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Graphical Compositions of Semirigid Equivalence Relations

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Abstract

A system of equivalence relations on a base set $A=\{1,\ldots,n\}$ is semirigid if only the trivial functions (i.e., projections and constant functions) preserve all the equivalence relation jointly. First we explore properties of semirigid systems R of equivalence relations, especially we show that the graph corresponding to a such system has doubly-connectedness property. We show that $R=\{\theta_{12},\theta_{23},\ldots,\theta_{n-1,n},\theta_{n1}\}$ (a loop of n nodes connected by distinctively colored n edges) is semirigid where θ_{ij} is a minimum partition of A which has sole non-singleton block $\{i,j\}$. In the last part of the paper we show that n is the minimum of $|R|$ such that R, a subset of $\{\theta_{ij}:1\leq i<j\leq n\}$, grows into semirigidity. For the proof we present a notion of compositions of two semirigid systems and show as a main theorem that it preserves semirigidity. We also exhibit that the compositions yield a variety of semirigid systems.

1 Introduction

Let n be a positive integer $n>2$ and set $A=\{1,\ldots,n\}$. For all positive integers m denote by $O^{(m)}$ all m-ary functions (operations) of the n-valued logic; i.e., the set of maps $f:\ A^m\rightarrow A$. In particular $O^{(1)}$ is the set of all selfmaps of A. Then $O=\cup_{m=1}O^m$ is the set of n-valued functions. An m-ary function $f(\boldsymbol{x})$ is called i-th projection if $f(\boldsymbol{x})=x_i$ for all $\boldsymbol{x}=(x_1,\ldots,\ldots,x_m)\in A^m$ for some $1\leq i\leq m$. In particular, the identity selfmap e of A, defined by setting $e(x):=x$ for all $x\in A$, belongs to $O^{(1)}$. An m-ary function $f(\boldsymbol{x})$ is a constant if $f(\boldsymbol{x})=c$ for all $\boldsymbol{x}\in A^m$, where c is an element from A. The set K of projections and all constant functions are called *trivial functions.* A *clone* $C\subseteq O$ is a composition closed subset of O and by definition it contains all projections. The sets K and O are clones. For a clone C its unary part $C^{(1)}:=C\cap O^{(1)}$ is the *foundation* of C. We call a set R of relations on A *semirigid* if the set of functions which preserves every relation in R coincides with the trivial functions. The study of semirigidity of relational system has important theoretical applications [4]. For example, every semirigid relational system from equivalence relations is a generator of the congruence lattices on A [8]. We focus our attention on equivalence relations. In [8] it is shown that the minimum number of the elements of a semirigid system R of equivalence relations is three and a system of three concrete partitions is given. A new proof of its semirigidity by using a purely relational method [1, 2] is given in [5]. We represent a system of equivalence relations by a colored graph where the set of nodes is the base set and edges represent binary relations whereby each relation is distinguished by its color. In this paper we explore properties of semirigid systems of equivalence relations. We show that if the system is semirigid then every edge of the graph is doubly connected (i.e., every two nodes at the ends of an edge are connected by a second path which uses only edges colored differently from the color of the edge). We show that $R = \{\theta_{12}, \theta_{23}, \dots, \theta_{n-1,n}, \theta_{n1}\}$ (a loop of n nodes connected by distinctively colored n edges) is semirigid, where θ_{ij} is a minimum partition of A which has sole non-singleton block $\{i, j\}$. Then we introduce an intuitive notion of graphical compositions of systems of equivalence relations. As a main theorem we show that the composition preserves semirigidity (for the proof we use the existence of a second path). As a consequence of compositions we show that n is the minimum of $|R|$ such that R, a subset R of $\{\theta_{ij} : 1 \le i < j \le n\}$, is semirigid. This answers the question of determining a minimum number of different $\{\theta_{ij}\}$ so that the subset grows into a semirigid system. We also show that a variety of semirigid systems can be obtained through the composition of semirigid systems.

The main idea was given in [5]. Here we present a full proof of the main theorem. We also give further problems which involve the notion of compositions.

2 Semirigid equivalence relations

Let A be a finite set ($|A| > 2$). Denote A by the numbers $\{1, \dots, n\}$. An equivalence relation on A is a reflexive, symmetric and transitive relation on A and it is a partition of A. We denote the set of all equivalence relations on A by E. Let $\alpha, \beta \in E$. Call $\alpha, \beta \in E$ *orthogonal* if $\alpha \cap \beta = \Delta_A = \{(a, a) : a \in A\}$, i.e., no subset other than a singleton node belongs to two blocks each from the two partitions.

