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Existence of eigenvalues and eigenfunctions for radially symmetric fully nonlinear elliptic operators

Norihisa Ikoma (生駒 典久) [1]

Department of Pure and Applied Mathematics,
School of Science and Engineering, Waseda University.

1 Introduction

This note is based on a joint work [13] with H. Ishii and we take a slightly different approach in the radial case from the one in [13]. See also the comments after Theorem 1.2.

In this note, we consider the eigenvalue problem for fully nonlinear elliptic operator F:

$$(1) \qquad \begin{cases} F(D^2u, Du, u, x) + \mu u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Here $\Omega \subset \mathbb{R}^N$ is an open interval (a,b) with $-\infty < a < b < \infty$ when $N = 1$, or an open ball $B_R = B_R(0)$ when $N \geq 2$, $u : \bar{\Omega} \to \mathbb{R}$ and $\mu \in \mathbb{R}$ represent the unknown function (eigenfunction) and constant (eigenvalue), respectively, and $F : \mathbb{S}^N \times \mathbb{R}^N \times \mathbb{R} \times \Omega \to \mathbb{R}$ is a given function, where $\mathbb{S}^N$ denotes the space of real symmetric $N \times N$ matrices.

The study of the eigenvalue problem for fully nonlinear elliptic operator goes back to the work of P.-L. Lions [16] and for the developments we refer to [1, 4, 5, 14, 17, 20] and to [2, 8, 9] for some earlier related works.

Recently, Busca, Esteban and Quaas [5] and Esteban, Felmer and Quaas [11] showed the existence of higher eigenvalues and of the corresponding eigenfunctions in the one-dimensional or the radially symmetric problem. In this note we extend the results of [11] into the L^q framework.

Before giving our assumptions (F1)-(F4) on the function F, we introduce the Pucci operators $P^{\pm}$. Given constants $\lambda \in (0, \infty)$ and $\Lambda \in [\lambda, \infty]$, $P^{\pm}$ denote the Pucci operators defined as the functions on $\mathbb{S}^N$ given, respectively, by $P^+(M) \equiv P^+(M; \lambda, \Lambda) = \sup\{ \operatorname{tr} AM : A \in \mathbb{S}^N, \lambda I_N \leq A \leq \Lambda I_N\}$ and $P^-(M) = -P^+(-M)$, where I_N denotes the $N \times N$ identity matrix and the relation, $X \leq Y$, is the standard order relation between $X, Y \in \mathbb{S}^N$. We remark that in the case $\Lambda = \infty$, $P^+(M) = \infty$ if $M \not\leq 0$ and $P^+(M) = \lambda \sum_{j=1}^N \nu_j$ if $M \leq 0$.

(F1) The function $F : \mathbb{S}^N \times \mathbb{R}^N \times \mathbb{R} \times \Omega \to \mathbb{R}$ is a Carathéodory function, i.e., the function $x \mapsto F(M, p, u, x)$ is measurable for any $(M, p, u) \in \mathbb{S}^N \times \mathbb{R}^{N+1}$ and the function $(M, p, u) \mapsto F(M, p, u, x)$ is continuous for a.a. $x \in \Omega$.

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(F2) There exist constants $\lambda \in (0, \infty)$, $\Lambda \in [\lambda, \infty]$, $q \in [1, \infty]$ and functions β, $\gamma \in L^q(\Omega)$ such that

$$\begin{aligned} &F(M_1, p_1, u_1, x) - F(M_2, p_2, u_2, x) \\ &\leq P^+(M_1 - M_2) + \beta(x)|p_1 - p_2| + \gamma(x)|u_1 - u_2| \end{aligned}$$

for all $(M_1, p_1, u_1), (M_2, p_2, u_2) \in \mathbb{S}^N \times \mathbb{R}^{N+1}$ and a.a. $x \in \Omega$.

(F3) $F(tM, tp, tu, x) = tF(M, p, u, x)$ for all $t \geq 0$, all $(M, p, u) \in \mathbb{S}^N \times \mathbb{R}^{N+1}$ and a.a. $x \in \Omega$.

Here we remark that if $\Lambda = \infty$ and $M_1 \not\leq M_2$, then the inequality in condition (F2) is trivially satisfied since $P^+(M_1 - M_2) = \infty$.

The next condition concerns the radial symmetry in the multi-dimensional case.

(F4) The function F is radially symmetric in the sense that for any $(m, l, q, u) \in \mathbb{R}^4$ and a.a. $r \in (0, R)$, the function

$$\omega \mapsto F(m\omega \otimes \omega + l(I_N - \omega \otimes \omega), q\omega, u, r\omega)$$

is constant on the unit sphere $S^{N-1} \subset \mathbb{R}^N$. Here and henceforth $x \otimes x$ denotes the matrix in $\mathbb{S}^N$ with the (i, j) entry given by $x_i x_j$ if $x \in \mathbb{R}^N$.

We study the eigenvalue problem (1) in the Sobolev space $W^{2,q}(\Omega)$. For any pair $(\mu, \varphi) \in \mathbb{R} \times (W^{2,1}(\Omega) \cap W_0^{1,1}(\Omega))$ which satisfies (1) in the almost everywhere sense, we call μ and ϕ an *eigenvalue* and *eigenfunction* of (1), respectively, provided $\varphi(x) \not\equiv 0$. We call such a pair an *eigenpair* of (1).

We state our main results in this note.

Theorem 1.1. *Let* $N = 1$ *and* $\Omega = (a, b)$, *and assume that* (F1), (F2) *with* $\Lambda = \infty$, *and* (F3) *hold. Then*

(i) *For any* $n \in \mathbb{N}$, *there exist eigenpairs* $(\mu_n^\pm, \varphi_n^\pm) \in \mathbb{R} \times W^{2,q}(a, b)$ *of* (1) *and sequences* $(x_{n,j}^\pm)_{j=0}^n \subset [a, b]$ *such that*

$$\begin{cases} a = x_{n,0}^\pm < x_{n,1}^\pm < \cdots < x_{n,n}^\pm = b, \\ (-1)^{j-1}\varphi_n^+(x) > 0 \text{ in } (x_{n,j-1}^+, x_{n,j}^+) \text{ for } j = 1, \dots, n, \\ (-1)^j \varphi_n^-(x) > 0 \text{ in } (x_{n,j-1}^-, x_{n,j}^-) \text{ for } j = 1, \dots, n. \end{cases}$$

(ii) *The eigenpairs* $\{(\mu_n^\pm, \varphi_n^\pm)\}_{n=1}^\infty$ *are complete in the sense that for any eigenpair* $(\mu, \varphi) \in \mathbb{R} \times W^{2,q}(a, b)$ *of* (1), *there exist* $n \in \mathbb{N}$ *and* $\theta > 0$ *such that either* $(\mu, \varphi) = (\mu_n^+, \theta\varphi_n^+)$ *or* $(\mu, \varphi) = (\mu_n^-, \theta\varphi_n^-)$ *holds.*

For $q \in [1, \infty]$, let $W_r^{2,q}(0, R)$ denote the space of those functions $\varphi \in W^{2,q}(B_R)$ which are radially symmetric. We may identify any function f in $W_r^{2,q}(0, R)$ with a function g on $[0, R]$ such that $f(x) = g(|x|)$ for a.a. $x \in B_R$ and we employ the standard abuse of notation: $f(x) = f(|x|)$ for $x \in B_R$. We set $\lambda_* = \lambda/\Lambda$ and $q_* = N/(\lambda_* N + 1 - \lambda_*)$ if $\Lambda < \infty$. Note that $0 < \lambda_* \leq 1$ and $q_* \in [1, N)$.

Theorem 1.2. *Let $N \geq 2$, $\Omega = B_R$, and assume that* (F1)-(F4) *with $\Lambda < \infty$ hold. Assume also $q \in (\max\{N/2, q_*\}, \infty]$ and $\beta \in L^N(B_R)$ if $q < N$. Then:*
(i) *For each $n \in \mathbb{N}$, there exist eigenpairs $(\mu_n^\pm, \varphi_n^\pm) \in \mathbb{R} \times W_r^{2,q}(0,R)$ of* (1) *and sequences $(r_{n,j}^\pm)_{j=0}^n \subset [0,R]$ such that*

$$\begin{cases} 0 = r_{0,n}^\pm < r_{n,1}^\pm < \cdots < r_{n,n}^\pm = R, \\ (-1)^{j-1}\varphi_n^+(r) > 0 \quad \text{in } (r_{n,j-1}^+, r_{n,j}^+) \text{ for } j = 1, \ldots, n, \\ (-1)^j \varphi_n^-(r) > 0 \quad \text{in } (r_{n,j-1}^-, r_{n,j}^-) \text{ for } j = 1, \ldots, n, \\ \varphi_n^+(0) > 0 > \varphi_n^-(0). \end{cases}$$

(ii) *The eigenpairs $\{(\mu_n^\pm, \varphi_n^\pm)\}_{n=1}^\infty$ are complete in the sense that for any eigenpair $(\mu, \varphi) \in \mathbb{R} \times W_r^{2,q}(0,R)$ of* (1)*, there exist $n \in \mathbb{N}$ and $\theta > 0$ such that either $(\mu, \varphi) = (\mu_n^+, \theta\varphi_n^+)$ or $(\mu, \varphi) = (\mu_n^-, \theta\varphi_n^-)$ is valid.*

In this note we only treat the case where $N \geq 2$, i.e., Theorem 1.2. As mentioned before, we will give a slightly different approach from the one in [13]. In [13], we take the following approach. For any $\varepsilon > 0$ and $n \geq 1$, first we show the existence of solutions of

$$\begin{cases} F(D^2 u_{n,\varepsilon}^\pm, Du_{n,\varepsilon}^\pm, u_{n,\varepsilon}^\pm, x) + \mu_\varepsilon^\pm u_{n,\varepsilon}^\pm = 0 \quad \text{in } A(\varepsilon, R), \\ u_{n,\varepsilon}^\pm \in W_r^{2,q}(\varepsilon, R), \ (u_{n,\varepsilon}^\pm)'(\varepsilon) = 0, \ u_{n,\varepsilon}^\pm(R) = 0, \ \pm u_{n,\varepsilon}^\pm(\varepsilon) > 0 \end{cases}$$

which have $n-1$ zeroes in $[\varepsilon, R)$. Here $A(\varepsilon, R) := \{x \in \mathbb{R}^N : \ \varepsilon < |x| < R\}$ and $W_r^{2,q}(\varepsilon, R)$ denotes the set consisting of all radial functions in $W^{2,q}(A(\varepsilon, R))$. Then let $\varepsilon \to 0$ and observe that we can extract a subsequence whose limit is an eigenpair of (1) with the desired properties.

However, in this note, we will show the existence of eigenpairs through the unique solvability of

$$F(D^2u, Du, u, x) - \kappa u + f(x) = 0 \quad \text{in } B_R(0), \ u \in W_r^{2,q}(0,R) \cap W_0^{1,q}(B_R(0)),$$

for some $\kappa \in \mathbb{R}$ and any radial function $f \in L^q(B_R(0))$. See, for instance, sections 5 and 6 (Theorems 5.1 and 6.1).

Lastly, we give a remark about the condition on β in Theorem 1.2. Our requirement on β in Theorem 1.2 is only that $\beta \in L^q(B_R) \cap L^N(B_R)$. This condition seems relatively sharp from the known results in a priori estimates of solutions to (1). We refer to [6, 7, 10, 12, 15, 18]. See also Proposition 3.6 in this connection.

2 Preliminaries

Throughout this note, we suppose $N \geq 2$. First, we introduce the notations. For $0 \leq a < b \leq R$ and $q \in [1, \infty]$,

$$A(a,b) := \{x \in \mathbb{R}^N : a < |x| < b\} \quad \text{if } a > 0 \quad \text{and} \quad A(0,b) := B_b(0),$$

$$L_r^q(a,b) := \{u \in L^q(A(a,b)) : u \text{ is radial}\},$$

$$W_r^{2,q}(a,b) := \{u \in W^{2,q}(A(a,b)) : u \text{ is radial}\},$$

$$\|u\|_{L_r^q(a,b)}^q := \int_a^b r^{N-1}|u(r)|^q dr \quad \text{if } q \in [1,\infty) \quad \text{and} \quad \|u\|_{L_r^\infty(a,b)} := \|u\|_{L^\infty(a,b)}.$$

Note that $C_r^\infty(\overline{A(a,b)}) := \{u \in C^\infty(\overline{A(a,b)}) : u \text{ is radial}\}$ is dense in $W_r^{2,q}(a,b)$.

