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Report on Centralizing Monoids on a Three-Element Set

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Abstract

For a set A with $|A| > 1$, a centralizing monoid on A is a set of unary functions defined on A which commute with all members of some set of multi-variable functions on A. In this paper we restrict ourselves to the case where A is a three-element set and present the list of all centralizing monoids on A. There are 192 centralizing monoids on a three-element set, which are divided into 48 conjugate classes.

Keywords: clone; centralizer; centralizing monoid

1 Introduction

Let A be a set with $|A| > 1$. For $n > 0$ denote by $\mathcal{O}_A^{(n)}$ the set of all n-variable functions defined over A having the range A, that is, the set of maps from A^n into A. Let $\mathcal{O}_A$ be the set of all functions defined over A, i.e., $\mathcal{O}_A = \bigcup_{n=1}^{\infty} \mathcal{O}_A^{(n)}$. The notion of commutation for multi-variable functions is defined as a natural generalization of commutation for unary functions. The centralizer F^* for a subset F of $\mathcal{O}_A$ is the set of functions which commute with all functions in F. A centralizing monoid is the unary part of some centralizer. For more than thirty years centralizers and centralizing monoids have been studied under various names (e.g., [Da79], [Sza85]). For our previous works on centralizing monoids refer to [MR09], [MR10] and [MR11].

The purpose of this paper is to present the list of all centralizing monoids on a three-element set. There exist 192 centralizing monoids on a three-element set, which are divided into 48 conjugate classes.

2 Definitions and Basic Facts

For functions $f \in \mathcal{O}_A^{(n)}$ and $g \in \mathcal{O}_A^{(m)}$, we say that f *commutes* with g, or f and g *commute*, if

$$f(g({}^t\boldsymbol{c}_1), \ldots, g({}^t\boldsymbol{c}_n)) \quad = \quad g(f(\boldsymbol{r}_1), \ldots, f(\boldsymbol{r}_m))$$

holds for every $m \times n$ matrix M over A with rows $\boldsymbol{r}_1, \ldots, \boldsymbol{r}_m$ and columns $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_n$.

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For example, $f \in \mathcal{O}_A^{(2)}$ and $g \in \mathcal{O}_A^{(3)}$ commute if

$$f\,(g(x_1,x_2,x_3),\, g(y_1,y_2,y_3)) \;=\; g\,(f\,(x_1,y_1),\, f\,(x_2,y_2),\, f\,(x_3,y_3))$$

holds for all $x_1, x_2, x_3,\ y_1, y_2, y_3 \in A$.

We write $f \perp g$ when f commutes with g. The binary relation $\perp$ on $\mathcal{O}_A$ is obviously a symmetric relation.

Remark

(1) For unary functions $f, g \in \mathcal{O}_A^{(1)}$, f and g commute if $f(g(x)) = g(f(x))$ holds for every $x \in A$. Thus, the commutation defined above is a natural generalization of the ordinary commutation for unary functions.

(2) For an algebra $\mathcal{A} = (A;\, F)$ and $g \in \mathcal{O}_A^{(1)}$, g is an *endomorphism* of $\mathcal{A}$ if

$$f(g(x_1), \ldots, g(x_n)) \;=\; g(f(x_1, \ldots, x_n))$$

holds for all $f \in F$ and $(x_1, \ldots, x_n) \in A^n$. This equation is equivalent to saying that f commutes with g in our terminology. Hence g is an endomorphism of $\mathcal{A}$ if and only if $f \perp g$ holds for every $f \in F$. Denote by $\mathrm{End}\,(\mathcal{A})$ the set of endomorphisms of $\mathcal{A}$, i.e., $\mathrm{End}\,(\mathcal{A}) = \{g \in \mathcal{O}_A^{(1)} \mid f \perp g \text{ for } \forall f \in F\}$.

Definition 2.1 *For $F \subseteq \mathcal{O}_A$ the* centralizer F^* *of F is defined by*

$$F^* \;=\; \{\, g \in \mathcal{O}_A \;\mid\; g \perp f \;\text{ for all }\; f \in F \,\}.$$

For any subset $F \subseteq \mathcal{O}_A$ the centralizer F^* is a clone, that is, F^* contains all projections and is closed under composition. (Note: A projection $e_i^n \in \mathcal{O}_A^{(n)}$, $1 \le i \le n$, is an n-variable function which always takes the i-th argument as its value.) When $F = \{f\}$ we often write f^* instead of F^*. We also write F^{**} for $(F^*)^*$. The map $F \longmapsto F^{**}$ is a closure operator on $\mathcal{O}_A$.

