} \frac{\partial \varphi}{\partial t} \, d\mu \, dt + \int_{t_1}^{t_2} \int_{\Omega'} |Du|^p \, dx \, dt$$
$$\le \int_{t_1}^{t_2} \int_{\Omega'} |Du|^{p-1} |Du + D\varphi| \, d\mu \, dt$$
$$\le \Big(1 - \frac{1}{p}\Big) \int_{t_1}^{t_2} \int_{\Omega'} |Du|^p \, d\mu \, dt + \frac{1}{p} \int_{t_1}^{t_2} \int_{\Omega'} |Du + D\varphi|^p \, d\mu \, dt.$$

In the last step we used Young's inequality. By absorbing terms, we seee that u is a K-quasiminimizer with $K = 1$

On the other hand, if u is a K-quasiminimizer with $K = 1$, $\varphi \in C_0^\infty(\Omega \times (0, T))$ and $\varepsilon > 0$, then $\varepsilon\varphi$ belongs to $C_0^\infty(\Omega' \times (t_1, t_2))$ for some $\Omega' \Subset \Omega$ and $0 < t_1 < t_2 < T$. By the quasiminimizing property, we have

$$\varepsilon \int_{t_1}^{t_2} \int_{\Omega'} u^{p-1} \frac{\partial \varphi}{\partial t} \, d\mu \, dt + \frac{1}{p} \int_{t_1}^{t_2} \int_{\Omega'} |Du|^p \, d\mu \, dt$$
$$\le \frac{1}{p} \int_{t_1}^{t_2} \int_{\Omega'} |Du + \varepsilon D\varphi|^p \, d\mu \, dt.$$

This implies that

$$\int_{t_1}^{t_2} \int_{\Omega'} u^{p-1} \frac{\partial \varphi}{\partial t} \, d\mu \, dt$$
$$+ \frac{1}{p} \int_{t_1}^{t_2} \int_{\Omega'} \frac{1}{\varepsilon} (|Du|^p - |Du + \varepsilon D\varphi|^p) \, d\mu \, dt \le 0.$$

Since

$$\frac{1}{\varepsilon} \big(|Du|^p - |Du + \varepsilon D\varphi|^p\big) \to -p |Du|^{p-1} \frac{Du}{|Du|} \cdot D\varphi$$

as $\varepsilon \to 0$, by the dominated convergence theorem we arrive at

$$\int_0^T \int_\Omega u^{p-1} \frac{\partial \varphi}{\partial t} \, d\mu \, dt - \int_0^T \int_\Omega |Du|^{p-2} Du \cdot D\varphi \, d\mu \, dt \le 0.$$

The reverse inequality follows by choosing $-\varepsilon\varphi$ as the test function.
□

Thus if $K = 1$ every quasiminimizer is a weak solution to a partial differential equation. In contrast, when $K > 1$, then being a quasiminimizer is not a local property. This can be easily seen already in the elliptic case by one dimensional examples. Indeed, consider a function which is defined on the positive axis and assumes the value $1/i$ on the interval $(i-1, i]$ for $i = 1, 2, \dots$ This function is an elliptic quasiminimizer with $p = 2$ when tested on intervals of lenght less than two. However, it fails to be a quasiminimizer on the whole positive axis.

The advantage of the notion of a quasiminimizer is that it makes sense also in metric spaces and this enables us to develop the theory of nonlinear parabolic partial differential equations also in the metric context, we refer to [KMPP], [MM] and [MS].

4. Regularity results

4.1. Harnack's estimates. A natural geometry that respects the scaling is that r in the spatial direction corresponds to r^p in the time direction.

Let $0 < \sigma < 1$ and $\tau \in \mathbb{R}$. We denote

$$Q = B(x, r) \times (\tau - r^p, \tau + r^p),$$

$$\sigma Q^+ = B(x, \sigma r) \times \left(\tau + \frac{1}{2} r^p - \frac{1}{2}(\sigma r)^p, \tau + \frac{1}{2} r^p + \frac{1}{2}(\sigma r)^p\right)$$

and

$$\sigma Q^- = B(x, \sigma r) \times \left(\tau - \frac{1}{2} r^p - \frac{1}{2}(\sigma r)^p, \tau - \frac{1}{2} r^p + \frac{1}{2}(\sigma r)^p\right).$$

The main result of [KK] is the following scale and location invariant version of the parabolic Harnack estimate.