Let u be a smooth radial function and we identify $u(x)$ with $u(|x|)$. Then it is easy to see

$$(2) \qquad Du(x) = u'(|x|)\frac{x}{|x|}, \quad D^2u(x) = u''(|x|)P_x + \frac{u'(|x|)}{|x|}(I_N - P_x) \quad \text{for } x \neq 0$$

where P_x denotes the matrix $x \otimes x/|x|^2 = (x_ix_j/|x|^2)$ which represents the orthogonal projection in $\mathbb{R}^N$ onto the one-dimensional space spanned by the vector x.

Next, we introduce a norm in $W_r^{2,q}(a,b)$ which is equivalent to the usual norm $\|\cdot\|_{W^{2,q}(A(a,b))}$.

Lemma 2.1. *The following norm is equivalent to* $\|\cdot\|_{W^{2,q}(A(a,b))}$ *in* $W_r^{2,q}(a,b)$*:*

$$\|u\|_{W_r^{2,q}(a,b)} := \|u\|_{L_r^q(a,b)} + \|u'/r\|_{L_r^q(a,b)} + \|u''\|_{L_r^q(a,b)}.$$

Proof. First, noting that $C_r^\infty(\overline{A(a,b)})$ is dense in $W_r^{2,q}(a,b)$, (2) holds for any $u \in W_r^{2,q}(a,b)$ and a.a. $x \in A(a,b)$. On the other hand, we have

$$|D^2u(x)| := \left(\sum_{i,j}\left|\frac{\partial^2 u}{\partial x_i \partial x_j}(x)\right|^2\right)^{1/2} = \left(|u''(|x|)|^2 + (N-1)\frac{|u'(|x|)|^2}{|x|^2}\right)^{1/2}.$$

Thus it is easy to see that $\|\cdot\|_{W_r^{2,q}(a,b)}$ and $\|\cdot\|_{W^{2,q}(A(a,b))}$ is equivalent. ☐

In the rest of this note, we use $\|\cdot\|_{W_r^{2,q}(a,b)}$ instead of the usual norm $\|\cdot\|_{W^{2,q}(A(a,b))}$.

Next, we rewrite (1) in the radial form and give some remarks. Assume that F satisfies (F1), (F2) with $\Lambda < \infty$ and (F4). We fix a point $\omega_0 \in S^{N-1}$ and define the function $\mathcal{F} : \mathbb{R}^4 \times (0,R) \to \mathbb{R}$ by

$$\mathcal{F}(m,l,p,u,r) := F(m\omega_0 \otimes \omega_0 + (I_N - \omega_0 \otimes \omega_0)l, p\omega_0, u, r\omega_0).$$

We write $\mathcal{F}[u](r)$ for $\mathcal{F}(u''(r), u'(r)/r, u'(r), u(r), r)$. Thanks to (F4) and (2), (1) is equivalent to

$$(3) \qquad \mathcal{F}[u] + \mu u = 0 \quad \text{a.e. in } (0,R), \ u \in W_r^{2,q}(0,R), \ u(R) = 0.$$

We also introduce radial versions $\mathcal{P}^\pm : \mathbb{R}^2 \to \mathbb{R}$ of the Pucci operators by

$$(4) \qquad \begin{aligned} \mathcal{P}^+(m,l) :=& P^+(m\omega_0 \otimes \omega_0 + (I_N - \omega_0 \otimes \omega_0)l) \\ =& \Lambda\,(m_+ + (N-1)l_+) - \lambda\,(m_- + (N-1)l_-) \end{aligned}$$

and $\mathcal{P}^-(m,l) = -\mathcal{P}^+(-m,-l)$. Here $m_\pm := \max\{\pm m, 0\}$. By (F2), we have

$$(5) \qquad \begin{aligned} &\mathcal{F}(m_1,l_1,p_1,u_1,r) - \mathcal{F}(m_2,l_2,p_2,u_2,r) \\ \leq& \mathcal{P}^+(m_1-m_2, l_1-l_2) + \beta(r\omega)|p_1-p_2| + \gamma(r\omega)|u_1-u_2| \end{aligned}$$

for all $(m_i,l_i,p_i,u_i) \in \mathbb{R}^4$, $i = 1,2$, and a.a. $(r,\omega) \in (0,R) \times S^{N-1}$. In view of Fubini's theorem in the polar coordinates, we can choose a $\omega \in S^{N-1}$ which has the properties that the inequality (5) holds for all $(m_i, l_i, p_i, u_i) \in \mathbb{R}^4$, $i = 1, 2$, and a.a. $r \in (0, R)$, and the functions $r \mapsto r^{N-1}(\beta(r\omega))^q$, $r \mapsto r^{N-1}(\gamma(r\omega))^q$ are integrable in $(0, R)$. We fix such an ω, call it ω_1, and, with abuse of notation, we write β and γ the functions $r \mapsto \beta(r\omega_1)$ and $r \mapsto \gamma(r\omega_1)$, respectively. In other words, under the assumptions (F1), (F2) and (F4), we conclude the following:

(F5) There exist functions β, $\gamma \in L^q_{\mathrm{r}}(0, R)$ such that

$$\begin{aligned}&\mathcal{F}(m_1, l_1, p_1, u_1, r) - \mathcal{F}(m_2, l_2, p_2, u_2, r)\\ \le&\mathcal{P}^+(m_1 - m_2, l_1 - l_2) + \beta(r)|p_1 - p_2| + \gamma(r)|u_1 - u_2|\end{aligned}$$

for all $(m_i, l_i, p_i, u_i) \in \mathbb{R}^4$, $i = 1, 2$, and a.a. $r \in (0,\ R)$.

Since $\mathcal{P}^-(m, l) = -\mathcal{P}^+(-m, -l)$, it holds from (F5) that for all $(m_i, l_i, p_i, u_i) \in \mathbb{R}^4$ and a.a. $r \in (0, R)$,

$$\begin{aligned}&\mathcal{F}(m_1, l_1, p_1, u_1, r) - \mathcal{F}(m_2, l_2, p_2, u_2, r)\\ \ge&\mathcal{P}^-(m_1 - m_2, l_1 - l_2) - \beta(r)|p_1 - p_2| - \gamma(r)|u_1 - u_2|.\end{aligned} \tag{6}$$

For later use, we rewrite the conditions in terms of $\mathcal{F}$:

(r-F1) The function $\mathcal{F}$ is a Carathéodory function.

(r-F2) There exist $\beta, \gamma \in L^q_{\mathrm{r}}(0, R)$ such that

$$\begin{aligned}&\mathcal{F}(m_1, p_1, u_1, r) - \mathcal{F}(m_2, p_2, u_2, r)\\ \le&\mathcal{P}^+(m_1 - m_2, p_1 - p_2, r) + \beta(r)|p_1 - p_2| + \gamma(r)|u_1 - u_2|\end{aligned}$$

for all $(m_i, p_i, u_i) \in \mathbb{R}^3$, $i = 1, 2$, and a.a. $r \in (0,\ R)$.

(r-F3) $\mathcal{F}(tm, tl, tp, tu, r) = t\mathcal{F}(m, l, p, u, r)$ for every $(m, l, p, u) \in \mathbb{R}^4$ and a.a. $r \in (0, R)$.

In what follows, we shall prove the existence of solutions to (3) under (r-F1)–(r-F3). In order to show the existence of eigenpairs to (3), the solvability of the following equations plays an important role under (r-F1), (r-F2) and $\mathcal{F}[0] \in L^q_{\mathrm{r}}(0, R)$: for each $0 \le a < b \le R$,

$$\mathcal{F}_\kappa[u] = 0 \quad \text{a.e. in } (a, b),\ u \in W^{2,q}_{\mathrm{r}}(a, b),\ u(b) = 0,\ u'(a) = 0 \text{ if } a > 0 \tag{7}$$

where $\mathcal{F}_\kappa(m, l, p, u, r) := \mathcal{F}(m, l, p, u, r) - \kappa u$ and $\kappa \in \mathbb{R}$. The constant κ is fixed later.

To rewrite (7) in the normal form, we use the following lemma (See Lemma 2.1 in [11]).

Lemma 2.2. *Under the conditions* (r-F1) *and* (r-F2), *the following hold:*
(i) *There is a unique* $g = g_{\mathcal{F}}(l, p, u, d, r) \in \mathbb{R}$ *such that* $\mathcal{F}(g, l, p, u, r) = d$ *for any* $(l, p, u, d) \in \mathbb{R}^4$ *and a.a.* $r \in (0, R)$.

(ii) *For all* $(m, l, p, u, d) \in \mathbb{R}^5$ *and a.a.* $r \in (0, R)$, $m < g_{\mathcal{F}}(l, p, u, d, r)$ *(resp.* $m > g_{\mathcal{F}}(l, p, u, d, r)$ *) if and only if* $\mathcal{F}(m, l, p, u, r) < d$ *(resp.* $\mathcal{F}(m, l, p, u, r) > d$ *).*
(iii) *The function* $g_{\mathcal{F}}$ *satisfies the following Lipschitz condition:*

$$\begin{aligned}&|g_{\mathcal{F}}(l_1, p_1, u_1, d_1, r) - g_{\mathcal{F}}(l_2, p_2, u_2, d_2, r)|\\ \leq &\lambda^{-1} L(r)(|l_1 - l_2| + |p_1 - p_2| + |u_1 - u_2| + |d_1 - d_2|)\end{aligned}$$

for every $(l_i, p_i, u_i, d_i) \in \mathbb{R}^4$ *and a.a.* $r \in (0, R)$ *where* $L(r) := \max\{\Lambda(N - 1), \beta(r), \gamma(r), 1\}$. *Furthermore, it holds that for any* $d \in \mathbb{R}$,

$$|g_{\mathcal{F}}(0, 0, 0, d, r)| \leq \lambda^{-1} |\mathcal{F}(0, 0, 0, 0, r) - d|$$

Proof. (i) Let $m_1 < m_2$. Then for each $(l, p, u) \in \mathbb{R}^3$ and a.a. $r \in (0, R)$, it follows from (4) and (r-F2) that

$$\mathcal{F}(m_1, l, p, u, r) - \mathcal{F}(m_2, l, p, u, r) \leq \mathcal{P}^+(m_1 - m_2, 0) = -\lambda(m_2 - m_1) < 0. \tag{8}$$

Thus for any $(l, p, u) \in \mathbb{R}^3$ and a.a. $r \in (0, R)$, we see from (8) that the function $m \mapsto \mathcal{F}(m, l, p, u, r)$ is strictly increasing in m and $\lim_{m \to \pm\infty} \mathcal{F}(m, l, p, u, r) = \pm\infty$. By the intermediate value theorem yields that for all $d \in \mathbb{R}$ there exists a unique $g = g_{\mathcal{F}}(l, p, u, d, r) \in \mathbb{R}$ satisfying $\mathcal{F}(g, l, p, u, r) = d$.

The assertion (ii) holds from the strict monotonicity of $\mathcal{F}(m, l, p, u, r)$ in m.

Next we show the assertion (iii). Let $(l_1, p_1, u_1, d_1), (l_2, p_2, u_2, d_2) \in \mathbb{R}^4$, $g_i = g_{\mathcal{F}}(l_i, p_i, u_i, d_i, r)$ and $g_1 < g_2$. Then it follows from (r-F2) that

$$\begin{aligned}d_1 - d_2 &\leq \mathcal{P}^+(g_1 - g_2, l_1 - l_2) + \beta(r)|p_1 - p_2| + \gamma(r)|u_1 - u_2|\\ &= \lambda(g_1 - g_2) + \Lambda(N - 1)|l_1 - l_2| + \beta(r)|p_1 - p_2| + \gamma(r)|u_1 - u_2|.\end{aligned}$$

Therefore we obtain $0 < g_2 - g_1 \leq \lambda^{-1} L(r)(|l_1 - l_2| + |p_1 - p_2| + |u_1 - u_2| + |d_1 - d_2|)$. This ensures the Lipschitz continuity of $g_{\mathcal{F}}$. Moreover if $g = g_{\mathcal{F}}(0, 0, 0, d, r) > 0$, then by (6) we have $\mathcal{P}^-(g, 0) \leq \mathcal{F}(g, 0, 0, 0, r) - \mathcal{F}(0, 0, 0, 0, r) = d - F(0, 0, 0, 0, r)$. Hence $0 < g \leq \lambda^{-1}|d - \mathcal{F}(0, 0, 0, 0, r)|$. We can also prove in the case where $g = g_{\mathcal{F}}(0, 0, 0, d, r) < 0$. □

By Lemma 2.2, it is easy to see that $\mathcal{F}[u](r) = 0$ for a.e. $r \in (a, b)$ is equivalent to $u''(r) = g_{\mathcal{F}}(u'(r)/r, u'(r), u(r), 0, r)$ for a.e. $r \in (a, b)$. Since $g_{\mathcal{F}}$ satisfies the Lipschitz continuity, by the contraction mapping argument, we can show

Proposition 2.3. *Under the assumptions* (r-F1), (r-F2) *and* $\mathcal{F}[0] \in L^q_{\mathrm{r}}(0, R)$, *for each* $0 < a < b \leq R$, $\alpha_1, \alpha_2 \in \mathbb{R}$, $q \geq 1$, *there is a unique solution* $u \in W^{2,q}_{\mathrm{r}}(a, b)$ *of* $\mathcal{F}[u](r) = 0$ *a.e. in* (a, b) *with* $u(a) = \alpha_1$ *and* $u'(a) = \alpha_2$.