A non-empty subset M of $\mathcal{O}_A^{(1)}$ is a *monoid* if it is closed under composition and contains the identity *id*. The set $\mathcal{O}_A^{(1)}$ is the largest monoid on A and the singleton $\{id\}$ is the smallest monoid on A. For any centralizer F^* the unary part of F^*, i.e., $F^* \cap \mathcal{O}_A^{(1)}$, is a monoid.

We give the definition of a centralizing monoid with its equivalent properties.

Definition 2.2 *For $M \subseteq \mathcal{O}_A^{(1)}$, M is a* centralizing monoid *if M satisfies the equation*

$$M \;=\; M^{**} \cap \mathcal{O}_A^{(1)}.$$

Lemma 2.1 *For $M \subseteq \mathcal{O}_A^{(1)}$ the following conditions are equivalent.*

(1) *M is a centralizing monoid.*

(2) *For some subset $F \subseteq \mathcal{O}_A$, $M = F^* \cap \mathcal{O}_A^{(1)}$*

(3) *For some algebra $\mathcal{A} = (A;\, F)$, $M = \mathrm{End}\,(\mathcal{A})$*

The proof is straightforward. Note that Lemma 2.1 (2) asserts that a centralizing monoid is the unary part of some centralizer.

The following lemma, which we call the *Witness Lemma*, is equivalent to Lemma 2.1 (2).

Lemma 2.2 *For a monoid $M \subseteq \mathcal{O}_A^{(1)}$ and a subset $S \subseteq \mathcal{O}_A$, if the following conditions* (i) *and* (ii) *hold then M is a centralizing monoid.*

(i) *For any $f \in M$ and any $u \in S$, f and u commute, i.e., $f \perp u$.*

(ii) *For any $g \in \mathcal{O}_A^{(1)} \setminus M$ there exists $w \in S$ such that g does not commute with w, i.e., $g \not\perp w$.*

A subset S in the lemma will be called a *witness* for a centralizing monoid M. We denote by $M(S)$ the centralizing monoid M with S as its witness, i.e., $M(S) = S^* \cap \mathcal{O}_A^{(1)}$. In particular, when $S = \{f\}$ we write $M(f)$ instead of $M(\{f\})$.

Proposition 2.3 *([MR11]) Every centralizing monoid has a finite subset of $\mathcal{O}_A$ as its witness.*

A centralizing monoid M is *maximal* if there is no centralizing monoid M' satisfying $M \subset M' \subset \mathcal{O}_A^{(1)}$. (Here $\subset$ denotes the proper inclusion.)

A function f ($\in \mathcal{O}_A$) is called a *minimal function* if (i) f generates a minimal clone C and (ii) f has the minimum arity among functions generating C.

Proposition 2.4 *Every maximal centralizing monoid has a singleton set as its witness. Moreover, for every maximal centralizing monoid M there exists a minimal function f ($\in \mathcal{O}_A$) which serves as a witness of M, i.e., $M = M(f)$.*

3 On a Three-Element Set

In the sequel, we consider only the case where the base set A is $E_3 = \{0, 1, 2\}$. Following [La06], each unary function on E_3 will be denoted as in Table 1.

3.1 Review on Maximal Centralizing Monoids on E_3

Proposition 2.4 asserts that all maximal centralizing monoids can be obtained via minimal functions. Due to B. Csákány ([Cs83]) all minimal clones on E_3 are known. There are 84 minimal clones on E_3. We know all maximal centralizing monoids on E_3 from [MR11].