Theorem 4.1. *Let $1 < p < \infty$ and assume that the measure μ is doubling and supports a $(1,p)$-Poincaré inequality. Let u be a nonnegative weak solution and let $0 < \sigma < 1$. Then we have*

$$\operatorname*{ess\,sup}_{\sigma Q^-} u \le C \operatorname*{ess\,inf}_{\sigma Q^+} u,$$

where the constant C depends only on p, D_0, P_0 and σ.

The original proof with the Lebesgue measure can be found in [T]. For different approaches we refer to [GV], [FS] and [V1]. See also [DGV2], [K] and [Su] for the corresponding results for the p-parabolic equation.

4.2. **Comments on the proof.** The proof of Harnack's inequality is based on the Moser iteration scheme, which in turn is based on a successive use of Caccioppoli type energy estimates and the parabolic Sobolev inequality. In estimates, we may have quantities which are not a priori finite. Nevertheless, we can make our calculations with truncated functions and we obtain the result by passing the level of truncation to infinity. Finally, the estimates for super and subsolutions are glued together by an abstract real analysis lemma of Bombieri in [BG] and [B]. See also Lemma 2.2.6 in [SC1]. This avoids the delicate problems with the parabolic John-Nirenberg lemma.

4.3. **Local Hölder continuity.** Harnack's inequality does not immediately imply local Hölder continuity, since we cannot add constants. Indeed, consider a one dimensional example of a function, which is constant one on the negative side and constant two on the nonnegative side. Clearly, it satisfies Harnack's inequality, but it fails to be continuous at the origin.

The papers [KSU] and [KLSU] give a Hölder continuity proof for nonnegative solutions of the doubly nonlinear equation. Their main result is the following.

Theorem 4.2. *Let* $1 < p < \infty$ *and assume that the measure is doubling and supports a* $(1, p)$*-Poincaré inequality. Then every nonnegative weak solution* u *of the doubly nonlinear equation is locally Hölder continuous, in symbols,*

$$u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega \times (0, T)).$$

When $p = 2$, then the local Hölder continuity follows from Harnack estimates, since we can add constants to solutions, but the case $p \neq 2$ seems too be more challenging.

4.4. **Comments on the proof.** It is somewhat unexpected that there are several difficulties that are not present in the case of the p-parabolic equation. The original proof for the p-parabolic equation in [DB] introduces an intrinsic scaling, which absorbs the inhomogenuity of the equation. In this case, the geometry depends in a delicate way on the solution itself. The main idea of the proof is to show a reduction of oscillation by considering two alternatives. This means that the oscillation of the solution, in the intrinsic space-time cylinder, is reduced by a controlled factor when the cylinder is shrinked. The proof gives a measure estimate for distribution sets, which after a suitable iteration implies that if the set where the solution is small or large, occupies small enough portion of a subcylinder, then the solution is large or small, respectively, in a smaller cylinder. Finally, the local Hölder continuity follows from an iterative argument.

The doubly nonlinear equation has a different character compared to the p-parabolic equation. Indeed, it seems to have a dichotomic behaviour. In large scales, when the oscillation of the solution is large, the equation behaves like the heat equation. In this case, the scaling property and Harnack's inequality dominate and the reduction of oscillation follows easily. On the other hand, in small scales the oscillation is small. Consequently, the nonlinear term containing the time derivative formally looks like

$$\frac{\partial(u^{p-1})}{\partial t} = (p-1)u^{p-2}\frac{\partial u}{\partial t} \approx C\,\frac{\partial u}{\partial t}.$$

This indicates a p-parabolic type behavior and in this case DiBenedetto's approach can be applied. From the technical point of view, the nonlinearity of the term containing the time derivative causes problems in proving Caccioppoli type energy estimates. This problem has been settled in [KSU] by introducing an integral term which absorbs the nonlinearity. A similar idea has been previously used, for example, in connection with the porous medium equation.

The next step is to show that the information can be forwarded in time. If the infimum is small, the fact that in Harnack's inequality the infimum is taken at a later time than the supremum provides us a natural way to forward information in time. In the remaing case, after a suitable energy estimate and a logarithmic lemma have been proved the claim follows DiBenedetto's argument.