Remark 2.4. The similar results to Lemma 2.2 and Proposition 2.3 hold for $\mathcal{F}_\kappa$.

3 Estimates on radial functions

In this section we establish a priori type estimates on functions in $W^{2,q}_{\mathrm{r}}(a,b)$, motivated by the boundary value problem (7) under (r-F1), (r-F2) and $\mathcal{F}[0] \in L^q_{\mathrm{r}}(0,R)$.

Throughout this note we set $\lambda_* = \lambda/\Lambda \in (0,1]$ and $q_* = N(1+\lambda_*(N-1)) = N/(\lambda_* N + (1-\lambda_*)) < N$.

The following two lemmas play important roles to derive a priori estimates of (7). For a proof, see [13].

Lemma 3.1. *Let $0 \le a < b \le R$, $q \in (q_*, \infty]$, $\beta \in L^N_{\mathrm{r}}(0,R)$ and $f \in L^q_{\mathrm{r}}(a,b)$. Let v be a measurable function on $[a,b]$ such that for each $c \in (a,b)$, v is absolutely continuous on $[c,b]$. Assume that $f \ge 0$ a.e. in (a,b), $v/r \in L^q_{\mathrm{r}}(a,b)$, $v \ge 0$ in $[a,b]$, $v(a) = 0$ if $a > 0$ and*

$$v'(r) + \lambda_*(N-1)\frac{v(r)}{r} \le \lambda^{-1}\beta(r)v(r) + \lambda^{-1}f(r) \quad \textit{for a.a. } r \in (a,b).$$

Then there exists a constant $C_1 > 0$, depending only on λ_*, q, $\|\lambda^{-1}\beta\|_{L^N_{\mathrm{r}}(0,R)}$ and N, such that

$$\|v/r\|_{L^q_{\mathrm{r}}(a,b)} \le C_1\lambda^{-1}\|f\|_{L^q_{\mathrm{r}}(a,b)}. \tag{9}$$

An important point of the above estimate is that the constant C can be chosen independently of the parameter a.

Lemma 3.2. *Let $q \in (N/2, \infty]$ and $0 \le a < b \le R$. Let u be a function on $[a,b]$ such that for each $c \in (a,b]$, the function u is absolutely continuous on $[c,b]$, $u(b) \le 0$ and $\|(u')_-/r\|_{L^q_{\mathrm{r}}(a,b)} < \infty$. Then there exists a constant $C_2 > 0$, depending only on q and N, such that*

$$\sup_{(a,b]} u \le C_2\left(b^{(2q-N)/(q-1)} - a^{(2q-N)/(q-1)}\right)^{(q-1)/q}\|(u')_-/r\|_{L^q_{\mathrm{r}}(a,b)}.$$

The next lemma concerns the embedding $W^{2,q}_{\mathrm{r}}(0,b) \subset C^1([0,b])$. Note that if $a > 0$, then $W^{2,q}_{\mathrm{r}}(a,b) \subset C^1([a,b])$ for any $q \ge 1$. For instance, see Berestycki and Lions [3], Strauss [19].

Lemma 3.3. *Let $q \ge N$, $0 \le a < b \le R$ and $u \in W^{2,q}_{\mathrm{r}}(a,b)$. Assume in addition that $u'(a) = 0$ if $a > 0$. Then*

$$\|u'\|_{L^\infty(a,b)} \le R^{1-N/q}q^{1/q}\|u'/r\|^{1-1/q}_{L^q_{\mathrm{r}}(a,b)}\|u''\|^{1/q}_{L^q_{\mathrm{r}}(a,b)}.$$

In particular, $W^{2,N}_{\mathrm{r}}(0,b) \subset C^1([a,b])$ and $u'(0) = 0$ hold for all $u \in W^{2,N}_{\mathrm{r}}(0,b)$.

Proof. It is enough to show the above inequality when u is smooth by the density of $C^\infty_{\mathrm{r}}(\overline{A(a,b)})$ in $W^{2,q}_{\mathrm{r}}(a,b)$. Thus we may assume $u'(a) = 0$.

For any $a \le r \le R$, we have

$$\begin{aligned}|u'(r)|^q &\le \int_a^r q|u'(t)|^{q-1}|u''(t)|\,\mathrm{d}t \le R^{q-N}q\int_a^r |u'(t)/t|^{q-1}|u''(t)|t^{N-1}\,\mathrm{d}t \\ &\le R^{q-N}q\|u'/r\|^{q-1}_{L^q_{\mathrm{r}}(a,b)}\|u''\|_{L^q_{\mathrm{r}}(a,b)}.\end{aligned}$$

Thus the conclusion follows. □

The next lemma is about the estimate of $\|\beta u'\|_{L^q_r(a,b)}$.

Lemma 3.4. *Let $1 < q$, $0 \le a < b \le R$ and $u \in W^{2,q}_r(a,b)$. Assume that $u'(a) = 0$ if $a > 0$ and $\beta \in L^N_r(0,R)$. Then there exists a constant $C > 0$, depending only on q, N and R, such that*

$$\begin{aligned}&\|\beta u'\|_{L^q_r(a,b)}\\ &\le C \max\{\|\beta\|_{L^q_r(0,R)}, \|\beta\|_{L^N_r(0,R)}\} \left(\|u'/r\|^{1-1/q}_{L^q_r(a,b)} \|u''\|^{1/q}_{L^q_r(a,b)} + \|u'/r\|_{L^q_r(a,b)} \right).\end{aligned}$$

Proof. When $1 < q < N$, see [13]. In the case where $q \ge N$, the claim holds from Lemma 3.3 since $u' \in L^\infty(a,b)$. □

The following lemma is an Alexandrov-Bakelman-Pucci type inequality.

Lemma 3.5. *Let $q \in (\max\{N/2, q^*\}, \infty]$, $0 \le a < b \le R$, $\beta \in L^q_r(0,R) \cap L^N_r(0,R)$, $u \in W^{2,q}_r(a,b)$ and $f \in L^q_r(a,b)$. Assume that $u(b) = 0$, $u'(a) = 0$ if $a > 0$ and u satisfies*

$$\mathcal{P}^+[u](r) + \beta(r)|u'(r)| + f(r) \ge 0 \quad \text{a.e. in } (a,b).$$

Then there exists a constant $C_3 > 0$, depending only on λ, Λ, q, N and $\|\beta\|_{L^N_r(0,R)}$, such that

$$\max_{[a,b]} u \le C_3 \left(b^{(2q-N)/(q-1)} - a^{(2q-N)/(q-1)} \right)^{(q-1)/q} \|f_+\|_{L^q_r(a,b)}.$$

Proof. Fix any $(m,l,d) \in \mathbb{R}^3$ such that $\mathcal{P}^+(m,l) + d \ge 0$ and $d \ge 0$. Assume that $l \le 0$. We have $0 \le \lambda m + \lambda(N-1)l + d$ if $m \le 0$ and $0 \le \Lambda m + \lambda(N-1)l + d$ if $m > 0$. Noting $l \le 0$, we obtain

$$m + \lambda_*(N-1)l + \lambda^{-1} d \ge 0 \quad \text{for any } (m,l,d) \in \mathbb{R}^3 \text{ with } l \le 0 \text{ and } d \ge 0. \tag{10}$$

If we set $v = (u')_-$, then we have $v(r) = -u'(r)$ and $v'(r) = -u''(r)$ a.e. if $v(r) > 0$, and $v(r) = 0$ and $v'(r) = 0$ a.e. if $v(r) \le 0$. Using (10), we get

$$-v' - \lambda_*(N-1)\frac{v}{r} + \lambda^{-1}\beta v + \lambda^{-1} f_+(r) \ge 0 \quad \text{a.e. in } (a,b).$$

By Lemma 3.1, there exists a constant $C_1 > 0$, depending only on λ_*, q, N and $\|\lambda^{-1}\beta\|_{L^N_r(0,R)}$, such that

$$\|(u')_-/r\|_{L^q_r(a,b)} \le C_1 \|\lambda^{-1} f_+\|_{L^q_r(a,b)}.$$

On the other hand, by Lemma 3.2 and $u \in C([a,b])$, there is a $C_2 > 0$ such that

$$\max_{[a,b]} u(r) \le C_2 (b^{(2q-N)/(q-1)} - a^{(2q-N)/(q-1)})^{(q-1)/q} \|(u')_-/r\|_{L^q_r(a,b)}$$

Combining the above two inequalities, we can show our claim. □

Proposition 3.6. *Let $0 \le a < b \le R$, $q \in (\max\{N/2, q_*\}, \infty]$, $\beta \in L^q_{\mathrm{r}}(0,R) \cap L^N_{\mathrm{r}}(0,R)$, $f^1, f^2 \in L^q_{\mathrm{r}}(a,b)$ and $u \in W^{2,q}_{\mathrm{r}}(a,b)$. Assume that*

$$\begin{cases} \mathcal{P}^+[u](r) + \beta|u'| + f^1 \ge 0 \quad \text{a.e. in } (a,b), \\ \mathcal{P}^-[u](r) - \beta|u'| - f^2 \le 0 \quad \text{a.e. in } (a,b), \\ u'(a) = 0 \quad \textit{if} \quad a > 0, \qquad \text{and} \qquad u(b) = 0. \end{cases}$$

Then there exists a constant $C > 0$, depending only on q, λ, Λ, N, R, $\|\beta\|_{L^N_{\mathrm{r}}(0,R)}$ and $\|\beta\|_{L^q_{\mathrm{r}}(0,R)}$ such that

$$\|u\|_{W^{2,q}_{\mathrm{r}}(a,b)} \le C\left(\|f^1_+\|_{L^q_{\mathrm{r}}(a,b)} + \|f^2_+\|_{L^q_{\mathrm{r}}(a,b)}\right).$$

Proof. First note that by the assumption, we have

$$\mathcal{P}^-[-u](r) + \beta(r)|u'(r)| + f^2(r) \ge 0.$$

Thus as in the proof of Lemma 3.5, it holds that

$$\|(u')_+/r\|_{L^q_{\mathrm{r}}(a,b)} \le C_1\|\lambda^{-1} f^2_+\|_{L^q_{\mathrm{r}}(a,b)}$$

where C_1 depends only on λ_*, q, N and $\|\lambda^{-1}\beta\|_{L^N_{\mathrm{r}}(0,R)}$. Hence, setting $M = \|\lambda^{-1} f^1_+\|_{L^q_{\mathrm{r}}(a,b)} + \|\lambda^{-1} f^2_+\|_{L^q_{\mathrm{r}}(a,b)}$, we have

$$\|u'/r\|_{L^q_{\mathrm{r}}(a,b)} \le C_1 M. \tag{11}$$

Secondly, for each $(m,l,d) \in \mathbb{R}^3$ with $m \le 0$ and $\mathcal{P}^+(m,l) + d \ge 0$, we have

$$m + \lambda_*^{-1}(N-1)|l| + \lambda^{-1} d \ge 0. \tag{12}$$

Using (12), $\mathcal{P}^+[u](r) + \beta(r)|u'(r)| + f_1(r) \ge 0$ and $\mathcal{P}^-[-u](r) + \beta(r)|u'(r)| + f_2(r) \ge 0$, we observe that

$$|u''| \le \lambda_*^{-1}(N-1)\frac{|u'|}{r} + \lambda^{-1}\beta|u'| + \lambda^{-1}(f^1_+ + f^2_+) \quad \text{a.e. in } (a,b). \tag{13}$$

By Lemma 3.2 and (11), we can choose a constant $C_2 > 0$, depending only on q, R and N, for which we have

$$\|u\|_{L^\infty(a,b)} \le C_1 C_2 M. \tag{14}$$

Also, by Lemmas 3.3, 3.4, (11) and Young's inequality, for each $\varepsilon > 0$, we find a constant $C_4 > 0$, depending only on ε, q, N, R, $\|\lambda^{-1}\beta\|_{L^N_{\mathrm{r}}(0,R)}$ and $\|\lambda^{-1}\beta\|_{L^q_{\mathrm{r}}(0,R)}$, for which we have

$$\|\lambda^{-1}\beta u'\|_{L^q_{\mathrm{r}}(a,b)} \le \varepsilon\|u''\|_{L^q_{\mathrm{r}}(a,b)} + C_1 C_4 M. \tag{15}$$

Combining this, with $\varepsilon = 1/2$, and (13), we get

$$\begin{aligned} \frac{1}{2}\|u''\|_{L^q_{\mathrm{r}}(a,b)} &\le \lambda_*^{-1}(N-1)\|u'/r\|_{L^q_{\mathrm{r}}(a,b)} + C_1 C_4 M + \|\lambda^{-1}(f_+ + g_+)\|_{L^q_{\mathrm{r}}(a,b)} \\ &\le (\lambda_*^{-1}(N-1)C_1 + C_1 C_4 + 1)M. \end{aligned}$$

This inequality together with (14) and (15) yields an estimate on $\|u\|_{W^{2,q}_{\mathrm{r}}(a,b)}$ with the desired properties. □

Next, for $\kappa \in \mathbb{R}$, we recall the definition of $\mathcal{F}_\kappa$: $\mathcal{F}_\kappa(r) := \mathcal{F}(m,l,p,u,r) - \kappa u$. By the definition, we remark that $\mathcal{F}[0](r) = \mathcal{F}_\kappa[0](r)$ holds. Noting (r-F2), if $u(r) - v(r) \geq 0$, then we have

$$\mathcal{F}_\kappa[u](r) - \mathcal{F}_\kappa[v](r) \leq \mathcal{P}^+[u-v](r) + \beta(r)|u'(r) - v'(r)| + (\gamma(r) - \kappa)_+(u(r) - v(r)). \tag{16}$$

Next we define a constant σ_κ by

$$\sigma_\kappa := C_3 \lambda^{-1} R^{2-N/q} \|(\gamma - \kappa)_+\|_{L^q_r(0,R)}. \tag{17}$$

Here C_3 appears in Lemma 3.5 and we remark that $\sigma_\kappa \to 0$ as $\kappa \to \infty$.