Proposition 3.1 *([MR11]) On E_3, there are 10 maximal centralizing monoids. Among them,* 3 *maximal centralizing monoids have unary constant functions as their witnesses, and* 7 *maximal centralizing monoids have ternary majority functions which generate minimal clones as their witnesses.*

	j_0	j_1	j_2	j_3	j_4	j_5	u_0	u_1	u_2	u_3	u_4	u_5	v_0	v_1	v_2	v_3	v_4	v_5
0	1	0	0	1	1	0	2	0	0	2	2	0	2	1	1	2	2	1
1	0	1	0	1	0	1	0	2	0	2	0	2	1	2	1	2	1	2
2	0	0	1	0	1	1	0	0	2	0	2	2	1	1	2	1	2	2

	s_1	s_2	s_3	s_4	s_5	s_6
0	0	0	1	1	2	2
1	1	2	0	2	0	1
2	2	1	2	0	1	0

	c_0	c_1	c_2
0	0	1	2
1	0	1	2
2	0	1	2

Table 1: Unary Functions in $\mathcal{O}_3^{(1)}$

Clearly, every constant function c_i, taking value i, ($i \in E_3$) generates a minimal clone. It is known ([Cs83]) that there are 7 minimal clones on E_3 generated by ternary majority functions. Hence, interestingly, every ternary majority function generating a minimal clone serves as a witness of a maximal centralizing monoid.

The following is the set of ternary majority functions generating minimal clones. (The numbering of majority functions is borrowed from [Cs83].) As noted above, each of the following majority functions serves as a witness of some maximal centralizing monoid.

Majority functions generating minimal clones (showing values only for mutually distinct x, y and z):

$$
\begin{aligned}
m_0\,(x,y,z) &= 0 \quad \text{if } |\{x,y,z\}| = 3 \\
m_{364}(x,y,z) &= 1 \quad \text{if } |\{x,y,z\}| = 3 \\
m_{728}(x,y,z) &= 2 \quad \text{if } |\{x,y,z\}| = 3 \\
m_{624}(x,y,z) &= y \quad \text{if } |\{x,y,z\}| = 3 \\
m_{109}(x,y,z) &= \begin{cases} 0 & \text{if } (x,y,z) \in \sigma \\ 1 & \text{if } (x,y,z) \in \tau \end{cases} \\
m_{473}(x,y,z) &= \begin{cases} 1 & \text{if } (x,y,z) \in \sigma \\ 2 & \text{if } (x,y,z) \in \tau \end{cases} \\
m_{510}(x,y,z) &= \begin{cases} 2 & \text{if } (x,y,z) \in \sigma \\ 0 & \text{if } (x,y,z) \in \tau \end{cases}
\end{aligned}
$$

Here σ and τ are the sets of triples: $\sigma = \{0,1,2),(1,2,0),(2,0,1)\}$ and $\tau = \{(0,2,1),(1,0,2),(2,1,0)\}$.

3.2 Strategy to Determine All Centralizing Monoids

The goal of this paper is to determine *all* centralizing monoids on $\{0,1,2\}$. The strategy to achieve this goal is the following.

(1) Choose a maximal centralizing monoid M_{max}.

(2) Find all submonoids of M_{max}.

(3) For each submonoid M of M_{max}, decide if M is a centralizing monoid or not.

(4) Repeat the above procedure for all maximal centralizing monoid.

(5) Finally, collect all submonoids, with repetitions deleted, which have been verified to be centralizing monoids.

A key step in the above strategy is, of course, the step (3). We use the following positive and negative tools to carry out the step (3).

POSITIVE TOOLS

In order to verify that a submonoid M is a centralizing monoid:

(P1) Find a subset $S \subseteq \mathcal{O}_3$ which serves as a witness for M. In other words, find a subset $S \subseteq \mathcal{O}_3$ which satisfies $M = M(S)$.

(P2) Find two centralizing submonoids M_1 and M_2 which satisfy $M_1 \cap M_2 = M$. (Remark: Let S_1 and S_2 be witnesses of M_1 and M_2, respectively. Then, obviously, $S_1 \cup S_2$ is a witness of $M_1 \cap M_2$.)

NEGATIVE TOOLS

In order to verify that a submonoid M is *not* a centralizing monoid:

(N1) Use Kuznetsov Criterion, which will be explained later (3.2.2 (N1)), to construct some specific function $g \in \mathcal{O}_A^{(1)}$ from functions $f_1, \ldots, f_m \in M$. If such function g does not belong to M then M is not a centralizing monoid.