It is shown in [8]:

Theorem 1. *The following three equivalence relations ρ, σ, τ on A are semirigid.*

Case $n = 2k + 1$ for $k \ge 1$.

$$\begin{aligned} \rho &= (\{1, 3, \dots, 2k+1\}, \{2, 4, \dots, 2k\}) \\ \sigma &= (\{1, 2\}, \{3, 4\} \dots, \{2k-1, 2k\}) \\ \tau &= (\{2, 3\}, \{4, 5\} \dots, \{2k, 2k+1\}) \end{aligned}$$

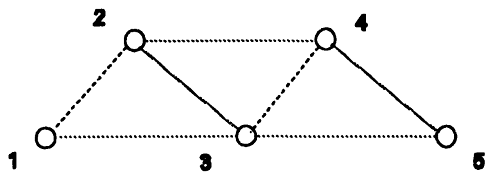


Figure 1: Zadori's semirigid system z_k of three equivalence relations ($n = 5, k = 2$)

Case $n = 2k + 2$ *for* $k > 1$.

$$\begin{aligned}\rho &= (\{1,3,\ldots,2k+1\},\{2,4,\ldots,2k\})\\ \sigma &= (\{1,2,2k+2\},\{3,4\}\ldots,\{2k-1,2k\})\\ \tau &= (\{2,3\},\{4,5\}\ldots,\{2k,2k+1,2k+2\})\end{aligned}$$

In Fig 1 we show z_k the three relations $\rho, \sigma.\tau$ for $n = 5, k = 2$ (in the figure $\{1,3,5\}$ is a block). Note that ρ separates A into the two large components and the blocks from both σ and τ bridge the components. The 3 relations are orthogonal one another. The bridges are conveniently represented by edges in the graph (where each element from A is a node).

Thus we may represent an equivalence relation by a non-directed graph: the set of nodes (vertices) of the graph is the set A and each two distinct nodes (pair) is connected by an edge if and only if it is included in a block of a partition. So each block forms a clique in this graph. We distinguish two partitions by different colors and for a set of equivalence relations, we can superpose them on a single graph. Then, the orthogonality of two partitions means that no edge belongs to two blocks with different colors.

We give a lemma (recall that K is the clone consisting only of trivial functions).

Lemma 2. *(Reduction lemma [6]) Let $A > 2$. Then $K^{(1)}$ is the foundation of a clone C on A if and only if $C = K$.*

So, semirigidity property of relations can be tested only on unary functions. It is easy to prove the following simple property of semirigid system.

Lemma 3. *If a system $R = (A, \rho_1, \ldots, \rho_k)$ of equivalence relations on A is semirigid, then each pair of distinct nodes is covered by some block of a partition from R.*

Proof. Assume that node 1 is not covered by any relation in our graph. Define a unary function f by setting $f(a) = 1$ if $a \neq 1$ for all $a \in A$ and $f(1) = 2$. It is easy to check that any relation ρ_i preserves f and f is not a trivial function. □

So if we draw a graph of semirigid system R on the node set A (each partition is distinguished by a color), there is no isolated node in our graph. Thus our graph is a connected colored graph.

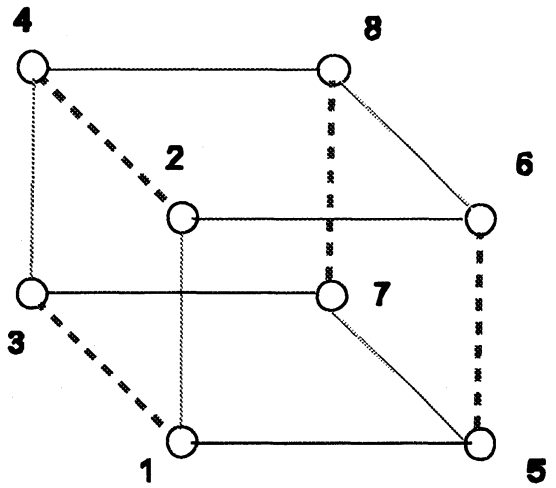


Figure 2: There is no second path between nodes 2 and 6.

3 Properties of semirigid systems of relations

In this section we assume that every relation is on a finite set A. For a set of equivalence relations $\rho_1, \dots, \rho_k$ we denote the equivalence relation θ generated by the union $\rho_1 \cup \dots \cup \rho_k$ by $\theta = \rho_1 \vee \dots \vee \rho_k$, i.e., $x\theta y$ if $x = x_0, \dots, x_m = y$ and $(x_i, x_{i+1}) \in \rho_j$ for some $1 \le j \le m$.

Lemma 4. *(Existence of a second path) Assume $(A, \rho_1, \dots, \rho_k)$ is semirigid. Then for every j we have*

$$\rho_j \subseteq \bigvee_{i \neq j} \rho_i \tag{1}$$

i.e., if $x\rho_j y$ then there is a second path connecting x and y via ρ_j, $j \neq i$, meaning $x = x_0, x_1, \dots, x_m = y$ with $x_{t-1}\rho_{j_t}x_t$, $j_t \neq i$ for every $t = 1, \dots, m$.