Proposition 3.7. *Suppose* (r-F1), (r-F2) *and* $\mathcal{F}[0] \in L^q_r(0,R)$. *Assume also that* $q \in (\max\{N/2, q_*\}, \infty]$, $\sigma_\kappa < 1$, $0 \leq a < b \leq R$ *and* $u \in W^{2,q}_r(a,b)$ *is a solution of* (7). *Then there exists a* C *depending only on* q, λ, Λ, N, R, $\|\beta\|_{L^N_r(0,R)}$, $\|\beta\|_{L^q_r(0,R)}$, $\|\gamma\|_{L^q_r(0,R)}$, κ *such that*

$$\|u\|_{W^{2,q}_r(a,b)} \leq C \|\mathcal{F}[0]\|_{L^q_r(a,b)}.$$

Proof. If $u_+ \not\equiv 0$, then let $r^+ \in [a,b)$ be a maximum point of u_+, respectively. Furthermore, let

$$b^+ := \inf\{r \in (r^+, b] : u_+(r) = 0\} > r^+.$$

Noting $u \geq 0$ in $[r^+, b^+]$, it follows from (16) that for a.a. $r \in (r^+, b^+)$,

$$\begin{aligned} 0 =& \mathcal{F}_\kappa[u](r) = \mathcal{F}_\kappa[u](r) - \mathcal{F}_\kappa[0](r) + \mathcal{F}[0](r) \\ \leq& \mathcal{P}^+[u](r) + \beta(r)|u'(r)| + (\gamma(r) - \kappa)_+ u_+(r) + |\mathcal{F}[0](r)|. \end{aligned}$$

By Lemma 3.5, we have

$$\begin{aligned} u(r^+) = \max_{r^+ \leq r \leq b^+} u(r) &\leq C_3 R^{2-N/q} \|(\gamma - \kappa)_+ u_+ + |\mathcal{F}[0]|\|_{L^q_r(r^+,b^+)} \\ &\leq \sigma_\kappa \max_{[r^+,b^+]} u + C_3 R^{2-N/q} \|\mathcal{F}[0]\|_{L^q_r(a,b)}. \end{aligned}$$

From $\sigma_\kappa < 1$, it holds that

$$\|u_+\|_{L^\infty(a,b)} \leq C_3 R^{2-N/q} (1 - \sigma_\kappa)^{-1} \|\mathcal{F}[0]\|_{L^q_r(a,b)}.$$

Similarly, if $u_- \not\equiv 0$, then we set $u_-(r^-) = \max_{a \leq r \leq b} u_-(r) > 0$, $u_-(b^-) = 0$ and $-u \geq 0$ in $[r^-, b^-]$. Furthermore we can show

$$0 \leq \mathcal{P}^+[-u](r) + \beta(r)|u'(r)| + (\gamma - \kappa)_- u_- + |\mathcal{F}[0](r)| \quad \text{a.e. in } (r^-, b^-).$$

Repeating the argument in the above, one obtains

$$\|u_-\|_{L^\infty(a,b)} \leq C_3 R^{2-N/q} (1 - \sigma_\kappa)^{-1} \|\mathcal{F}[0]\|_{L^q_r(a,b)}.$$

Thus it holds that

$$\|u\|_{L^\infty(a,b)} \leq C_3 R^{2-N/q} (1 - \sigma_\kappa)^{-1} \|\mathcal{F}[0]\|_{L^q_r(a,b)}. \tag{18}$$

Next, by (r-F2), we have

$$0=\mathcal{F}_\kappa[u](r)\le\mathcal{P}^+[u](r)+\beta(r)|u'(r)|+(\gamma(r)+|\kappa|)|u(r)|+|\mathcal{F}[0](r)| \quad \text{a.e. in } (a,b),$$
$$0\ge\mathcal{P}^-[u](r)-\beta(r)|u'(r)|-(\gamma(r)+|\kappa|)|u(r)|-|\mathcal{F}[0](r)| \quad \text{a.e. in } (a,b).$$

Therefore, Proposition 3.6 and (18) ensure

$$\begin{aligned}\|u\|_{W^{2,q}_{\mathrm{r}}(a,b)}&\le\tilde{C}(\|(\gamma+|\kappa|)u\|_{L^q_{\mathrm{r}}(a,b)}+\|\mathcal{F}[0]\|_{L^q_{\mathrm{r}}(a,b)})\\&\le\tilde{C}(\|u\|_{L^\infty(a,b)}\|\gamma+|\kappa|\|_{L^q_{\mathrm{r}}(a,b)}+\|\mathcal{F}[0]\|_{L^q_{\mathrm{r}}(a,b)})\le C\|\mathcal{F}[0]\|_{L^q_{\mathrm{r}}(a,b)}\end{aligned}$$

where C depends only on q, λ, Λ, N, R, $\|\beta\|_{L^N_{\mathrm{r}}(0,R)}$, $\|\beta\|_{L^q_{\mathrm{r}}(0,R)}$, $\|\gamma\|_{L^q_{\mathrm{r}}(0,R)}$ and κ. □

4 Comparison theorem

In this section, we prove a weak maximum principle and strong maximum principle, respectively. A weak maximum principle for $\mathcal{F}_\kappa$ is stated as follows.

Proposition 4.1. *Let $q\in(\max\{N/2,q^*\},\infty]$, $\sigma_\kappa<1$ appearing in (17), $0\le a<b\le R$, $u,v\in W^{2,q}_{\mathrm{r}}(a,b)$ and $f,g\in L^q_{\mathrm{r}}(a,b)$. Furthermore, suppose that u,v,f,g satisfy*

$$\mathcal{F}_\kappa[v]+g\le\mathcal{F}_\kappa[u]+f \quad \text{a.e. in } (a,b)$$

and $v'(a)\le u'(a)$ and $u(b)\le v(b)$. Then it follows that

$$\max_{[a,b]}(u-v)\le C_3(1-\sigma_\kappa)^{-1}\left(b^{(2q-N)/(q-1)}-a^{(2q-N)/(q-1)}\right)^{(q-1)/q}\|(f-g)_+\|_{L^q_{\mathrm{r}}(a,b)}.$$

Proof. Set $w(r):=u(r)-v(r)$. We may assume $\max_{[a,b]}w(r)>0$. Let $r_0\in[a,b)$ be a maximum point of w. Furthermore, set $r_1=\min\{r\in[r_0,b]:\ w(r)=0\}$. By the assumptions, $u'(r_0)=0$ holds.

On the other hand, it follows from (16) that

$$0\le\mathcal{P}^+[w]+\beta|w'|+(\gamma-\kappa)_+w+(f-g)_+ \quad \text{a.e. in } (r_0,r_1).$$

Applying Lemma 3.5, we obtain

$$\begin{aligned}\max_{[a,b]}w&\le C_3\left(b^{(2q-N)/(q-1)}-a^{(2q-N)/(q-1)}\right)^{(q-1)/q}\|(\gamma-\kappa)_+w+(f-g)_+\|_{L^q_{\mathrm{r}}(r_0,r_1)}\\&\le\sigma_\kappa\max_{[a,b]}w+C_3\left(b^{(2q-N)/(q-1)}-a^{(2q-N)/(q-1)}\right)^{(q-1)/q}\|(f-g)_+\|_{L^q_{\mathrm{r}}(a,b)}.\end{aligned}$$

Since $\sigma_\kappa<1$, we have the conclusion. □

The next proposition is a version of the strong maximum principle for radial functions.

Proposition 4.2. *Let $0\le a<b\le R$, $q\in(\max\{N/2,q_*\},\infty]$, $u\in W^{2,q}_{\mathrm{r}}(a,b)$, $\beta\in L^N_{\mathrm{r}}(a,b)$ and $\gamma\in L^q_{\mathrm{r}}(a,b)$. Assume that $u\ge0$ in $[a,b]$ and*

$$\mathcal{P}^-[u]-\beta|u'|-\gamma u\le0 \quad \text{a.e. in } (a,b).$$

Then either $u\equiv0$ in $[a,b]$ or $u>0$ in (a,b). Furthermore, $\max\{u(b),-u'(b)\}>0$ and $\max\{u(a),u'(a)\}>0$ holds if $a>0$. When $a=0$, $u(0)>0$ holds.

Proof. First we show that if $u'(r_0) = 0$ and $u(r_0) = 0$ for some $r_0 \in [a,b]$ with $r_0 > 0$, then $u \equiv 0$ in $[a,b]$. Set $v = (u')_-$ and $w = (u')_+$. Since u satisfies $\mathcal{P}^+[-u] + \beta|u'| + \gamma u \geq 0$ a.e. in (a,b), we observe that

$$-\hat{\gamma}u - \hat{\beta}v \leq v' \quad \text{and} \quad w' \leq \hat{\beta}w + \hat{\gamma}u \quad \text{a.e. in } (a,b)$$

where $\hat{\beta}(r) = \lambda^{-1}(\beta + \Lambda(N-1)/r)$ and $\hat{\gamma}(r) = \lambda^{-1}\gamma(r)$. Thus by Gronwall's inequality, we have

$$(19) \qquad (u')_-(t) \leq \int_t^{r_0} \hat{\gamma}(s)u(s)\exp\left(\int_t^s \hat{\beta}(\tau)d\tau\right)ds \quad \text{for all } t \in (a, r_0],$$

$$(20) \qquad (u')_+(t) \leq \int_{r_0}^{t} \hat{\gamma}(s)u(s)\exp\left(\int_s^t \hat{\beta}(\tau)d\tau\right)ds \quad \text{for all } t \in [r_0, b].$$

We fix $\varepsilon \in (a, r_0)$ arbitrarily. Then for each $r \in [\varepsilon, r_0]$, it follows from (19) that

$$u(r) = u(r) - u(r_0) \leq \int_r^{r_0} (u')_-(t)dt \leq (r_0 - \varepsilon)\exp(\|\hat{\beta}\|_{L^1(\varepsilon,r_0)}) \int_r^{r_0} \hat{\gamma}(s)u(s)ds.$$

Using Gronwall's inequality again, we get $u \equiv 0$ in $[\varepsilon, r_0]$. Since $\varepsilon > 0$ is arbitrary, $u \equiv 0$ in $[a, r_0]$. Similarly $u \equiv 0$ in $[r_0, b]$ holds from (20). Hence $u \equiv 0$ in $[a,b]$. Moreover, by the above arguments, we see that if $u \not\equiv 0$, then $\max\{u(b), -u'(b)\} > 0$. Furthermore, $\max\{u(a), u'(a)\} > 0$ holds if $a > 0$.

Next we treat the case where $a = 0$. In this case, it is enough to show that $u \equiv 0$ provided $u(0) = 0$. We choose $a > 0$ so small that $C_1C_2a^{2-N/q}\|\gamma\|_{L^q_r(0,a)} < 1$ where C_1 and C_2 appear in Lemmas 3.1 and 3.2.