(N2) Find a submonoid M' which is known to be a centralizing monoid and a submonoid N which is known *not* to be a centralizing monoid. If M' and N satisfy $M \cap M' = N$ then M is not a centralizing monoid.

3.2.1 Witness (P1)

From the witness lemma, $M\ (= M(S)) = \{ f \in \mathcal{O}_3^{(1)} \mid \forall g \in S,\ f \perp g \}$ for any subset $S \subseteq \mathcal{O}_3$ is a centralizing monoid with S as its witness. There are ample number of examples.

Example 1-1 Let $b_0(x,y)$ and $b_8(x,y)$ be binary functions given by the following tables:

$b_0 =$

$x \backslash y$	0	1	2
0	0	0	0
1	0	1	0
2	0	0	2

$b_8 =$

$x \backslash y$	0	1	2
0	0	0	0
1	0	1	0
2	2	2	2

(In passing, we note that b_0 and b_8 are minimal functions.) Then, we see that both

$$M(b_0) = \{ j_1, j_2, u_1, u_2, s_1, s_2, c_0, c_1, c_2 \} \quad \text{and} \quad M(b_8) = \{ j_1, u_2, u_3, s_1, c_0, c_1, c_2 \}$$

are centralizing monoids.

Example 1-2 Let $g(x,y)$ be defined by

$$g(x,y) = \begin{cases} x & \text{if } x+y \neq 3 \\ y & \text{if } x+y = 3 . \end{cases}$$

Cayley table of g is

$g =$

$x \backslash y$	0	1	2
0	0	0	0
1	1	1	2
2	2	1	2

Then, we see that

$$M(g) = \{ j_0, j_5, u_0, u_5, s_1, s_2, c_0, c_1, c_2 \}$$

is a centralizing monoid.

3.2.2 Intersection (P2)

Example 2 Consider three submonoids of $\mathcal{O}_3^{(1)}$:

$$\begin{aligned} M_1 &= \{ j_0, j_5, v_0, v_5, s_1, c_0, c_1, c_2 \} \\ M_2 &= \{ j_5, u_5, v_5, s_1, c_0, c_1, c_2 \} \\ M &= \{ j_5, v_5, s_1, c_0, c_1, c_2 \} \end{aligned}$$

Suppose that it has already been verified that both M_1 and M_2 are centralizing monoids. Then, since $M = M_1 \cap M_2$, it follows that M is also a centralizing monoid.

3.2.3 Kuznetsov Criterion (N1)

For $F \subseteq \mathcal{O}_k$ and p-ary relation ρ on E_k, ρ is *equational* over F if there exist $q \geq p$ and a system Σ of equations over F with variables $x_1, \ldots, x_q$ such that

$$\rho = \{(a_1, \ldots, a_p) \mid (\exists\, a_{p+1}) \ldots (\exists\, a_q)\ (a_1, \ldots, a_q) \text{ is a solution of } \Sigma\}.$$

Moreover, for $f \in \mathcal{O}_k$ and $F \subseteq \mathcal{O}_k$, f is *p-expressible* by F if the graph $f^{\square}$ is equational over F.

Theorem 3.2 *(Kuznetsov criterion)*
For $f \in \mathcal{O}_k$ and $F \subseteq \mathcal{O}_k$, f is p-expressible by F if and only if $f \in F^{**}$.

We can make use of this theorem to verify that some monoid is not a centralizing monoid.

Example 3-1 Take unary functions j_0 and j_2 from Table 1. Consider two equations

$$j_0(x) = j_2(y) \qquad \text{and} \qquad j_2(x) = j_0(y)$$

on variables x, y. The sets of solutions for these equations are $\{ (x,y) \mid j_0(x) = j_2(y) \} = \{ (0,2), (1,0), (1,1), (2,0), (2,1) \}$ and $\{ (x,y) \mid j_2(x) = j_0(y) \} = \{ (0,1), (0,2), (1,1), (1,2), (2,0) \}$, respectively. Hence the set of solutions for the system of these equations is:

$$\{ (x,y) \mid j_0(x) = j_2(y),\ j_2(x) = j_0(y) \} \;=\; \{ (0,2), (1,1), (2,0) \}$$

Since $\{ (0,2), (1,1), (2,0) \}$ is the graph of s_6, where s_6 is the permutation $(0\,2)$, Kuznetsov Criterion asserts that

$$s_6 \in \{ j_0, j_2 \}^{**}.$$

Hence, we conclude that if $j_0, j_2 \in M$ and $s_6 \notin M$ for any submonoid M, then M is not a centralizing monoid.