Proof. Suppose that $(A, \rho_1, \dots, \rho_k)$ is semirigid. Let $(x, y) \in \rho_1$ with $x \neq y$. Let $\theta = \rho_2 \vee \dots \vee \rho_k$. We are to prove $(x, y) \in \theta$. Suppose not. Let $E_\theta(y)$ be the equivalence class of y with respect to θ, i.e., $E_\theta(y) = \{z : z\theta y\}$. Let $f : A \to A$ be a map defined by

$$f(z) = \begin{cases} x & : \text{ if } z \notin E_\theta(y) \\ y & : \text{ if } z \in E_\theta(y) \end{cases}$$

The map f is not a trivial function provided $|A| > 2$.

Claim. f preserves $\rho_1, \dots, \rho_k$. Let i, $i = 1 \dots, k$ and $(a, b) \in \rho_i$. We have to prove that $(f(a), f(b)) \in \rho_i$. Suppose not, then first $f(a) \neq f(b)$. Next $i \neq 1$. With no loss of generality we may suppose $f(a) = x$ and $f(b) = y$. By definition $b \in E_\theta(y)$, $a \notin E_\theta(y)$, hence $(a, b) \notin \theta$. But $(a, b) \in \rho_i$ and $i \neq 1$, and $\theta = \rho_2 \vee \dots \vee \rho_k$, thus $(a, b) \in \theta$. A contradiction. □

In Fig. 2 we give an example of a graph (with 3 colors) having no second path (there is no second path between nodes 2 and 6). In Fig. 3 we give an example of doubly connected graph

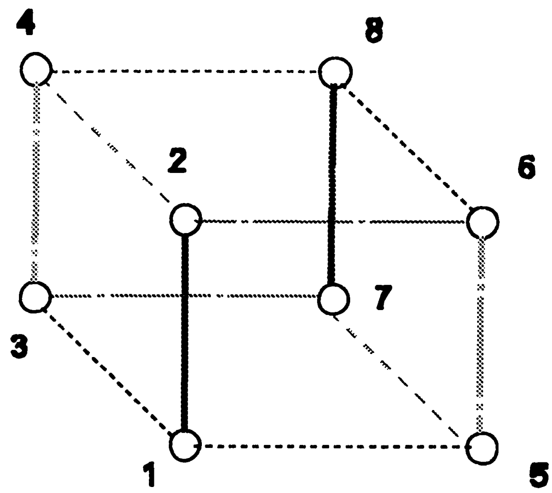


Figure 3: Non semirigid doubly connected graph (6 colors)

(with 6 colors) that is not semirigid (so doubly connectedness is not a sufficient condition for semirigidity).

Corollary 5. *If $(A, \rho_1, \ldots, \rho_k)$ is semirigid, then there is no node with the degree (i.e., the number of edges connected to the node) = 1.*

Proof. . Assume that $(a, b) \in \rho_1$ and assume that there is no element of the form (b, x), $x \neq a, b$ in ρ_i, $i = 2, \ldots, k$ (the degree of the node b is 1 and it is a pendant node in the graph made from this semirigid system). Then obviously there is no path which connect b and a not via ρ_1. This contradicts to Lemma 4. □

Theorem 6. *An n-loop $R = \{\theta_{12}, \theta_{23}, \ldots, \theta_{n-1,n}, \theta_{n1}\}$ for every integer $n \geq 3$ which consists of n nodes with n edges distinctly colored is semirigid. Further, n is the minimum number of $\theta_{a,b}$ in $R = \{\theta_{a,b} : (a, b) \in I\}$ such that R for the subset $I \subseteq \{(i, j) : 1 \leq i \leq j \leq n\}$ grows into a semirigid system.*

Proof. From $(1, 2) \in \theta_{12}$ we have $(f(1), f(2)) \in \theta_{12}$. We have three cases: case 1. $f(1) = 1, f(2) = 2$: From $f \in \theta_{23}$ we have $f(3) = 3$, and eventually we have $f(i) = i$, $1 \leq i \leq n$. Case 2. $f(1) = 2, f(2) = 1$: From $f \in \theta_{23}$ we have $f(3) = 1$, and eventually we have $f(i) = 1$, $2 \leq i \leq n$, but this contradicts $f \in \theta_{n1}$. Case 3. $f(1) = f(2) = m$, $m \neq 1, 2$: From $f \in \theta_{23}$ we have $f(3) = m$, and eventually we have $f(i) = m$, $1 \leq i \leq n$, a constant function. Thus R is semirigid. The proof of the second part is given in Section 4 as we need a notion of compositions. □

In Fig.4 we show an example of a semirigid loop for $k = 6$.

Let $\rho_1, \ldots, \rho_k$ be a set of distinct equivalence relation on set A. Denote $\mathbf{0} := \Delta_A = \{(x, x) : x \in A\}$ and $\mathbf{1} := A \times A = \{(x, y) : x, y \in A\}$.