Recently, new approaches have been found for the regularity argument, see in [GSV]. These ideas are based on methods which were developed for Harnack estimates in [DGV2]. It would be interesting to know whether these new ideas would provide a more direct way to obtain regularity results also for the doubly nonlinear equation.

4.5. **Higher regularity.** By the elliptic regularity theory, the gradient of a weak solution of the p-Laplace equation is locally Hölder continuous. In general, this is the highest degree of regularity that we can expect also in the parabolic case. The first step towards this goal is to show that the gradient of a weak solution is locally bounded and thus the solution is locally Lipschitz continuous in the space variable. This is the main result of [Si1]. See also [Si2].

Theorem 4.3. *Let $1 < p < \infty$ and assume that the measure is doubling and supports a $(1,p)$-Poincaré inequality. Then the gradient of a positive weak solution u of the doubly nonlinear equation is locally bounded in the space variable, in symbols,*

$$u \in L^p_{\mathrm{loc}}(0,T,H^{1,\infty}_{\mathrm{loc}}(\Omega,\mu)).$$

In particular, the function u is locally Lipschitz continuous in the space variable.

4.6. **Comments on the proof.** For the p-parabolic equation, the local Hölder continuity of the gradient has been proved by DiBenedetto and Friedman in [DF1]. See also [DF2] and [DF3]. Their argument is based on the differentiation of the equation. After this, they use standard techniques to prove Caccioppoli inequalities for the differentiated equation and employ Moser's iteration to show that the gradient of the solution is locally integrable to every positive power. Finally, they conclude the boundedness of the gradient by a De Giorgi type argument.

The difficulty with the doubly nonlinear equation comes again from the nonlinearity in the time derivative term. More precisely, the differentiated equation formally looks like

$$\begin{aligned}&\frac{\partial}{\partial t}\left((p-1)u^{p-2}u_{x_i}\right)\\ &\qquad - \operatorname{div}\Big(|Du|^{p-2}Du_{x_i} + \frac{\partial}{\partial x_i}\left(|Du|^{p-2}\right)Du\Big) = 0,\\ &\qquad i = 1,2,\dots,n.\end{aligned}$$

Observe, that there is an extra factor u^{p-2} in front of the time derivative compared to the p-parabolic equation. However, this factor can be dealt with a freezing argument.

The next step is to show that the gradient is locally integrable to any positive power. In the final step, DiBenedetto and Friedman use a De Giorgi type argument to conclude the local boundedness of the gradient. This point has been simplified in [Si1] by a Moser type iteration scheme. It was long thought that the Moser iteration cannot be used for nonhomogeneous parabolic equations, like the equation for the gradient. However, a careful analysis of Moser's method shows that the constants do not blow up in the iteration procedure. Otherwise the argument in [Si1] follows the same lines as in [DF1].

The drawback of the argument in [Si1] is that it is uses intrinsic scaling related to the p-parabolic equation. As a consequence, the final estimate is nonhomogeneous although the original equation is homogeneous with respect to scaling. It would be interesting to find a more direct argument which would give homogeneous estimates also for the gradient.

References

[AS] D.G. Aronsson and J. Serrin, Local behaviour of solutions of quasilinear parabolic equations, Arch. Rat. Mech. Anal 25 (1967), 81–122

[BCLS] D. Bakry, T. Coulhon, M. Ledoux and L. Saloff-Coste, Sobolev inequalities in disguise, Indiana Univ. Math. J. 44 (1995), 1033–1074

[BBK] M.T. Barlow, R.F. Bass and T. Kumagai, Stability of parabolic Harnack inequalities on metric measure spaces, J. Math. Soc. Japan 58 (2006), no. 2, 485–519.

[BB] J. Björn, and A. Björn, Nonlinear potential theory on metric spaces, in preparation.

[B] E. Bombieri, Theory of minimal surfaces and a counterexample to the bernstein conjecture in high dimension, Mimeographed Notes of Lectures Held at Courant Institute, New York University (1970)

[BG] E. Bombieri and E. Giusti, Harnack's inequality for elliptic differential equations on minimal surfaces, Invent. Math. 15 (1972), 24–46

[BM] M. Biroli and U. Mosco, Saint-Venant type principle for Dirichlet forms on discontinuous media, Ann. Mat. Pura Appl. 169 (1995), 125–181

[CF] F. Chiarenza and M. Frasca, A note on weighted Sobolev inequality, Proc. Amer. Math. Soc. 93 (1985), 703–704

[CS] F. Chiarenza and R. Serapioni, A Harnack inequality for degenerate parabolic equations, Comm. Partial Differential Equations 9(8) (1984), 719–749

[D] T. Delmotte, Parabolic Harnack inequality and estimates of markov chains on graphs, Rev. Math, Iberoamericana 15 (1999), 181–232.