Example 3-2 In some cases, constant functions are useful to produce non-constant functions. Consider equations

$$j_1(x) = j_5(y) \qquad \text{and} \qquad c_0(x) = j_1(y)$$

where c_0 is the constant function taking value 0. The set of solutions for the system of these equations is:

$$\{ (x,y) \mid j_1(x) = j_5(y),\ c_0(x) = j_1(y) \} \;=\; \{ (0,0), (1,2), (2,0) \}$$

From the fact that

$$\{ (0,0), (1,2), (2,0) \} \;=\; u_1{}^{\square}$$

where u_1 is a two-valued function in Table 1, we claim by Kuznetsov Criterion that

$$u_1 \in \{j_1, j_5, c_0\}^{**}.$$

Hence, for example,

$$M = \{j_1, j_5, s_1, c_0\}$$

is not a centralizing monoid.

Example 3-3 Extra variables bound by the existential quantifier are also helpful. Consider the following:

$$(\exists z)\ s_1(y) = j_1(z)$$

This is equivalent to saying that

$$y \in \{0, 1\}.$$

On the other hand, we have

$$\{(x,y) \mid u_1(x) = u_4(y)\} = \{(0,1), (1,0), (1,2), (2,1)\}.$$

So, we obtain

$$\{(x,y) \mid (\exists z)\,[u_1(x) = u_4(y) \wedge s_1(y) = j_1(z)]\} = \{(0,1), (1,0), (2,1)\} = j_4{}^{\square}$$

It follows from Kuznetsov Criterion that

$$j_4 \in \{j_1, u_1, u_4, s_1\}^{**}.$$

Hence, for example,

$$M = \{j_1, u_1, u_4, v_4, s_1, c_0, c_1, c_2\}$$

is not a centralizing monoid.

The following is an example of more general results obtained from Kuznetsov Criterion ([MR09]).

Lemma 3.3 *Let M be a submonoid of $\mathcal{O}_A^{(1)}$. Suppose that there exists a non-empty subset N of M such that the intersection of the set of the fixed points of f for all $f \in N$ is a singleton set $\{a\}$ for some $a \in A$. Then the constant function $c_a \in \mathcal{O}_A^{(1)}$ taking value a belongs to M^{**}.*

Example 3-4 An immediate consequence of Lemma 3.3 is, for example, that if M contains s_2 $(= (1\,2))$ then the constant function c_0 must belong to M^{**}.

3.2.4 Intersection (N2)

It is Immediate to see the following: For monoids M, M', $N \subseteq \mathcal{O}_3^{(1)}$, if M' is a centralizing monoid, N is not a centralizing monoid and $N = M \cap M'$ then M is not a centralizing monoid.

Example 4 Take the following submonoids of $\mathcal{O}_3^{(1)}$:

$$\begin{aligned} M &= \{j_5, u_5, s_1, s_2\} \\ M' &= \{s_1, s_2, c_0\} \\ N &= \{s_1, s_2\} \end{aligned}$$

It is easily verified that M' is a centralizing monoid (because $M' = M(s_2)$), and N is not a centralizing monoid (because $c_0 \in N^{**}$, due to Example 3.4). Then, the equality $N = M \cap M'$ implies that M is not a centralizing monoid.

3.3 Main Result

As stated above, there are 10 *maximal* centralizing monoids on E_3. They are divided into four conjugate classes. (Two submonoids are *conjugate* to each other if one can be obtained from the other by renaming of the elements in the base set E_3.) Each of three conjugate classes consists of 3 maximal centralizing monoids and one conjugate class consists of 1 maximal centralizing monoid. Namely, four conjugate classes are $\{M(c_t) \mid t = 0, 1, 2\}$, $\{M(m_i) \mid i = 0, 364, 728\}$, $\{M(m_j) \mid j = 109, 473, 510\}$ and $\{M(m_{624})\}$ in the notation from Subsection 3.1.