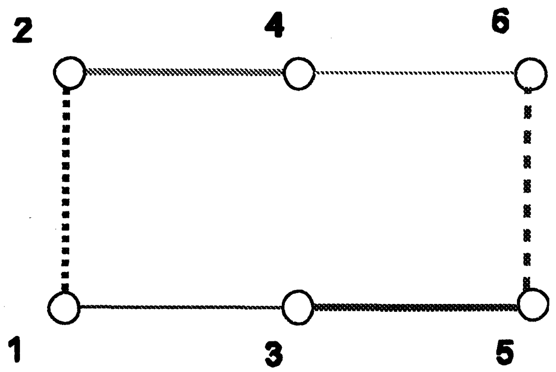


Figure 4: N-loop with distinctly colored n-edges is semirigid.

Lemma 7. *Assume that $R := (A, \rho_1, \ldots, \rho_k)$ is semirigid. Then for every j, $j = 1, \ldots, k$:*

$$\vee_{i \neq j}\ \rho_i = \mathbf{1} \tag{2}$$

$$\wedge_{i \neq j}\ \rho_i = \mathbf{0}. \tag{3}$$

Proof of (2): in fact we showed in Lemma 4 that $\rho_j \subseteq \vee_{i \neq j}\ \rho_i$. To conclude observe that $\vee \rho_i = \mathbf{1}$ as this means that the graph is connected (Lemma ??).

Proof of (3). We suppose $j = 1$. We show that $\wedge_{i \neq 1} \rho_i = 0$. Suppose not. That means $\cap_{i \neq 1} \rho_i \neq \Delta_A$. Let

$$(x, y) \in \cap_{i \neq 1} \rho_i \setminus \Delta_A$$

Case 1. $(x, y) \in \rho_1$. We have $(x, y) \in \cap_i \rho_i$. In this case, the map $f : A \to A$ defined by $f(z) := x$ if $z \neq y$ and $f(z) := x$ if $z = y$ preserves each ρ_i, hence preserves R, Its image is a 2-element set. We have forbidden this. Thus Case 1 does not occur.

Case 2. $(x, y) \notin \rho_1$. Let $f : A \to A$ be the map defined by

$$f(z) = \begin{cases} x & : \text{ if } (z, y) \notin \rho_1 \\ y & : \text{ otherwise.} \end{cases}$$

Claim. The map f preserves the set of relations R.

Proof of the claim. Let $i \in \{1, \ldots, k\}$ and $(u, v) \in \rho_i$. We prove that $(f(u), f(v)) \in \rho_i$. If $f(u) = f(v)$, this holds since $\Delta_A \subseteq \rho_i$. Hence we may suppose $f(u) \neq f(v)$. Then either

$$(f(u) = x \text{ and } f(v) = y) \quad \text{or} \quad (f(u) = y \text{ and } f(v) = x).$$

If $i \neq 1$, then, since $(x, y) \in \rho_i$, $(f(u), f(v)) \in \rho_i$ follows. The case $i = 1$ is impossible. Indeed, in this case, if $f(u) = x$ and $f(v) = y$ then $(u, y) \notin \rho_1$ and $(v, y) \notin \rho_1$, but, since $(u, v) \in \rho_1$ we get $(u, y) \in \rho_1$, a contradiction. If $f(u) = y$ and $f(v) = x$ we get a contradiction, too. □

Note. We could get (2) and (3) under the weaker condition that there is no unary operation which preserve R and whose image has two elements.

As we have seen that the case $k = 3$ (3 colors) is a minimum for semirigidity, we give the proof of the orthogonality for this case.

Corollary 8. *If $(A, \rho_1, \rho_2, \rho_3)$ is semirigid, then ρ_i is orthogonal with ρ_j for $i \neq j$.*

Proof. By Lemma 7 we have that if $(A, \rho_1, \dots, \rho_k)$ is semirigid, then intersection of $k-1$ of these relations is the equality. Set $k = 3$. Then we have $\rho_1 \cap \rho_2 = \Delta_A$, $\rho_1 \cap \rho_3 = \Delta_A$, $\rho_2 \cap \rho_3 = \Delta_A$. □

It is shown [8] that, if $(A, \rho_1, \rho_2, \rho_3)$ is semirigid, then the system of relations $\{\Delta_A, \rho_1, \rho_2, \rho_3, A \times A\}$ forms a sublattice of the lattice of the equivalences on A.

4 Semirigidity extension by identifying subsets of the nodes

In this section we discuss a system of relations obtained by two systems of relations through identifying subsets of the nodes. We show that many systems of semirigid relations can be obtained by this method.

Assume that we are given two colored graphs G_1 and G_2 on respective sets of vertices A and B ($A \cap B = \phi$). For simplicity we assume in this section that every system of equivalence relations is orthogonal, i.e., if $\alpha, \beta \in R$ then $\alpha \cap \beta = \Delta_A = \{(a, a) : a \in A\}$.

We state a definition of **graphical composition** of the two graphs given by identifying subsets of the nodes.