As in the above, if we set $v = (u')_+$, then we have

$$v' + \lambda_*(N-1)\frac{v}{r} \leq \lambda^{-1}(\beta v + \gamma u) \quad \text{a.e. in } (0,b).$$

By Lemma 3.1, we get

$$\|(u')_+/r\|_{L^q_r(0,a)} \leq C_1\|\gamma u\|_{L^q_r(0,a)} \leq C_1\|\gamma\|_{L^q_r(0,a)} \max_{[0,a]} u$$

where $C_1 > 0$ is a constant independent of a. Applying Lemma 3.2 to the function $r \mapsto u(c) - u(r)$, with $0 < c \leq a$, we get

$$\max_{0\leq r\leq c}(u(c) - u(r)) \leq C_2c^{(2q-N)/q}\|(u')_+/r\|_{L^q_r(0,c)},$$

where $C_2 > 0$ is a constant independent of c and a. In particular, since $u(0) = 0$, we have

$$\max_{0\leq c\leq a} u(c) \leq C_2a^{(2q-N)/q}\|(u')_+/r\|_{L^q_r(0,a)}.$$

Thus, we get

$$\max_{[0,a]} u \leq C_1C_2a^{(2q-N)/q}\|\gamma\|_{L^q_r(0,a)} \max_{[0,a]} u.$$

Since $C_1C_2a^{(2q-N)/q}\|\gamma\|_{L^q_r(0,a)} < 1$, we find $\max_{[0,a]} u = 0$, which implies $u \equiv 0$ in $[0,a]$. Using the previous argument, we can conclude $u \equiv 0$ in $[0,b]$. $\square$

5 Solvability of (7)

This section is devoted to proving that (7) has a unique solution in $W_{\mathrm{r}}^{2,q}(a,b)$ under (r-F1), (r-F2) and $\mathcal{F}[0] \in L_{\mathrm{r}}^q(0,R)$.

Theorem 5.1. *Assume $\mathcal{F}$ satisfies* (r-F1), (r-F2) *and $\mathcal{F}[0] \in L_{\mathrm{r}}^q(0,R)$. Let $\sigma_\kappa < 1$ and $q \in (\max\{2/N, q_*\}, \infty]$. Then for each $0 \le a < b \le R$, the equation* (7) *has a unique solution u and u satisfies*

$$\|u\|_{W_{\mathrm{r}}^{2,q}(a,b)} \le C\|\mathcal{F}_\kappa[0]\|_{L_{\mathrm{r}}^q(0,R)}$$

where C depends only on $q, N, \lambda, \Lambda, R, \kappa, \|\beta\|_{L_{\mathrm{r}}^N(0,R)}, \|\beta\|_{L_{\mathrm{r}}^q(0,R)}$ and $\|\gamma\|_{L_{\mathrm{r}}^q(0,R)}$.

To prove Theorem 5.1, we prepare the next lemma concerning a supersolution to $\mathcal{P}^+$.

Lemma 5.2. *Let $0 < a < b \le R$, $q \in (\max\{2/N, q_*\}, \infty]$ and $f \in L_{\mathrm{r}}^q(a,b)$. Then there exists a $\phi \in W_{\mathrm{r}}^{2,q}(a,b)$ such that $\phi \ge 0$ in $[a,b]$ and*

$$\mathcal{P}^+[\psi] + \beta|\phi'| + \gamma\phi + |f| \le 0 \quad \text{a.e. in } (a,b), \quad \phi(b) = 0, \; \phi'(r) < 0.$$

Proof. Let $\eta > 0$ and define

$$\phi(r) := \int_r^b e^{A(t)}dt \quad \text{where} \quad A(t) := \int_a^r \eta(\beta(s) + \gamma(s) + |f(s)|)ds.$$

Then it is easy to see

$$\phi(b) = 0, \quad \phi'(r) = -e^{A(r)} < 0, \quad \phi(r) \le (b-a)e^{A(b)},$$
$$\phi''(r) = -\eta(\beta(r) + \gamma(r) + |f(r)|)e^{A(r)}.$$

Thus $\phi \in W_{\mathrm{r}}^{2,q}(a,b)$ and it holds that

$$\begin{aligned}&\mathcal{P}^+[\phi](r) + \beta(r)|\phi'(r)| + \gamma(r)\phi(r) + |f(r)| \\ &\le (1-\eta\lambda)\beta(r)e^{A(r)} + ((b-a)e^{A(b)} - \eta\lambda)\gamma(r) + (1-\eta\lambda)|f(r)|.\end{aligned}$$

Hence, taking $\eta > 0$ sufficiently large, we obtain $\mathcal{P}^+[\phi] + \beta|\phi'| + \gamma\phi + |f| \le 0$ a.e. in (a,b), which completes the proof. □

Proof of Theorem 5.1. The uniqueness follows from Proposition 4.1. Furthermore, the estimates for u also hold from Proposition 3.7. So it is sufficient to show the existence.

First we assume $a > 0$. Let ϕ be the function appearing in Lemma 5.2 with $f(r) = |\mathcal{F}[0](r)|$ and set $v^\pm(r) := \pm\phi(r)$. Then we see that $\mathcal{F}_\kappa[v^+] \le 0 \le \mathcal{F}_\kappa[v^-]$ a.e. in (a,b), $v^-(a) < 0 < v^+(a)$ and $(v^+)'(a) < 0 < (v^-)'(a)$.

For any $d \in \mathbb{R}$, we denote by $u(r:d)$ the unique solution of $\mathcal{F}_\kappa[u] = 0$ a.e. in (a,b) with $u(a:d) = d$ and $u'(a:d) = 0$ where u' stands for $\partial u/\partial r$. Such a solution exists from Remark 2.4. Next we shall prove the following claim:

$$v^+(r) < u(r:d) \text{ (resp. } u(r:d) < v^-(r)) \text{ in } [a,b] \quad \text{if } d > v^+(a) \text{ (resp. } d < v^-(a)).$$

First we suppose $d > v^+(a)$. Then we can take a neighborhood U of a such that $u(r : d) > v^+(r)$ for all $r \in U$. Next set $r_0 := \inf\{r \in (a,b] : u(r : d) = v(r)\}$. We argue by contradiction and assume $r_0 \in (a,b]$. Since $\mathcal{F}_\kappa[u] = 0 \geq \mathcal{F}_\kappa[v^+]$ a.e. in (a, r_0) and $v'(a) < 0 = u'(a)$, $v(r_0) = u(r_0)$, it follows from Proposition 4.1 that $u - v \leq 0$ in $[a, r_0]$, which is a contradiction. Thus $v^+(r) < u(r : d)$ in $[a,b]$ if $d > v^+(a)$. For the other claim, one can prove similarly.

Noting that the function $d \mapsto u(b : d)$ is continuous, we can choose a $d_0 \in [v^-(a), v^+(a)]$ such that $u(a : d_0) = 0$. Thus the existence result holds in the case where $a > 0$.

Next we consider the case where $a = 0$. Let $(u_k) \subset W^{2,q}_{\mathrm{r}}(1/k, b)$ be a solution of (7) in $(1/k, b)$. Furthermore, we extend u_k by

$$v_k(r) := \begin{cases} u_k(r) & \text{if } 1/k \leq r \leq b, \\ u_k(1/k) & \text{if } 0 \leq r < 1/k. \end{cases}$$

Then $v_k \in W^{2,q}_{\mathrm{r}}(0,b)$ since $v_k'(1/k) = 0$. Moreover, by Proposition 3.7 and Lemma 3.2, (v_k) is bounded in $W^{2,q}_{\mathrm{r}}(0,b)$.

Now suppose $q \neq \infty$. Taking a subsequence if necessary, we may assume $v_{k_\ell} \rightharpoonup v_0$ weakly in $W^{2,q}_{\mathrm{r}}(0,b)$. Note also that $v_{k_\ell} \to v_0$ strongly in $C^1([\varepsilon, b])$ for each $\varepsilon \in (0,b)$. Let $0 < s \leq t$ and $1/k_\ell \leq s$. Then the from the property of $g_{\mathcal{F}_\kappa}$, we have

$$v_{k_\ell}'(t) - v_{k_\ell}'(s) = \int_s^t g_{\mathcal{F}_\kappa}(v_{k_\ell}'(\tau)/\tau, v_{k_\ell}'(\tau), v_{k_\ell}(\tau), 0, \tau)d\tau.$$

Let $k_\ell \to \infty$, then we observe from Lemma 2.2 that

$$v_0'(t) - v_0'(s) = \int_s^t g_{\mathcal{F}_\kappa}(v_0'(\tau)/\tau, v_0'(\tau), v_0(\tau), 0, \tau)d\tau$$

for every $0 < s < t \leq b$. This means

$$v_0''(r) = g_{\mathcal{F}_\kappa}(v_0'(r)/r, v_0'(r), v_0(r), 0, r) \quad \text{a.a. } r \in (0,b).$$

Therefore, v_0 is a solution of (7).

In the case where $q = \infty$, then for any $p < \infty$, (v_k) is bounded in $W^{2,p}_{\mathrm{r}}(0,b)$. Thus we may assume $v_{k_\ell} \rightharpoonup v_0$ weakly in $W^{2,p}_{\mathrm{r}}(0,b)$. Then as in the above, we can show v_0 is a solution of (7). Moreover, since $\|v_0\|_{W^{2,p}_{\mathrm{r}}(0,b)} \leq C_b \sup_{k\geq 1} \|v_k\|_{W^{2,p}_{\mathrm{r}}(0,b)}$ holds for all $p \in (N, \infty)$, we have $v_0 \in W^{2,\infty}_{\mathrm{r}}(0,b)$. Thus we complete the proof. □

6 Existence of Principal Eigenpairs

In this section, we prove the existence of principal eigenpairs for (3).

Theorem 6.1. *Let $\mathcal{F}$ satisfy* (r-F1)–(r-F3), $q \in (\max\{N/2, q_*\}, \infty]$ *and* $0 \leq a < b \leq R$. *Then there exist pairs $(\mu_{\mathrm{N}}^\pm, \varphi_{\mathrm{N}}^\pm) \in \mathbb{R} \times W^{2,q}_{\mathrm{r}}(a,b)$ satisfying $\mathcal{F}[\varphi_{\mathrm{N}}^\pm] + \mu_{\mathrm{N}}^\pm \varphi_{\mathrm{N}}^\pm = 0$ a.e. in (a,b), $\pm\varphi_{\mathrm{N}}^\pm > 0$ in $[a,b)$, $\varphi_{\mathrm{N}}^\pm(b) = 0$ and $(\varphi_{\mathrm{N}}^\pm)'(a) = 0$ if $a > 0$.*

First we fix a $\kappa \in \mathbb{R}$ so that

$$\sigma_\kappa = C_3 \lambda^{-1} R^{2-N/q} \|(\gamma - \kappa)_+\|_{L^q_{\mathrm{r}}(0,R)} < 1.$$

Next, for every $f \in L^q_{\mathrm{r}}(a,b)$, we consider

$$\mathcal{F}_\kappa[u] + f = 0 \quad \text{a.e. in } (a,b),\ u(b) = 0,\ u \in W^{2,q}_{\mathrm{r}}(a,b),\ u'(a) = 0 \text{ if } a > 0. \tag{21}$$

Put $\hat{\mathcal{F}}(m,l,p,u,r) := \mathcal{F}_\kappa(m,l,p,u,r) + f(r)$. Then it is easy to see that $\tilde{\mathcal{F}}$ satisfies (r-F1), (r-F2) and $\hat{\mathcal{F}}[0] \in L^q_{\mathrm{r}}(a,b)$. Hence according to Theorem 5.1, there is a unique solution $u \in W^{2,q}_{\mathrm{r}}(a,b)$ to (21). We introduce the solution mapping T_{N}: $L^q_{\mathrm{r}}(a,b) \to W^{2,q}_{\mathrm{r}}(a,b)$ by $T_{\mathrm{N}} f(r) := u(r)$. Noting $\hat{\mathcal{F}}[0] = f$, T_{N} satisfies

$$\|T_{\mathrm{N}} f\|_{W^{2,q}_{\mathrm{r}}(a,b)} \le C \|f\|_{L^q_{\mathrm{r}}(a,b)} \tag{22}$$

for every $f \in L^q_{\mathrm{r}}(a,b)$.