We start from choosing one maximal centralizing monoid M_{max} from each conjugate class and do the following:

Find all submonoids of M_{max}. For each submonoid M of M_{max}, use a positive tool or a negative tool described in Subsection 3.2, to decide if M is a centralizing monoid or not. For each centralizing monoid M, obtain all centralizing monoids which are conjugate to M.

Repeat the above procedure for every representative M_{max} of every conjugate class of maximal centralizing monoids.

Finally, collect all centralizing monoids and delete the repetitions.

Proposition 3.4 (1) *The number of centralizing monoids on* E_3 *is* 192. *There is no centralizing monoid consisting of* k *elements for* $12 \leq k \leq 16$ *and* $18 \leq k \leq 26$.

(2) *The number of the conjugate classes of the centralizing monoids is 48.*

The list of all centralizing monoids on E_3 is given in Tables 2–7. In the table, each row corresponds to a centralizing monoid M and each column corresponds to a unary function f where $\circ$ designates the membership of f in M.

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List of Centralizing Monoids on E_3

All centralizing monoids on E_3 are listed in the following 5 tables.

- There are 28 columns in each table. Each column, except the first one, corresponds to a unary function on E_3. An item in the first column indicates a "label (name)" and a property (stated below) of a centralizing monoid.
- Each row indicates a centralizing monoid consisting of those unary functions marked ○ .
- An item in the first column has the form "m-n t_1" or "m-n t_1, t_2". Denote by M the corresponding centralizing monoid. Then, m is the size $|M|$ of M, n means that M is the n-th centralizing monoid in the table among monoids with the same size m, and t_1, resp. t_2, shows that M is conjugate to the first centralizing monoid in the group by the permutation s_{t_1}, resp. s_{t_2}. For example, the monoid labeled "8-1 3" is conjugate to the monoid labeled "8-1 1" by the permutation $s_3 (= (0\ 1))$. Also, the monoid labeled "17-1 2, 4" is conjugate to the monoid labeled "17-1 1, 3" by the permutation $s_2 (= (1\ 2))$ as well as by the permutation $s_4 (= (0\ 1\ 2))$.

	j_0	j_1	j_2	j_3	j_4	j_5	u_0	u_1	u_2	u_3	u_4	u_5	v_0	v_1	v_2	v_3	v_4	v_5	s_1	s_2	s_3	s_4	s_5	s_6	c_0	c_1	c_2
27-1	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○
17-1 1,3	○	○			○	○	○	○			○	○	○	○			○	○	○		○				○	○	○
17-1 2,4	○		○	○		○	○		○	○		○	○		○	○		○	○					○	○	○	○
17-1 5,6		○	○	○	○			○	○	○	○			○	○	○	○		○	○					○	○	○
11-1 1,3			○	○					○	○					○	○			○		○				○	○	○
11-1 2,4		○			○			○			○			○			○		○					○	○	○	○
11-1 5,6	○					○	○					○	○					○	○	○					○	○	○
10-1 1,3			○	○					○	○					○	○			○						○	○	○
10-1 2,4		○			○			○			○			○			○		○						○	○	○
10-1 5,6	○					○	○					○	○					○	○						○	○	○
10-2 1,3		○				○		○				○	○				○		○						○	○	○
10-2 2,4			○			○			○			○			○			○	○						○	○	○
10-2 5,6		○		○					○		○				○		○		○						○	○	○
9-1 1,3									○	○					○	○			○		○				○	○	○
9-1 2,4		○			○									○			○		○					○	○	○	○
9-1 5,6	○					○	○					○							○	○					○	○	○
9-2 1,3											○	○					○	○	○		○				○	○	○
9-2 2,4				○		○							○		○				○					○	○	○	○
9-2 5,6		○	○					○	○										○	○					○	○	○
9-3 1,3									○		○	○			○		○	○	○		○						○
9-3 2,4		○		○		○							○		○		○		○					○		○	
9-3 5,6		○	○			○		○	○			○							○	○					○		
9-4																			○	○	○	○	○	○	○	○	○