Type 1 composition. $G_1 * G_2$ (identifying a single node): Two nodes, each from G_1 and G_2, are simply identified and the two graphs are connected through the identified node. Note that $|C| = |A| + |B| - 1$ for the new base set and the number of partitions on C is simply the sum of the numbers of the partitions of the graphs.

Type 2 composition. $G_1 + G_2$ (identifying two or more nodes): Two subsets S_A and S_B, each from the nodes of G_1 and G_2 with $|S_A| = |S_B| = t$ and with the property that both are single colored, are identified and the two graphs are connected through the identified nodes. Note that the two partitions confluent into a single partition on the new base set C: $|C| = |A| + |B| - t$. Hence the number of partitions is less than the sum of the numbers of the partitions of the graphs exactly by one.

To illustrate our definition, let us denote Zadori's construction on the base set A with $|A| = 2i + 1$ by z_i for $i > 0$. Thus z_1 shown in Fig. 5 represents the semirigid system of

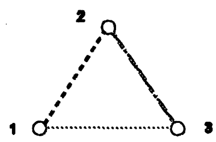


Figure 5: z_1: the semirigid triangle

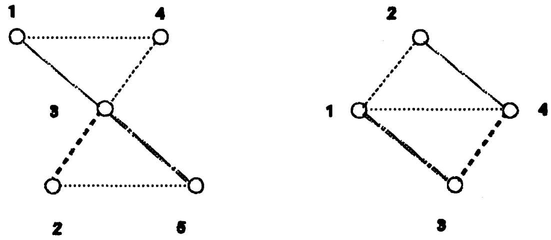


Figure 6: Compositions $z_1 * z_1$ and $z_1 + z_1$

three partitions on the 3-element set.In Fig. 6 we show two graphs a: $z_1 * z_1$ (6 colors) and b: $z_1 + z_1$ (5 colors) obtained by identifying a single node and by identifying two nodes, respectively.

Simple consideration leads to the following theorem.

Theorem 9. *Assume that both graphs G_1 and G_2 are semirigid. Then:*

The graph $G = G_1 * G_2$ is quasi semirigid. That is, it admits exactly two more functions as well as the trivial functions: one is the identity on A and takes the constant on B determined by the identity, and the other is obtained by interchanging the roles of A and B.

Proof. Assume that a function f admits G. Then it should be a trivial function on A (in this case we took only the set of partitions on G_1 into our consideration, leaving the nodes outside A as singletons). By the symmetry, f should be a trivial functions on B. As the consequence the case 1 results. As for the case 2, no freedom like the case 1 is admitted since an edge (a, b) among the identified nodes determines the values of f either as a constant or as those of the identity function. □

In order to state a theorem we formulate the definition of the composition in more detail.

Let G_1 be a graph defined by $G_1 := (A, \rho_1, \ldots, \rho_p)$, where $\rho_1, \ldots, \rho_p$ are equivalence relations on A, and let G_2 be a graph defined by $G_2 := (B, \sigma_1, \ldots, \sigma_q)$, where $\sigma_1, \ldots, \sigma_q$ are equivalence relations on B. Let the equivalence blocks of ρ_1 and σ_1 be $\rho_1 = \{r_1, \ldots, r_\ell\}$ and $\sigma_1 = \{s_1, \ldots, s_m\}$, respectively. We identify node $a_i \in A$ with node $b_i \in B$ for $i = 1, \ldots, t$. Assume further that the two subset $\{a_i\}$ and $\{b_i\}$ belong to r_1 and s_1, respectively:

$$S_A = \{a_1, \ldots, a_t\} \subset r_1, \; S_B = \{b_1, \ldots, b_t\} \subset s_1.$$

Put G the graph obtained from G_1 and G_2 by "gluing" subsets of the nodes (each subset from a single color). For convenience we write this operation in symbol $G = G_1 + G_2$. The base set of G is $C = A \cup B$ with the identities $a_i = b_i$, $i = 1, \ldots, t$. The relations ρ_1 and σ_1 become a confluent relation $\tau = \rho_1 \vee \sigma_1$ by gluing.

Lemma 10. *Assume that the graph G is obtained by G_1 and G_2 through gluing as described in the above procedure. Then the composed system of relations is*

$$G = (A \cup B, \rho_1 \vee \sigma_1, \rho_2, \ldots, \rho_p, \sigma_2, \ldots, \sigma_q)$$

where:

1) the blocks of $\tau := \rho_1 \vee \tau_1$ are

$$\{r_1 \cup s_1, r_2, \ldots, r_\ell, s_2, \ldots, s_m\}.$$

That is, the two blocks each containing the identifying subset merge into a single block and the other blocks from ρ_1 and σ_1 remain as blocks without changes in G.

2) The relations $\rho_2 \ldots, \rho_p$ and $\sigma_2, \ldots, \sigma_q$ are the equivalence relations naturally extended to on C, i.e., exactly the loops $\{(x, x) : x \in C\}$ are added to the original relations.