Lemma 6.2. *The following hold:*
(i) *If $f \ge 0$ a.e. (a,b), then $(T_{\mathrm{N}} f) \ge 0$ in $[a,b]$. Furthermore, if $f \not\equiv 0$, then $T_{\mathrm{N}} f > 0$ in $[a,b)$, $(T_{\mathrm{N}} f)'(b) < 0$.*
(ii) *Let $f_k \to f_0$ strongly in $L^q_{\mathrm{r}}(a,b)$. Then $T_{\mathrm{N}} f_k \to T_{\mathrm{N}} f_0$ strongly in $W^{2,q}_{\mathrm{r}}(a,b)$.*

Proof. (i) Set $u(r) = T_{\mathrm{N}} f$. Since f is nonnegative, $\mathcal{F}_\kappa[u] + f = 0 \le \mathcal{F}_\kappa[0] + f$ in (a,b). Thus by Proposition 4.1, we have $0 \le u$ in $[a,b]$. Furthermore, if $f \not\equiv 0$, then u satisfies $\mathcal{P}^-[u] - \beta|u'| - (\gamma + |\kappa|)u \le 0$ a.e. in (a,b). Thus Proposition 4.2 shows $u > 0$ in $[a,b)$ and $u'(b) < 0$.
(ii) Next let $f_k \to f_0$ strongly in $L^q_{\mathrm{r}}(a,b)$ and set $u_k(r) := (T_{\mathrm{N}} f_k)(r)$. For each $k, \ell \in \mathbb{N}$, we obtain, $u_k'(a) = u_l'(a) = 0$ if $a > 0$, $u_k(b) = u_l(b) = 0$ and

$$\begin{aligned}
0 &= \mathcal{F}_\kappa[u_k] + f_k - \mathcal{F}_\kappa[u_l] - f_l \\
&\le \mathcal{P}^+[u_k - u_l] + \beta|u_k' - u_l'| + (\gamma + \kappa)|u_k - u_l| + |f_k - f_l|, \\
0 &\ge \mathcal{P}^-[u_k - u_l] - \beta|u_k' - u_l'| - (\gamma + \kappa)|u_k - u_l| - |f_k - f_l| \quad \text{a.e. in } (a,b).
\end{aligned}$$

We apply Proposition 3.6 to get

$$\|u_k - u_l\|_{W^{2,q}_{\mathrm{r}}(a,b)} \le C(\|(\gamma + \kappa)\|_{L^q_{\mathrm{r}}(a,b)} \|u_k - u_l\|_{L^\infty(a,b)} + \|f_k - f_l\|_{L^q_{\mathrm{r}}(a,b)}). \tag{23}$$

It follows from (22) that (u_k) is bounded in $W^{2,q}_{\mathrm{r}}(a,b)$. Taking a subsequence, we may assume $u_{k_j} \rightharpoonup u$ weakly in $W^{2,q}_{\mathrm{r}}(a,b)$ and strongly in $L^\infty(a,b)$. Hence, by (23), $u_{k_j} \to u$ strongly in $W^{2,q}_{\mathrm{r}}(a,b)$.

Next we show u solves $\mathcal{F}_\kappa[u] + f_0 = 0$ in (a,b). If we showed this claim, then by the uniqueness, $u = u_0$ holds. Thus the uniqueness of the weak limit implies $u_k \rightharpoonup u_0$ weakly in $W^{2,q}_{\mathrm{r}}(a,b)$. Therefore $u_k \to u_0$ strongly in $W^{2,q}_{\mathrm{r}}(a,b)$ from (23).

Since $\mathcal{F}_\kappa[u_k] + f_k = 0$ in (a,b), we have

$$u_{k_j}''(r) = g_{\mathcal{F}_\kappa}(u_{k_j}'(r)/r, u_{k_j}'(r), u_{k_j}(r), f_{k_j}(r), r).$$

Thus for every $a < s < t < b$, it holds

$$u'_{k_j}(t) - u'_{k_j}(s) = \int_s^t g_{\mathcal{F}_\kappa}(u'_{k_j}(\tau)/\tau, u'_{k_j}(\tau), u_{k_j}(\tau), f_{k_j}(\tau), \tau)d\tau.$$

Noting that $u_{k_j} \to u$ strongly in $C^1_{\mathrm{loc}}(a,b)$, from Lemma 2.2 and Lebesgue's dominated convergence theorem, we obtain

$$u'(t) - u'(s) = \int_s^t g_{\mathcal{F}_\kappa}(u(\tau)/\tau, u'(\tau), u(\tau), f_0(\tau), \tau)d\tau$$

for each $a < s < t < b$. This means $u''(r) = g_{\mathcal{F}_\kappa}(u'(r)/r, u'(r), u(r), f_0(r), r)$, so does $\mathcal{F}_\kappa[u] + f_0 = 0$. □

Define $X_{\mathrm{N}} \subset W^{2,q}_{\mathrm{r}}(a,b)$ by

$$X_{\mathrm{N}} := \{f \in W^{2,q}_{\mathrm{r}}(a,b) : \ f > 0 \text{ in } [a,b), \ f(b) = 0, \ f'(b) < 0\}.$$

We equip $W^{2,q}_{\mathrm{r}}(a,b)$ norm into X_{N}. Then, in view of Lemma 6.2, we see that $T_{\mathrm{N}}f \in X_{\mathrm{N}}$ if $f \in X_{\mathrm{N}}$ and $T_{\mathrm{N}} : X_{\mathrm{N}} \to X_{\mathrm{N}}$ is continuous.

Next for each $f \in X_{\mathrm{N}}$, we define R_{N} by

$$R_{\mathrm{N}}f(r) := \begin{cases} T_{\mathrm{N}}f(r)/f(r) & \text{if } r \in [a,b), \\ (T_{\mathrm{N}}f)'(b)/f'(b) & \text{if } r = b, \end{cases}$$

It follows from (r-F3) that for any $t \geq 0$ and $f \in X_{\mathrm{N}}$,

$$R_{\mathrm{N}}(tf)(r) = R_{\mathrm{N}}f(r). \tag{24}$$

Lemma 6.3. *The following hold:*
(i) *If $f \in X_{\mathrm{N}}$, then $R_{\mathrm{N}}f \in C([a,b])$ and $0 < \min_{[a,b]} R_{\mathrm{N}}f \leq \max_{[a,b]} R_{\mathrm{N}}f < \infty$.*
(ii) *The map $R_{\mathrm{N}} : X_{\mathrm{N}} \to C([a,b])$ is continuous.*

Proof. Noting L'Hôpital's rule, it is easy to see that the assertion (i) holds. We turn to the assertion (ii). Let $f_n, f_0 \in X_{\mathrm{N}}$ satisfy $f_n \to f_0$ strongly in $W^{2,q}_{\mathrm{r}}(a,b)$. By Lemma 6.2, $T_{\mathrm{N}}f_n \to T_{\mathrm{N}}f_0$ strongly in $W^{2,q}_{\mathrm{r}}(a,b)$. In particular, we have $f_n \to f_0$ and $T_{\mathrm{N}}f_n \to T_{\mathrm{N}}f_0$ strongly in $C^1_{\mathrm{loc}}((a,b])$ and $C([a,b])$. Since $f_0(0) > 0$, $R_{\mathrm{N}}f_n \to R_{\mathrm{N}}f_0$ uniformly in $[a, a+\delta]$ for some $\delta > 0$.

On the other hand, we see that

$$R_{\mathrm{N}}f_n(r) = \left(\int_{a+\delta}^r (T_{\mathrm{N}}f_n)'(s)ds + T_{\mathrm{N}}f_n(a+\delta)\right) \Big/ \left(\int_{a+\delta}^r f'_n(s)ds + f_n(a+\delta)\right),$$
$$R_{\mathrm{N}}f_0(r) = \left(\int_{a+\delta}^r (Tf_0)'(s)ds + T_{\mathrm{N}}f_0(a+\delta)\right) \Big/ \left(\int_{a+\delta}^r f'_0(s)ds + f_0(a+\delta)\right).$$

From these expressions, $R_{\mathrm{N}}f_n \to R_{\mathrm{N}}f_0$ uniformly in $[a+\delta, b]$. Thus we complete the proof. □

Lemma 6.4. *Let $f \in X_{\mathrm{N}}$ and $u = T_{\mathrm{N}} f$. Then*

$$\min_{[a,b]} R_{\mathrm{N}} f \leq \min_{[a,b]} R_{\mathrm{N}} u \leq \max_{[a,b]} R_{\mathrm{N}} u \leq \max_{[a,b]} R_{\mathrm{N}} f.$$

Moreover, if $\min_{[a,b]} R_{\mathrm{N}} f = \min_{[a,b]} R_{\mathrm{N}} u$, then

$$T_{\mathrm{N}} u = \left(\min_{[a,b]} R_{\mathrm{N}} f \right) u \quad \text{in } [a,b].$$

Proof. Set $v := T_{\mathrm{N}} u$ and $\theta = \min_{[a,b]} R_{\mathrm{N}} f$. Since $\theta f \leq u$ in $[a,b]$, it follows from (r-F3) that $\mathcal{F}_\kappa[v] + \theta f \leq 0 = \mathcal{F}_\kappa[\theta u] + \theta f$ in (a,b). Thus Proposition 4.1 yields $\theta u(r) \leq v(r)$ for all $r \in [a,b]$, which implies $\min_{[a,b]} R_{\mathrm{N}} u = \theta \leq \min_{[a,b]} R_{\mathrm{N}} u$. In a similar way, one can show $\max_{[a,b]} R_{\mathrm{N}} u \leq \max_{[a,b]} R_{\mathrm{N}} f$.

Next we suppose $\theta = \min_{[a,b]} R_{\mathrm{N}} f = \min_{[a,b]} R_{\mathrm{N}} u$. Setting $v := T_{\mathrm{N}} u$, then we have $\theta u \leq v$.

On the other hand, by (r-F2) and $\theta f \leq u$ in $[a,b]$, we can prove

$$0 = \mathcal{F}_\kappa[v] + u - \mathcal{F}_\kappa[\theta u] - \theta f \geq \mathcal{P}^-[w] - \beta |w'| - (\gamma + |\kappa|) w \quad \text{in } (a,b)$$

where $w(r) := v(r) - \theta u(r) \geq 0$. Thus by Proposition 4.2, it holds either $w \equiv 0$ in $[a,b]$ or $w(r) > 0$ for any $r \in [a,b)$ and $w'(b) < 0$. If the latter case happens, then we obtain $\theta < \min_{[a,b]} R_{\mathrm{N}} u$. This is a contradiction, hence $v \equiv \theta u$ holds. □

Proof of Theorem 6.1. First we remark that it is sufficient to prove for $(\mu_{\mathrm{N}}^+, \varphi_{\mathrm{N}}^+)$. Indeed, set $\mathcal{G}(m,l,p,u,r) := -\mathcal{F}(-m,-l,-p,-u,r)$. Then $\mathcal{G}$ satisfies (r-F1)–(r-F3) if and only if $\mathcal{F}$ satisfies (r-F1)–(r-F3). Furthermore, let $(\nu^+, \psi^+) \in \mathbb{R} \times W_{\mathrm{r}}^{2,q}(a,b)$ satisfy $\mathcal{G}[\psi^+] + \nu^+ \psi^+ = 0$ in (a,b) with $\psi(b) = 0$ and $\psi'(a) = 0$ if $a > 0$. Then it is easily seen that $(\nu^+, -\psi^+)$ is a negative eigenpair of $\mathcal{F}$. Therefore, it is enough to show for $(\mu_{\mathrm{N}}^+, \varphi_{\mathrm{N}}^+)$.

Now we prove the existence of $(\mu_{\mathrm{N}}^+, \varphi_{\mathrm{N}}^+)$. Let $f_0 \in X_{\mathrm{N}}$ satisfy $\|f\|_{L^\infty(a,b)} = 1$ and define u_n and f_n as follows:

$$u_n(r) := T_{\mathrm{N}} f_{n-1}(r) \quad \text{and} \quad f_n(r) := u_n(r)/\|u_n\|_{L^\infty(a,b)}.$$

Set also $\theta_n := \min_{[a,b]} R_{\mathrm{N}} u_n$ and $\Theta_n := \max_{[a,b]} R_{\mathrm{N}} u_n$. First, note that (u_n) is bounded in $W_{\mathrm{r}}^{2,q}(a,b)$ from (22). Second, by Lemma 6.4, we have $0 < \theta_n \leq \theta_{n+1} \leq \Theta_{n+1} \leq \Theta_n$. So we assume $\theta_n \to \theta > 0$. Furthermore, noting $R_{\mathrm{N}} u_n = R_{\mathrm{N}} f_n$ by (24), it holds that

$$\theta_n f_n(r) \leq u_{n+1}(r) \leq \Theta_n f_n(r) \quad \text{for all } r \in [a,b],$$

which implies $\theta_n \leq \|u_n\|_{L^\infty(a,b)} \leq \Theta_n$.