Table 2: Centralizing Monoids on E_3 (Size: 27–9)

	j_0	j_1	j_2	j_3	j_4	j_5	u_0	u_1	u_2	u_3	u_4	u_5	v_0	v_1	v_2	v_3	v_4	v_5	s_1	s_2	s_3	s_4	s_5	s_6	c_0	c_1	c_2
8-1 1			○	○					○	○									○						○	○	○
8-1 3			○	○											○	○			○						○	○	○
8-1 4								○			○			○			○		○						○	○	○
8-1 2		○			○			○			○								○						○	○	○
8-1 5	○					○							○					○	○						○	○	○
8-1 6							○					○	○					○	○						○	○	○
8-2 1,3									○	○					○	○			○						○	○	○
8-2 2,4		○			○									○			○		○						○	○	○
8-2 5,6	○					○	○					○							○						○	○	○
7-1 1												○		○			○		○						○	○	○
7-1 3							○					○					○		○						○	○	○
7-1 4	○					○									○				○						○	○	○
7-1 2						○									○	○			○						○	○	○
7-1 5		○							○	○									○						○	○	○
7-1 6		○			○				○										○						○	○	○
7-2 1		○						○									○		○						○	○	○
7-2 3						○						○	○						○						○	○	○
7-2 4						○						○						○	○						○	○	○
7-2 2			○						○						○				○						○	○	○
7-2 5				○					○						○				○						○	○	○
7-2 6		○									○						○		○						○	○	○
7-3 1,3									○						○				○		○				○	○	○
7-3 2,4		○															○		○					○	○	○	○
7-3 5,6						○						○							○	○					○	○	○
7-4 1,3			○	○															○		○				○	○	○
7-4 2,4								○			○								○					○	○	○	○
7-4 5,6													○					○	○	○					○	○	○
7-5 1,3											○	○					○	○	○		○						○
7-5 2,4				○		○							○		○				○					○		○	
7-5 5,6		○	○					○	○										○	○					○		
6-1 1														○			○		○						○	○	○
6-1 3							○					○							○						○	○	○
6-1 4	○					○													○						○	○	○
6-1 2															○	○			○						○	○	○
6-1 5									○	○									○						○	○	○
6-1 6		○			○														○						○	○	○
6-2 1								○									○		○						○	○	○
6-2 3												○	○						○						○	○	○
6-2 4						○												○	○						○	○	○
6-2 2			○												○				○						○	○	○
6-2 5				○					○										○						○	○	○
6-2 6		○									○								○						○	○	○

Table 3: Centralizing Monoids on E_3 (Size: 8–6)

	j_0	j_1	j_2	j_3	j_4	j_5	u_0	u_1	u_2	u_3	u_4	u_5	v_0	v_1	v_2	v_3	v_4	v_5	s_1	s_2	s_3	s_4	s_5	s_6	c_0	c_1	c_2
6-3 1		○						○											○						○	○	○
6-3 3						○							○						○						○	○	○
6-3 4												○						○	○						○	○	○
6-3 2			○						○										○						○	○	○
6-3 5				○											○				○						○	○	○
6-3 6											○						○		○						○	○	○
6-4 1												○		○			○		○							○	○
6-4 3							○					○					○		○						○		○
6-4 4	○					○									○				○						○	○	
6-4 2						○									○	○			○							○	○
6-4 5		○							○	○									○						○		○
6-4 6		○			○				○										○						○	○	
6-5 1		○				○		○				○							○						○		
6-5 3		○				○							○				○		○							○	
6-5 4									○			○			○			○	○								○
6-5 2			○			○			○			○							○						○		
6-5 5		○		○											○		○		○							○	
6-5 6									○		○				○		○		○								○
6-6 1,3									○						○				○						○	○	○
6-6 2,4		○															○		○						○	○	○
6-6 5,6						○						○							○						○	○	○
6-7 1,3												○					○		○						○	○	○
6-7 2,4						○									○				○						○	○	○
6-7 5,6		○							○										○						○	○	○
6-8 1,3			○	○															○						○	○	○
6-8 2,4								○			○								○						○	○	○
6-8 5,6													○					○	○						○	○	○
6-9																			○			○	○		○	○	○
5-1 1																	○		○						○	○	○
5-1 3												○							○						○	○	○
5-1 4						○													○						○	○	○
5-1 2															○				○						○	○	○
5-1 5									○										○						○	○	○
5-1 6		○																	○						○	○	○
5-2 1								○											○						○	○	○
5-2 3													○						○						○	○	○
5-2 4																		○	○						○	○	○
5-2 2			○																○						○	○	○
5-2 5				○															○						○	○	○
5-2 6											○								○						○	○	○