Proof of the lemma. 1) Let the two blocks be $r_1 \supseteq S_A = \{a_1, \ldots, a_t\}$ and $s_1 \supseteq S_B = \{b_1, \ldots, b_t\}$. For each distinct $x, y \in r_1 \cup s_1$ there is $x_0 = x, \ldots, x_m = y$ $(m > 1)$, $(x_i, x_{i+1}) \in r_1^2$ or $(x_i, x_{i+1}) \in s_1^2$ for $1 \leq i < m$. In particular, if $x \in r_1 \setminus S_A$ and $y \in s_1 \setminus S_B$ there is $z \in S_A = S_B$ such that $(x, z) \in r_1^2$ and $(z, y) \in s_1^2$, where $z = a_i = b_i$ for some $1 \leq i \leq t$. So $(x, y) \in \tau$.

2) For distinct $x, y \in C$, if $(x, y) \in \rho_i$ $(i > 1)$, then $(x, y) \in A^2$. Further, as the system $\{\rho_i\}$, $\{\sigma_i\}$ $(i > 0)$ are orthogonal in G_1 and G_2, respectively, ρ_i and τ, σ_i and τ and ρ_i and σ_j are orthogonal. Thus G is orthogonal. □

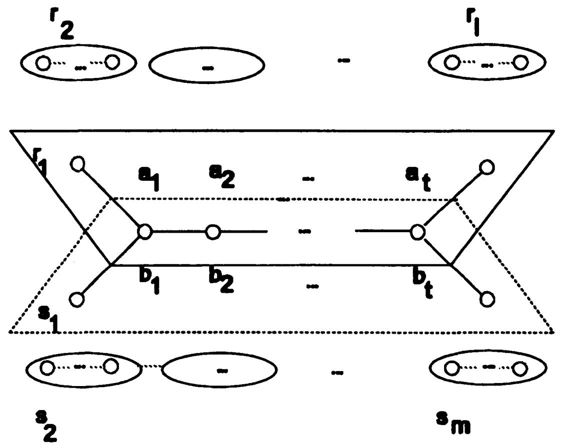
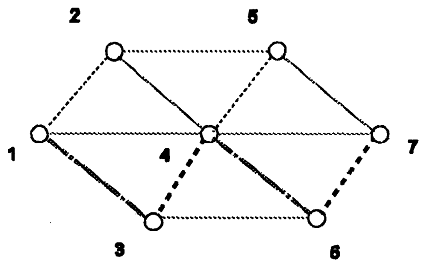


Figure 7: Confluence of the two relations ρ_1 and σ_1



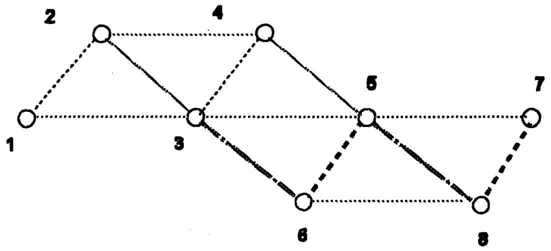


Figure 8: Two semirigid systems obtained by the composition (with 5 partitions).

Note. It is easy to check that the graphs G is connected and each two distinct node of C is doubly connected. From a node in A one reaches a node in B via a node in the shared nodes $S_A = S_B$.

Theorem 11. *If the graphs $G_1 := (A, \rho_1, \ldots, \rho_p)$ and $G_2 := (B, \sigma_1, \ldots, \sigma_q)$ are semirigid, then the graph $G = G_1 + G_2$ is semirigid, i.e., if a map $h: C \to C$ preserves the relations $\{\rho_1 \vee \sigma_1, \rho_2, \ldots, \rho_p, \sigma_2, \ldots, \sigma_q\}$ then h is a trivial function on C.*

We show a lemma.

Lemma 12. *Suppose that the function $h: C \to C$ is such given in the theorem. If the restriction $h_{|A}$ is not an injections into A, then*

$$h_{|A} = c \text{ (a constant)}, \ c \notin A.$$

Proof of the lemma. Assume that $f := h_{|A}$ is not an injection in A. Then there is $a \in A$ such that $h(a) = f(a) = c \in B \setminus A$. Assume $h_{|A}(a') = c'$ for $a' \in A$ $(a \neq a')$. We distinguish two cases: 1) $(a, a') \in \vee_{i=2}^{p} \rho_i$ and 2) there are distinct b, b' such that $(a, b) \in \vee_{i=2}^{p} \rho_i$, $(b, b') \in \rho_1$ and $(b', a') \in \vee_{i=2}^{p} \rho_i$.

Case 1. h preserves all ρ_i, $i > 1$, we have $(h(a)), h(a')) \in \vee_{i=2}^{p} \rho_i$. This means $c = c'$ as $c \notin A$.