Now we assume $q < \infty$. Taking a subsequence if necessary, we may suppose that there exists a $u \in W_{\mathrm{r}}^{2,q}(a,b)$ such that $u_{n_k} \rightharpoonup u$ weakly in $W_{\mathrm{r}}^{2,q}(a,b)$. Furthermore, $\theta \leq \|u\|_{L^\infty(a,b)}$ holds, which implies $f_{n_k} = u_{n_k}/\|u_{n_k}\|_{L^\infty(a,b)} \to u/\|u\|_{L^\infty(a,b)}$ strongly in $L^\infty(a,b)$. Thus $u_{n_k+1} = T_{\mathrm{N}} f_{n_k} \to T_{\mathrm{N}} u/\|u\|_{L^\infty(a,b)} =: v$ strongly in $W_{\mathrm{r}}^{2,q}(a,b)$ from Lemma 6.2. By Lemma 6.3, we obtain

$$\min_{[a,b]} R_{\mathrm{N}} v = \lim_{k \to \infty} \min_{[a,b]} R_{\mathrm{N}} u_{n_k+1} = \lim_{n_k \to \infty} \theta_{n_k+1} = \theta.$$

Since $R_{\mathrm{N}}(T_{\mathrm{N}}u_{n_k+1}) = R_{\mathrm{N}}(T_{\mathrm{N}}f_{n_k+1}) = R_{\mathrm{N}}u_{n_k+2}$ holds, we also have

$$\min_{[a,b]} R_{\mathrm{N}}(T_{\mathrm{N}}v) = \lim_{n_k\to\infty} \min_{[a,b]} R_{\mathrm{N}}u_{n_k+2} = \theta.$$

Hence, by Lemma 6.4, one can show $T_{\mathrm{N}}v \equiv \theta v$ in $[a,b]$, which implies that $(\mu^+,\varphi^+) = (\theta^{-1}+\kappa, v)$ is a positive eigenpair of (3).

When $q=\infty$, from the boundedness of (u_n) in $W^{2,\infty}_{\mathrm{r}}(a,b)$, there exist a subsequence (u_{n_k}) and u such that $u_{n_k} \rightharpoonup u$ weakly in $W^{2,m}_{\mathrm{r}}(a,b)$ for each $m \in \mathbb{N}$ with $m \geq N$. We remark that T_{N} and R_{N} depend on q and to stress it, here we write $R_{\mathrm{N},q}$ and $T_{\mathrm{N},q}$. If $f \in W^{2,q_1}_{\mathrm{r}}(a,b) \cap W^{2,q_2}_{\mathrm{r}}(a,b)$ with $q_1 < q_2$, then we can prove $T_{\mathrm{N},q_1}f = T_{\mathrm{N},q_2}f$. Thus repeating the above argument, the pair $(\theta^{-1}+\kappa,\ T_{\mathrm{N}}u/\|u\|_{L^\infty(a,b)})$ is a positive eigenpair in $\mathbb{R}\times W^{2,m}_{\mathrm{r}}(a,b)$ for every $m \geq N$. Moreover, since $\|u\|_{W^{2,m}_{\mathrm{r}}(a,b)} \leq C\sup_{n\geq 1}\|u_n\|_{W^{2,\infty}_{\mathrm{r}}(a,b)}$ for all $m \geq N$, we have $u \in W^{2,\infty}_{\mathrm{r}}(a,b)$, which completes the proof. □

Next, we prove the simplicity of the principal eigenpairs.

Proposition 6.5. *Let $0 < b \leq R$, $(\mu,\varphi) \in W^{2,q}_{\mathrm{r}}(0,b)$ satisfy $\mathcal{F}[\varphi] + \mu\varphi = 0$ a.e. in $(0,b)$, $\varphi \geq 0$, $\varphi \not\equiv 0$ and $\varphi(b) = 0$. Then there exists a $\theta > 0$ such that $(\mu,\varphi) = (\mu^+_{\mathrm{N}}, \theta\varphi^+_{\mathrm{N}})$ holds. Similarly, the simplicity of $(\mu^-_{\mathrm{N}},\varphi^-_{\mathrm{N}})$ also holds.*

Proof. First we remark that for any $\kappa \in \mathbb{R}$, (μ,φ) satisfies $\mathcal{F}_\kappa[\varphi] + (\kappa+\mu)\varphi = 0$ a.e. in $(0,b)$. Furthermore, taking $\kappa > 0$ sufficiently large, we may assume $\kappa+\mu > 0$, $\kappa+\mu^+_{\mathrm{N}} > 0$ and $\sigma_\kappa < 1$ defined in (17). Since $\varphi \not\equiv 0$, it follows from Lemma 6.2 that $\varphi > 0$ in $[0,b)$ and $\varphi'(b) < 0$.

Now we assume $\mu^+_{\mathrm{N}} \leq \mu$ and set $\theta := \inf_{[0,b)} \varphi/\varphi^+$. Noting $\theta\varphi^+ \leq \varphi$ in $[0,b)$ and (r-F3), we obtain

$$\mathcal{F}_\kappa[\varphi] = -(\kappa+\mu)\varphi \leq -(\kappa+\mu^+_{\mathrm{N}})\theta\varphi_{\mathrm{N}} = \mathcal{F}_\kappa[\theta\varphi^+_{\mathrm{N}}] \quad \text{a.e. in } (0,b).$$

Thus

$$\mathcal{P}^-[w] - \beta|w'| - (\gamma+\kappa)w \leq 0 \quad \text{a.e. in } (0,b)$$

where $w := \varphi - \theta\varphi^+_{\mathrm{N}}$. By Proposition 4.2, we see either $w \equiv 0$ in $[0,b]$ or $w > 0$ in $[0,b)$ and $w'(b) < 0$ holds. If the latter case happens, then $\theta < \inf_{[0,b)} \varphi/\varphi^+_{\mathrm{N}}$ holds, which is a contradiction. Thus $\varphi \equiv \theta\varphi^+_{\mathrm{N}}$ and $\mu = \mu^+_{\mathrm{N}}$ hold.

In the case where $\mu < \mu^+_{\mathrm{N}}$, exchanging the role of φ and φ^+_{N} in the above, we get the same conclusion. For the negative eigenpair, it is reduced to the positive case by using the function $\mathcal{G}(m,l,p,u,r) = -\mathcal{F}(-m,-l,-p,-u,r)$. □

By Proposition 6.5, the positive and negative eigenvalue of $\mathcal{F}$ in $[0,b]$ are unique for each $b \in (0,R]$. Thus we denote them by $\mu^+_{\mathrm{N}}(0,b)$ and $\mu^-_{\mathrm{N}}(0,b)$, respectively.

Proposition 6.6. *Let $0 < b_1 < b_2 \leq R$. Then $\mu^\pm_{\mathrm{N}}(0,b_2) < \mu^\pm_{\mathrm{N}}(0,b_1)$ holds. Furthermore, the functions $b \mapsto \mu^\pm_{\mathrm{N}}(0,b)$ are continuous in $(0,R]$ and $\mu^\pm_{\mathrm{N}}(0,b) \to \infty$ as $b \to 0$.*

Proof. We only show for $\mu_N^+(0,b)$. Now we argue by contradiction. Suppose $\mu_2 := \mu_N^+(0,b_2) \le \mu_N^+(0,b_1) := \mu_1$ and denote the corresponding eigenfunctions by φ_1 and φ_2, respectively. Put $\theta := \inf_{[0,b_1]} \varphi_2/\varphi_1$. Then $\theta\varphi_1 \le \varphi_2$ in $[0,b_1)$. Thus as in the above,

$$\mathcal{P}^-[w] - \beta|w'| - (\kappa+\gamma)w \le 0 \quad \text{a.e. in } (0,b_1)$$

and $w(b) > 0$ where $w := \varphi_2 - \theta\varphi_1$. So Proposition 4.2 tells us that $w > 0$ in $[0,b_1]$, which contradicts to the definition of θ. Thus we get $\mu_1 > \mu_2$.

Next, we show the continuity of μ_N^+. Let $b_n \to b_0 > 0$, $\mu_n := \mu_N^+(0,b_n)$ and φ_n be a corresponding positive eigenfunction with $\|\varphi_n\|_{L^\infty(0,b_n)} = 1$. Furthermore, by extending φ_n into $[b_n, R]$ appropriately, we suppose $\varphi_n \in W_r^{2,q}(0,R)$. We may also assume $b_0/2 \le b_n \le R$ without loss of generality.

By the monotonicity of μ_N^+, we have $\mu_N^+(0,R) \le b_n \le \mu_N^+(0,b_0/2)$. So it follows from (22) that (φ_n) is bounded in $W_r^{2,q}(0,R)$. Thus in the case where $q < \infty$, taking a subsequence if necessary, we may suppose $\varphi_{n_k} \rightharpoonup \varphi_0$ weakly in $W_r^{2,q}(0,R)$ and $\mu_{n_k} \to \mu_0$. As in the proof of Proposition 6.1, one can show that (μ_0,φ_0) is an eigenpair with $\|\varphi_0\|_{L^\infty(0,b_0)} = 1$ and $\varphi_0 > 0$ in $[0,b_0)$. Thus it holds from the simplicity of the positive eigenvalue, $\mu_0 = \mu_N^+(0,b_0)$ holds. Therefore the uniqueness of the limit implies $\mu_n \to \mu_N^+(0,b_0)$. The case $q = \infty$ can also be treated similarly.

Lastly, we show $\mu_N^+(0,b) \to \infty$ as $b \to 0$. Let (μ_b,φ_b) be a positive eigenpair with $\|\varphi_b\|_{L^\infty(0,b)} = 1$. Then we have $\mathcal{P}^+[\varphi_b] + \beta|\varphi_b'| + (\gamma + |\mu_b|)\varphi_b \ge 0$ a.e. in $(0,b)$. Using Lemma 3.5, we obtain

$$1 = \max_{[0,b]} \varphi_b \le C_3 b^{(2q-N)/q} \|(\gamma + |\mu_b|)\varphi_b\|_{L_r^q(0,b)} \le C_3 b^{(2q-N)/q} \|\gamma + |\mu_b|\|_{L_r^q(0,b)}.$$

The above inequality shows $|\mu_b| \to \infty$ as $b \to 0$. Furthermore, it follows the monotonicity of μ_b that $\mu_b \to \infty$ as $b \to 0$. □

7 Existence of general Eigenpairs

In this section, we shall prove Theorem 1.2. First we prove the existence and simplicity of general eigenpairs.

Theorem 7.1. *Assume* $N \ge 2$, $q \in (\max\{N/2, q_*\}, \infty]$, (r-F1)–(r-F3) *with* $\Lambda < \infty$ *and* $\beta \in L_r^N(0,R)$ *if* $q < N$.
(i) *For each* $n \in \mathbb{N}$, *there exist eigenpairs* $(\mu_n^\pm, \varphi_n^\pm) \in \mathbb{R} \times W_r^{2,q}(0,R)$ *of* (3) *and sequences* $(r_{n,j}^\pm)_{j=0}^n \subset [0,R]$ *such that*

$$\begin{cases} 0 = r_{n,0}^+ < r_{n,1}^+ < \cdots < r_{n,n}^+ = R, \ 0 = r_{n,0}^- < r_{n,1}^- < \cdots < r_{n,n}^- = R, \\ (-1)^{j-1}\varphi_n^+(r) > 0 \text{ in } (r_{n,j-1}^+, r_{n,j}^+) \text{ for } j = 1,\ldots,n, \\ (-1)^j \varphi_n^-(r) > 0 \text{ in } (r_{n,j-1}^-, r_{n,j}^-) \text{ for } j = 1,\ldots,n, \\ \varphi_n^+(0) > 0 > \varphi_n^-(0). \end{cases}$$

(ii) *Let* $(\mu,\varphi) \in \mathbb{R} \times W_r^{2,q}(0,R)$ *be an eigenpair of* (3) *and have* $n-1$ *zeroes* $(t_j)_{j=1}^{n-1}$ *in* $(0,R)$. *Then there exists a* $\theta > 0$ *such that either* $(\mu,\varphi) = (\mu_n^+, \theta\varphi)$ *or* $(\mu,\varphi) = (\mu_n^-, \theta\varphi)$ *holds.*

To prove Theorem 7.1, we introduce the following eigenvalue problems: for each $0 < a < b \le R$,

(25) $\mathcal{F}[u] + \mu u = 0$ a.e. in (a,b), $u \in W^{2,q}_{\mathrm{r}}(a,b)$, $u > 0$ in (a,b), $u(a) = u(b) = 0$.

Now we define $\mathcal{H}$ by

$$\mathcal{H}(m,p,u,x) := \mathcal{F}(m, p/x, p, u, x) : \ \mathbb{R}^3 \times (a,b) \to \mathbb{R}.$$

Note that $\mathcal{H}$ satisfies (F1)-(F3) in (a,b) and for $u(x) = u(|x|)$, $\mathcal{F}[u] + \mu u = 0$ in (a,b) if and only if $\mathcal{H}(u''(x), u'(x), u(x), x) + \mu u(x) = 0$ in (a,b). Thus we can apply Theorem 1.1 and obtain the following result.