Table 4: Centralizing Monoids on E_3 (Size: 6–5)

	j_0	j_1	j_2	j_3	j_4	j_5	u_0	u_1	u_2	u_3	u_4	u_5	v_0	v_1	v_2	v_3	v_4	v_5	s_1	s_2	s_3	s_4	s_5	s_6	c_0	c_1	c_2
5-3 1														○			○		○							○	○
5-3 3							○					○							○						○		○
5-3 4	○					○													○						○	○	
5-3 2															○	○			○							○	○
5-3 5									○	○									○						○		○
5-3 6		○			○														○						○	○	
5-4 1												○					○		○							○	○
5-4 3												○					○		○						○		○
5-4 4						○									○				○						○	○	
5-4 2						○									○				○							○	○
5-4 5		○							○										○						○		○
5-4 6		○							○										○						○	○	
5-5 1,3																			○		○				○	○	○
5-5 2,4																			○					○	○	○	○
5-5 5,6																			○	○					○	○	○
5-6 1,3									○						○				○		○						○
5-6 2,4		○															○		○					○		○	
5-6 5,6						○						○							○	○					○		
4-1 1																	○		○							○	○
4-1 3												○							○						○		○
4-1 4						○													○						○	○	
4-1 2															○				○							○	○
4-1 5									○										○						○		○
4-1 6		○																	○						○	○	
4-2 1												○							○							○	○
4-2 3																	○		○						○		○
4-2 4															○				○						○	○	
4-2 2						○													○							○	○
4-2 5		○																	○						○		○
4-2 6									○										○						○	○	
4-3 1		○						○											○						○		
4-3 3						○							○						○							○	
4-3 4												○						○	○								○
4-3 2			○						○										○						○		
4-3 5				○											○				○							○	
4-3 6											○						○		○								○
4-4 1,3									○						○				○								○
4-4 2,4		○															○		○							○	
4-4 5,6						○						○							○						○		
4-5 1,3												○					○		○								○
4-5 2,4						○									○				○							○	
4-5 5,6		○							○										○						○		
4-6																			○						○	○	○

Table 5: Centralizing Monoids on E_3 (Size: 5–4)

	j_0	j_1	j_2	j_3	j_4	j_5	u_0	u_1	u_2	u_3	u_4	u_5	v_0	v_1	v_2	v_3	v_4	v_5	s_1	s_2	s_3	s_4	s_5	s_6	c_0	c_1	c_2
3-1 1		○																	○						○		
3-1 3						○													○							○	
3-1 4												○							○								○
3-1 2									○										○						○		
3-1 5															○				○							○	
3-1 6																	○		○								○
3-2 1						○													○						○		
3-2 3		○																	○							○	
3-2 4									○										○								○
3-2 2												○							○						○		
3-2 5																	○		○							○	
3-2 6															○				○								○
3-3 1			○																○						○		
3-3 3				○															○							○	
3-3 4											○								○								○
3-3 2								○											○						○		
3-3 5													○						○							○	
3-3 6																		○	○								○
3-4 1														○			○		○								
3-4 3							○					○							○								
3-4 4	○					○													○								
3-4 2															○	○			○								
3-4 5									○	○									○								
3-4 6		○			○														○								
3-5 1,3		○				○													○								
3-5 2,4									○			○							○								
3-5 5,6															○		○		○								
3-6 1,3																			○						○	○	
3-6 2,4																			○						○		○
3-6 5,6																			○							○	○
3-7 1,3																			○		○						○
3-7 2,4																			○					○		○	
3-7 5,6																			○	○					○		
3-8																			○			○	○				
2-1 1		○																	○								
2-1 3						○													○								
2-1 4												○							○								
2-1 2									○										○								
2-1 5															○				○								
2-1 6																	○		○								
2-2 1,3																			○								○
2-2 2,4																			○							○	
2-2 5,6																			○						○		
1-1																			○								

Table 6: Centralizing Monoids on E_3 (Size: 3–1)