[DB] E. DiBenedetto, Degenerate parabolic equations, Springer-Verlag (1993)

[DGV1] E. DiBenedetto, U. Gianazza and V. Vespri, Degenerate and singular parabolic equations, in preparation.

[DGV2] E. DiBenedetto, U. Gianazza and V. Vespri, Harnack estimates for quasilinear degenerate parabolic differential equations, Acta Math. 200 (2008), no. 2, 181–209

[DGV3] E. DiBenedetto, U. Gianazza and V. Vespri, A geometric approach to the Hlder continuity of solutions to certain singular parabolic partial differential equations, preprint (2010)

[DF1] E. DiBenedetto and A. Friedman, Regularity of solutions of nonlinear degenerate parabolic systems, J. Reine Angew. Math. 349 (1984), 83–128

[DF2] E. DiBenedetto and A. Friedman, Hölder estimates for nonlinear degenerate parabolic systems, J. Reine Angew. Math. 357 (1985), 1–22

[DF3] E. DiBenedetto and A. Friedman, Hölder estimates for nonlinear degenerate parabolic systems, J. Reine Angew. Math. 363 (1985), 217–220

[DUV] E. DiBenedetto, J.M. Urbano and V. Vespri, Current issues on singular and degenerate evolution equations, Handbook of differential equations, Elsevier (2004), 169–286

[FG] E. Fabes and N. Garofalo, Parabolic B.M.O. and Harnack's inequality, Proc. Amer. Math. Soc. 50 (1985), no. 1, 63–69

[FKS] E. Fabes, C. Kenig and R. Serapioni, The local regularity of solutions of degenerate elliptic equations, Comm. Partial Differential Equations 7 (1982), no. 1, 77–116

[FS] S. Fornaro and M. Sosio, Intrinsic Harnack estimates for some doubly nonlinear degenerate parabolic equations, Adv. Differential Equations 13 (2008), no. 1-2, 139–168

[GV] U. Gianazza and V. Vespri, A Harnack inequality for solutions of doubly nonlinear parabolic equations, J. Appl. Funct. Anal. 1 (2006), no. 3, 271–284

[GSV] U. Gianazza, M. Surnachev and V. Vespri, A new proof of the Hölder continuity of solutions to p-Laplace type parabolic equations, preprint (2010)

[G] A. Grigor'yan, The heat equation on non-compact Riemannian manifolds, Matem. Sbornik 182 (1991), 55–87, Engl. Transl. Math. USSR Sb. 72 (1992), 47–77

[GW1] C.E. Gutierrez and R.L. Wheeden, Mean value and Harnack inequalities for degenerate parabolic equations, Colloq. Math. 60/61 (1) (1990), 157–194.

[GW2] C.E. Gutierrez and R.L. Wheeden, Harnack's inequality for degenerate parabolic equations, Comm. Partial Differential Equations 16 (485) (1991), 745–770

[HK] P. Hajłasz and P. Koskela, Sobolev met Poincaré, Mem. Amer. Math. Soc. 688 (2000)

[H] J. Heinonen, Lectures on analysis on metric spaces, Universitext, Springer-Verlag, New York (2001)

[HKM] J. Heinonen, T. Kilpeläinen and O. Martio, Nonlinear potential theory of degenerate elliptic equations, Oxford University Press, Oxford (1993)

[Is] K. Ishige, On the existence of solutoins of the Cauchy problem for a doubly nonlinear parabolic equation, SIAM J. Math. Anal. 27 (1996) No. 5,, 1235–1260

[Iv] A.V. Ivanov, Hölder estimates for equations of fast diffusion type, Algebra i Analiz, 6(4) (1994), 101–142

[KZ] S. Keith and X. Zhong, The Poincaré inequality is an open ended condition, Ann. of Math. (2) 167 (2008), no. 2, 575–599

[KKKP] J. Kinnunen, R. Korte, T. Kuusi and M. Parviainen, Nonlinear parabolic capacity and the singular set of a superparabolic function, in preparation

[KK] J. Kinnunen and T. Kuusi, Local behaviour of solutions to doubly nonlinear parabolic equations. Math. Ann. 337(3) (2007), 705–728

[KMPP] J. Kinnunen, M. Miranda, F. Paronetto and M. Parviainen, Regularity of parabolic quasiminimizers in metric spaces, in preparation.