Case 2. For $(b, b') \in \rho_1$ by Lemma 4 there is a second path from b to b' not via edge in ρ_1. So this case is reduce to the case 1. □

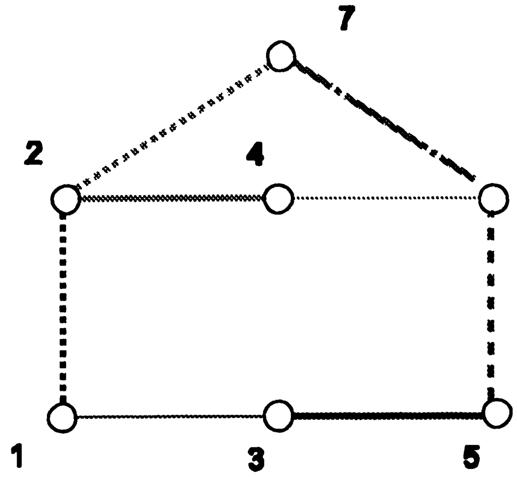


Figure 9: Gluing subsets with distinctively-colored edges yields a semirigid graph.

Lemma 13. *Suppose that the function* $h : C \to C$ *preserves the relations* $\{\rho_1 \vee \sigma_1, \rho_2, \ldots, \rho_p, \sigma_2, \ldots, \sigma_q\}$ *(given in the theorem). Then the restrictions* $h_{|A}$ $(h_{|B})$ *preserves the relations* $\rho_1, \ldots, \rho_p$ $(\sigma_1, \ldots, \sigma_q)$*, respectively.*

Proof of the lemma. If $f := h_{|A}$ is not an injection into A, then from the previous lemma $h = c \in B$ (a constant) on C and so, we are done. Thus we assume that f is an injection into A. Assume $(a, b) \in \rho_i$, $i > 1$. Then, since h preserves ρ_i we have $(f(a), f(b)) \in \rho_i$. Next assume that $(a, b) \in \rho_1$, $a, b \in A$. As h preserves $\tau = \rho_1 \vee \sigma_1$, assume that $(f(a), f(b)) \in \sigma_1 \backslash \tau$. Then we have $f(a), f(b) \in B$ and this contradicts that f is an injection. Thus we have $(f(a), f(b)) \in \rho_1$. □

Proof of the theorem. Suppose that the unary function $h : C \to C$ is a function given in the theorem. Then by the Lemmas 12 and 13, together with that G_1 is semirigid, $h_{|A}$ is a trivial function on A or a constant valued (from C) function. The same situation holds for $h_{|B}$. Thus the function h is the identity on C or a constant function $h = c \in C$. □

In Fig 6 we show examples of semirigid composition: in (a) three nodes $\{1, 4, 7\}$ are glued, while in (b) two nodes $\{3, 5\}$ are glued.

Applications

In the situation of the composition described just before Lemma 10, assume that all relations ρ_i and σ_j are from the set of $\Theta_A = \{\theta_{ij} : 1 \le i < j \le |A|\}$ and $\Theta_B = \{\theta_{ij} : 1 \le i < j \le |B|\}$, respectively, i.e., a distinctive smallest partition on A and on B. Further assume that the identifying nodes are from k consecutive chains $\rho_1, \ldots, \rho_{k-1}$ and $\sigma_1, \ldots, \sigma_{k-1}$ where $\rho_i = \theta_{a_i a_{i+1}}$ and $\sigma_i = \theta_{b_i b_{i+1}}$: $S_A = \{a_1, \ldots, a_k\}$ and $S_B = \{b_1, \ldots, b_k\}$ with identities $a_i = b_i$ $(i = 1, \ldots, k)$. Then we have a similar theorem

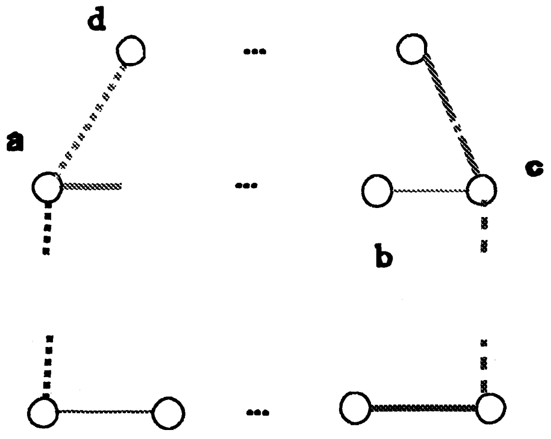


Figure 10: Two loops coalesced by distinctively colored edges

Theorem 14. *If the graphs $G_1 := (A, \rho_1, \ldots, \rho_p)$ and $G_2 := (B, \sigma_1, \ldots, \sigma_q)$ are semirigid, then the graph $G = G_1 + G_2$ is semirigid, i.e., if a map $h : C \to C$ preserves the relations $\{\rho_1 = \sigma_1, , \ldots, \rho_{k-1} = \sigma_{k-1}, \rho_k, \ldots, \rho_p, \sigma_k, \ldots, \sigma_q\}$ then h is a trivial function on C.*

It is easy to check that Lemmas 12 and 13 hold in this case, too. We omit the proof. In Fig.9 we give an example of this type of composition.