Proposition 7.2. *For any $0 < a < b \le R$, (25) has positive and negative eigenpairs $(\mu^{\pm}_{\mathrm{D}}, \varphi^{\pm}_{\mathrm{D}})$ which are simple. If we denote the unique positve and negative eigenvalues on $[a,b]$ by $\mu^{+}_{\mathrm{D}}(a,b)$ and $\mu^{-}_{\mathrm{D}}(a,b)$, then*
(i) *$\mu^{\pm}_{\mathrm{D}}(a_1,b_1) < \mu^{\pm}_{\mathrm{D}}(a_2,b_2)$ if $[a_2,b_2] \subset [a_1,b_1]$ and $[a_2,b_2] \ne [a_1,b_1]$.*
(ii) *The maps $(a,b) \mapsto \mu^{\pm}_{\mathrm{D}}(a,b) : \{(a,b) \in \mathbb{R}^2 : \ 0 < a < b < R\} \to \mathbb{R}$ are continuous.*
(iii) *As $\varepsilon \to 0$, $\inf\{\mu^{\pm}_{\mathrm{D}}(a,b) : 0 < a < b \le R, \ b - a < \varepsilon\} \to \infty$.*

The following two lemmas can be shown as in [13], so we omit a proof.

Lemma 7.3. *Let $h : (0,R) \to (0,R)$ be a nondecreasing continuous function such that $f(s) \le s$ in $(0,R)$. Then there exists unique functions $\tau^{\pm} : (0,R] \to (0,R)$ such that $\tau^{\pm}(t) < t$ and $\mu^{+}_{\mathrm{N}}(0, h(\tau^{\mp}(t))) = \mu^{\mp}_{\mathrm{D}}(\tau^{\mp}(t), t)$ for each $t \in (0,R]$. Furtheremore, the functions $\tau^{\pm}$ are continuous and strictly increasing in $(0,R]$.*

Lemma 7.4. *Let $n \in \mathbb{N}$ and $(r_j)_{j=0}^{n}, (s_j)_{j=0}^{n} \subset [0,R]$ be increasing sequences such that $r_0 = s_0 = 0$ and $r_n = s_n = R$. Then there exist $j,k \in \{1,\dots,n\}$ such that $[r_{j-1}, r_j] \subset [s_{j-1}, s_j]$ and $[s_{k-1}.s_k] \subset [r_{k-1}, r_k]$.*

Now we give a proof of Theorem 7.1.

Proof of Theorem 7.1. As in Proposition 6.1, it is enough to show only for $(\mu^{+}_{n}, \varphi^{+}_{n})$. First we treat the existence.

We show that for any $n \in \mathbb{N}$, there is a sequence $(r_{n,j}(t))_{j=1}^{n}$ of functions on $(0,R]$ such that

$$a < r_{n,1}(t) < r_{n,2}(t) < \ldots < r_{n,n}(t) = t \text{ for every } t \in (0,R], \tag{26}$$
$$r_{n,j}(t) \text{ is continuous and strictly increasing on } (0,R], \tag{27}$$
$$\mu^{s_j}_{\mathrm{D}}(r_{n,j-1}(t), r_{n,j}(t)) = \mu^{+}_{\mathrm{N}}(0, r_{n,1}(t)) \text{ for all } t \in (0,R] \text{ and } j \ge 2. \tag{28}$$

Here s_j stands for the symbol $+$ if j is odd and $-$ if j is even.

For $n = 1$, the function $r_{1,1}(t) = t$ clearly satisfies (26)–(28). We show by induction, so suppose that there is a sequence $(r_{n,j})_{j=1}^{n}$ satisfying (26)–(28). We apply Lemma 7.3 to obtain an increasing continuous function τ on $(0,R]$ such that $\tau(t) < t$ and $\mu^{+}_{\mathrm{N}}(0, r_{n,1}(\tau(t))) = \mu^{s_{n+1}}_{\mathrm{D}}(\tau(t), t)$ for all $t \in (0,R]$. Now define $r_{n+1,j}(t) = r_{n,j} \circ \tau(t)$ for every $1 \le j \le n$ and $r_{n+1,n+1}(t) = t$. Then it is easily seen that (26) and

(27) hold. Furthermore, since $r_{n,n}(t) = t$ and $\mu_{\mathrm{N}}^{+}(0, r_{n,1}(t)) = \mu_{\mathrm{D}}^{s_j}(r_{n,j-1}(t), r_{n,j}(t))$ for each $2 \leq j \leq n$ and $t \in (0, R]$, we have

$$\mu_{\mathrm{D}}^{s_{n+1}}(r_{n,n} \circ \tau(t), t) = \mu_{\mathrm{N}}^{+}(0, r_{n,1} \circ \tau(t)) = \mu_{\mathrm{D}}^{s_j}(r_{n,j-1} \circ \tau(t), r_{n,j} \circ \tau(t))$$

for any $t \in (0, R]$ and $2 \leq j \leq n$. Hence $(r_{n+1,j}(t))_{j=1}^{n+1}$ satisfies (26)–(28).

Now we prove the existence for $n \geq 2$. Set $r_{n,0}^{+} = 0$, $r_{n,j}^{+} = r_{n,j}(R)$ for each $j = 1, \ldots, n$ and $\mu_n^{+} = \mu_{\mathrm{N}}^{+}(0, r_{n,1}^{+})$. Then by (28), $\mu_n^{+} = \mu_{\mathrm{D}}^{s_j}(r_{n,j-1}^{+}, r_{n,j}^{+})$ holds for all $2 \leq j \leq n$. Let $\varphi_{n,1} \in W_{\mathrm{r}}^{2,q}(r_{n,0}^{+}, r_{n,1}^{+})$ be a positive eigenfunction corresponding to $\mu_{\mathrm{N}}^{+}(r_{n,0}^{+}, r_{n,1}^{+})$ and $\varphi_{n,j} \in W_{\mathrm{r}}^{2,q}(r_{n,j-1}^{+}, r_{n,j}^{+})$ an eigenfunction corresponding to $\mu_{\mathrm{D}}^{s_j}(r_{n,j-1}, r_{n,j})$. Then we obtain $(-1)^{j-1}\varphi_{n,j} > 0$ in $(r_{n,j-1}^{+}, r_{n,j}^{+})$ and

$$(29) \qquad (-1)^{j}\varphi_{n,j}'(r_{n,j}^{+} - 0) > 0 \quad \text{and} \quad (-1)^{k-1}\varphi_{n,k}'(r_{n,k}^{+} + 0) > 0$$

for every $1 \leq j \leq n$ and $2 \leq k \leq n$. Thus we can find a sequence $(\theta_j)_{j=1}^{n}$ of positive numbers such that

$$(30) \qquad \theta_1 = 1, \quad \theta_{j-1}\varphi_{j-1}'(r_{n,j-1}^{+} - 0) = \theta_j\varphi_j'(r_{n,j-1}^{+} + 0) \quad \text{for any } j = 2, \ldots, n.$$

Define φ_n^{+} by

$$\varphi_n^{+}(r) := \theta_j\varphi_{n,j}(r) \quad \text{if } r \in [r_{n,j-1}^{+}, r_{n,j}^{+}] \text{ and } 1 \leq r \leq n.$$

From (30), $\varphi \in W_{\mathrm{r}}^{2,q}(0, R)$ and $(\mu_n^{+}, \varphi_n^{+})$ is an eigenpair of (3) with $(-1)^{j-1}\varphi_n^{+}(r) > 0$ in $(r_{n,j-1}^{+}, r_{n,j}^{+})$ and $\varphi_n^{+}(0) > 0$.

Next we deal with the assertion (ii). When $n = 1$, the claim holds from Proposition 6.5, hence let $n \geq 2$ and $(\mu, \varphi) \in \mathbb{R} \times W_{\mathrm{r}}^{2,q}(0, R)$ be an eigenpair of (3) with $n - 1$ zeroes $0 < t_1 < \ldots < t_{n-1} < R$. Set $t_0 = 0$ and $t_n = R$. It is enough to show the claim in the case where $\varphi > 0$ in $[t_0, t_1)$.

By Lemma 7.4, there exist $j, k \in \{1, \ldots, n\}$ satisfying $[r_{n,j-1}, r_{n,j}] \subset [t_{j-1}, t_j]$ and $[t_{k-1}, t_k] \subset [r_{n,k-1}, r_{n,k}]$. Note that $(-1)^{m-1}\varphi_{n,m}^{+} > 0$ in $(r_{n,m-1}, r_{n,m})$ and $(-1)^{m-1}\varphi > 0$ in (t_{m-1}, t_m) for all $1 \leq m \leq n$. We also remark that $\mu = \mu_{\mathrm{N}}^{+}(0, t_1) = \mu_{\mathrm{D}}^{s_m}(t_{m-1}, t_m)$ and φ is an eigenfunction on $(0, t_1)$ and (t_{j-1}, t_j) corresponding to $\mu_{\mathrm{N}}^{+}(0, t_1)$ and $\mu_{\mathrm{D}}^{s_m}(t_{m-1}, t_m)$ for $2 \leq m \leq n$. Hence by Propositions 6.6 and 7.2, we obtain $\mu_n^{+} \leq \mu$ and $\mu \leq \mu_n^{+}$, which implies $\mu = \mu_n^{+}$. Furthermore, again by Propositions 6.6 and 7.2, we see that $r_{n,j}^{+} = t_j$ for all $1 \leq j \leq n$ and there exists a sequence $(\theta_j)_{j=1}^{n}$ of positive numbers satisfying $\varphi = \theta_j\varphi_{n,j}$ on $[r_{n,j-1}^{+}, r_{n,j}^{+}]$ for each $j = 1, \ldots, n$. Noting that φ is of class C^1 and (29), $\theta_j \equiv \theta > 0$ holds for $1 \leq j \leq n$. This completes the proof. □

Proof of Theorem 1.2. By Proposition 7.1, it is sufficient to prove the completeness of $\{(\mu_n^{\pm}, \varphi_n^{\pm})\}_{n=1}^{\infty}$. Let $(\mu, \varphi) \in \mathbb{R} \times W_{\mathrm{r}}^{2,q}(0, R)$ be an eigenpair of (3). Then in view of Proposition 7.1, we only show that $\varphi(0) \neq 0$ and φ has finitely many zeroes in $(0, R)$.

First we show that there is no accumulation point in $(0, R)$ of zeroes of φ. We argue by contradiction and suppose that $(r_n)_{n=1}^{\infty}$ is a sequence of zeroes of φ satisfying $r_n \neq r_m$ if $n \neq m$ and $r_n \to r_0 \in (0, R)$. Then by Rolle's theorem, we see that

$\varphi(r_0) = \varphi'(r_0) = 0$. Then $\varphi \equiv 0$ holds from Proposition 2.3, which is a contradiction. Hence there is no accumulation point of zeroes of φ in $(0, R)$.

Next we consider the case where 0 is an accumulation point of zeroes of φ. Let $(r_n)_{n=1}^{\infty}$ be a sequence of zeroes of φ. From the above argument, we may assume $r_1 > r_2 > \ldots > 0$. Now choose n so large that

$$C_3 r_n^{(2q-N)/(q-1)} \|(\gamma + |\mu|)\|_{L_r^q(0,R)} < 1$$

where C_3 appears in Lemma 3.5. We may also suppose that $\varphi > 0$ in (r_{n+1}, r_n) and $\varphi(t_n) = \max_{[r_{n+1}, r_n]} \varphi = 1$ for some $t_n \in (r_{n+1}, r_n)$. Then $\varphi'(t_n) = 0$, $\varphi(r_n) = 0$ and $\mathcal{P}^+[\varphi] + \beta|\varphi'| + (\gamma + |\mu|)\varphi \geq 0$ a.e. in (t_n, r_n). It follows from Lemma 3.5 and the choice of r_n that

$$\begin{aligned} 1 = \max_{[t_n, r_n]} \varphi &\leq C_3 r_n^{(2q-N)/(q-1)} \|(\gamma + |\mu|)\varphi\|_{L_r^q(t_n, r_n)} \\ &\leq C_3 r_n^{(2q-N)/q} \|(\gamma + |\mu|)\|_{L_r^q(0,R)} < 1, \end{aligned}$$

which is a contradiction. Thus φ has finitely many zeroes in $[0, R]$.

Lastly we show $\varphi(0) > 0$. If $\varphi(0) = 0$, then $\mathcal{P}^-[\varphi] - \beta|\varphi'| - (\gamma + |\mu|)\gamma\varphi \leq 0$ a.e. in $(0, s)$ for sufficiently small $s > 0$ since φ has finitely many zeroes. Then Proposition 4.2 yields $\varphi \equiv 0$ on $[0, s]$, which is a contradiction. Hence $\varphi(0) > 0$ holds and we complete the proof. □

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