[KS] J. Kinnunen and N. Shanmugalingam, Regularity of quasi-minimizers on metric spaces, Manuscripta Math. 105 (2001), 401–423

[K] T. Kuusi, Harnack estimates for weak supersolutions to nonlinear degenerate parabolic equations, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 7 (2008), no. 4, 673–716

[KLSU] T. Kuusi, R. Laleoglu, J. Siljander and J. M. Urbano, Regularity for doubly nonlinear parabolic equations: the singular case, in preparation

[KSU] T. Kuusi, J. Siljander and J. M. Urbano, Local Hölder continuity for doubly nonlinear parabolic equations, preprint (2010)

[LN] J.L. Lewis and Kaj Nyström, Boundary behaviour and the martin boundary problem for p-harmonic functions in Lipschitz domains, to appear in Ann. of Math.

[MM] N. Marola and M. Masson, The Harnack inequality for parabolic quasiminimizers in metric spaces, in preparation

[MS] M. Masson and J. Siljander, Hölder continuity for parabolic Q-minima in metric spaces, in preparation.

[M1] J. Moser, A Harnack inequality for parabolic differential equations, Comm. Pure Appl. Math. 17 (1964), 101–134, and correction in Comm. Pure Appl. Math. 20 (1967), 231–236

[M2] J. Moser, On a pointwise estimate for parabolic equations, Comm. Pure Appl. Math. 24 (1971), 727–740

[PV] M. M. Porzio and V. Vespri, Hölder estimates for local solutions of some doubly nonlinear degenerate parabolic equations, J. Differential Equations, 103(1) (1993), 146–178

[SC1] L. Saloff-Coste, Aspects of Sobolev-type inequalities, London Mathematical Society Lecture Note Series 289, Cambridge University Press (2002)

[SC2] L. Saloff-Coste, A note on Poincaré, Sobolev and Harnack inequalities, Duke Math. J. 65 (1992), IMRN 2, 27–38

[Se] S. Semmes, Finding curves on general spaces through quantitative topology, with applications for Sobolev and Poincaré inequalities, Selecta Math. (N.S.), 2 (1996), 155–296

[Si1] J. Siljander, Boundedness of the gradient for a doubly nonlinear parabolic equation, J. Math. Anal. Appl. 371 (2010), 158–167

[Si2] J. Siljander, Regularity for degenerate nonlinear parabolic partial differential equations, PhD thesis, Aalto University, School of Science and Technology (2010)

[St1] K.-T. Sturm, Analysis on local Dirichlet spaces II, Gaussian upper bounds for the fundamental solutions of parabolic equations, Osaka J. math, 32 (1995), 275–312

[St2] K.-T. Sturm, Analysis on local Dirichlet spaces III, The parabolic Harnack inequality, J. Math. Pures Appl. 75 (1996) no.9, 273-297

[Su] M. Surnachev, A Harnack inequality for weighted degenerate parabolic equations, J. Differential Equations 248 (2010), no. 8, 2092–2129

[T] N.S. Trudinger, Pointwise estimates and quasilinear parabolic equations, Comm. Pure Appl. Math. 21 (1968), 205–226

[V1] V. Vespri, Harnack type inequalities for solutions of certain doubly nonlinear parabolic equations, J. Math. Anal. Appl. 181 (1994), no. 1, 104–13

[V2] V. Vespri, On the local behaviour of solutions of a certain class of doubly nonlinear parabolic equations, Manuscripta Math. 75 (1992), 65–80

[W] W. Wieser, Parabolic Q-minima and minimal solutions to variational flow, Manuscripta Math. 59 (1987), no. 1, 63–107

[WZYL] Z. Wu, J. Zhao, J. Yin and H.Li, Nonlinear diffusion equations, World Scientific (2001)

[Z] S. Zhou, Parabolic Q-minima and their application, J. Partial Differential Equations, 7(4) (1994), 289–322

Address:
Aalto University, Department of Mathematics, P.O. Box 11100, FI-00076 AALTO, Finland.
E-mail: `juha.kinnunen@tkk.fi`