Now we give a proof of the second part of Theorem 6.

Proof of Theorem 6 (a sketch). Given a semirigid set R of relations of the type θ_{ab} on A, by Lemma 4 (the existence of second path) we may consider the graph of R as compositions either by $*$ or by $+$ of several loops. By Theorem 9 we can conclude that the composition $*$ cannot be in the graph of R. So we are enough to restrict ourselves on the composition $+$ where more than 1 nodes are shared by the compositions. Now consider the simplest case that there are only two loops. Suppose the situation on Fig. 10: the node a is a branching node and it is connected to the node c via edges (let the last node be the node b), and at the same time it is connected to the node d by an edge (a, b, c, d are all from A and a, c, d are distinct nodes). Assume the path from a to c via d and via b contains ℓ and m edges, respectively ($\ell, m > 0$). We show that there is a semirigid graph with less or equal number of edges to that in G but having a less by 1 number of loops. Indeed, we can reduce the number of loops by the following procedure. We attach the end point of the edge (b, c) (c's side) to the node d replacing the direct connection (a, d) by the chains of edges from a to c. Now the loop $a - d - c - a$ disappears and the path from a to c contains $\ell + m - 1$ (the case when there are two distinctive nodes a and b) or $\ell + m$ (the case $a = b$). Since all the edges are distinct, by the first part of Theorem 6 the resulting graph remains semirigid and has less than or equal number of edges (the number of nodes remains invariant). Applying this procedure to any loop which has coalesced part with other loops, by induction, we eventually

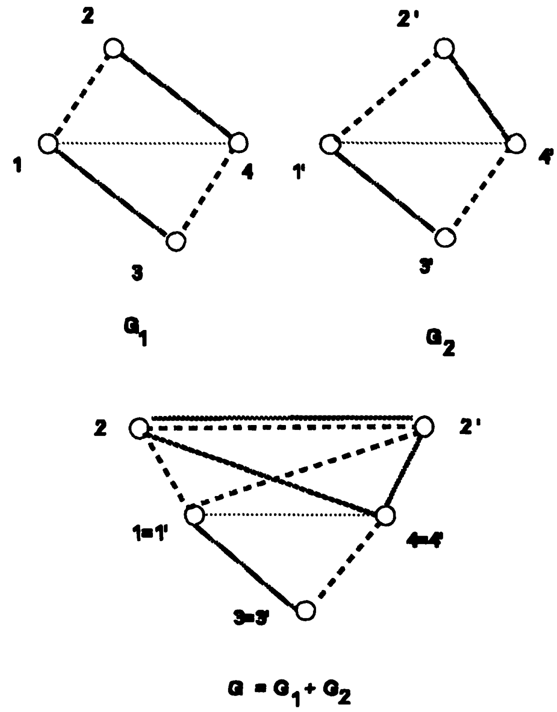


Figure 11: Non-orthogonality may occur by gluing.

obtain a single n-loop with distinctively colored edges. This completes the proof. □

5 Discussions

We have introduced the notion of graphical compositions by identifying subsets of the nodes, and have shown that, in restricted cases where the identifying subsets are single-colored, or all identifying edges bear distinct colors and these colors do not appear elsewhere, the composition preserves semirigidity. We have used the compositions to prove the minimality of n-loop semirigid systems. We have also shown that the compositions can yield a variety of semirigid systems.

As we have already seen we can define gluing of subsets with several colors, i.e., when S_A and S_B are isomorphic colored graphs. Indeed, as in Fig. 9 subsets with 2 colors may be glued to yield a semirigid graph. However, we show by an example that a complicated situation may arise in the general case. Consider colored subgraphs G_1 and G_2 in Fig. 11. For simplicity assume that each graph is three-colored, and assume for G_1 we have that σ includes $\{\{1,2\},\{3,4\}\}$, ρ includes $\{\{1,3\},\{2,4\}\}$ and τ includes $\{1,4\}$, and similarly, for G_2 we have that σ' includes $\{\{1',2'\},\{3',4'\}\}$, ρ' includes $\{\{1',3'\},\{2',4'\}\}$ and τ' includes $\{1',4'\}$. In G by gluing $\{1,3,4\}$ with $\{1',3',4'\}$ we have that $\sigma_1 = \sigma \vee \sigma'$ includes

$$\{\{1,2\},\{3,4\},\{1',2'\},\{2,2'\}\},$$

while $\rho_1 = \rho \vee \rho'$ includes

$$\{\{1,3\},\{2,4\},\{2',4'\},\{2,2'\}\}.$$

Thus $\rho_1 \cap \sigma_1$ includes $\{2,2'\}$, and this means that ρ_1 and σ_1 are not orthogonal. As we assumed 3-colors, every semirigid system should be orthogonal. This suggests that in the general case, gluing does not keep semirigidity and we need some more investigation.

We note that all the concrete semirigid systems given in this paper are checked by computer program.

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