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Hyperalgebras and hyperclones - different approaches

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Abstract

In the current paper, authors give a resume about results on hyperclone lattice on A, obtained by embedding a clone lattice into it and by embedding it into the lattice of clones on $P(A)\setminus\{\emptyset\}$. These results are used for characterization of minimal hyperclones on two element set and unary minimal hyperclones on an arbitrary finite set A.

1 Introduction

There are several synonims appearing for the class of algebraic systems whose clones have been studied and presented in this paper. They are hyperalgebras, multialgebras and polyalgebras.

The first paper on this topic was written by Marty in 1934. From that time, there have been published hundreds of papers devoted to special hyperalgebras. Some attention has also been given to hyperclones and partial hyperclones by a few authors.

Hyperoperations and hyperalgebras have both mathematical and nonmathematical applications. For example, they give possibility to describe nondeterministic processes. Rosenberg has studied hyperclones on a finite universe A via natural (not full) order embedding from the hyperclone lattice on A to the clone lattice on $P(A)\setminus\{\emptyset\}$.([14], [15]). He posed the question of further study of the lattice of hyperclones, in particular in the simplest two-element case $A=\{0,1\}$. It has been shown by Machida that the lattice on the two-element set has continuum cardinality ([8]). Drescher, in [5], presented an overview of partial hyperoperations and relations. Many questions which arise from the process of extending operations are studied there, too.

2 Definitions and preliminaries

Paper explores some definitions, claims and ideas mentioned in [5], [6],[8] and [11]. The authors site several of them in the current paper. Proofs of some claims are not presented and could be found in [9] and [10].

Let $\mathbb{N}$ be the set of positive integers, $n,m\in\mathbb{N}$, A a nonempty finite set, $P(A)$ the power set of A.

For a positive integer n, an n-ary operation on A is a mapping $f:A^n\to A$.

For a positive integer n, an n-ary hyperoperation on A is a mapping $f:A^n\to P(A)\setminus\{\emptyset\}$.

We will denote the set of all operations by O_A and the set of all hyperoperations on A by H_A.

For a positive integer n, an i-th projection on A of arity n, $1\le i\le n$, is an $n-$ary operation $\pi_i^n:A^n\to A$, $(x_1,\ldots,x_n)\mapsto x_i$.

For positive integers n and m, we define the composition $S_m^n:O_A^{(n)}\times(O_A^{(m)})^n\to O_A^{(m)}$, $(f,g_1,\ldots,g_n)\mapsto f(g_1,\ldots,g_n)$, where $f(g_1,\ldots,g_n):A^m\to A$, $(x_1,\ldots,x_m)\mapsto f(g_1(x_1,\ldots,x_m),\ldots,g_n(x_1,\ldots,x_m))$.

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Let $f \in O_A^{(n)}$ and $g \in O_A^{(m)}$, then $\varsigma f \in O_A^{(n)}$, $\tau f \in O_A^{(n)}$, $\Delta f \in O_A^{(n-1)}$, $f \circ g \in O_A^{(m+n-1)}$ and $\nabla f \in O_A^{(n+1)}$ are defined by

$(\varsigma f)(x_1, x_2, \ldots, x_n) = f(x_2, \ldots, x_n, x_1),\ n \geq 2$
$(\tau f)(x_1, x_2, x_3, \ldots, x_n) = f(x_2, x_1, x_3, \ldots, x_n),\ n \geq 2$
$(\Delta f)(x_1, \ldots, x_{n-1}) = f(x_1, x_1, \ldots, x_{n-1}),\ n \geq 2,$
$(\varsigma f) = (\tau f) = (\Delta f) = f,\ n = 1,$
$(f \circ g)(x_1, \ldots, x_m, x_{m+1}, \ldots, x_{m+n-1}) = f(g(x_1, \ldots, x_m), x_{m+1}, \ldots, x_{m+n-1})$
$(\nabla f)(x_1, x_2, \ldots, x_{n+1}) = f(x_2, \ldots, x_{n+1}),$

where $x_1, \ldots, x_{m+n-1} \in A$. The full algebra of operations is $\mathcal{O}_A = (O_A, \circ, \varsigma, \tau, \Delta, \pi_1^2)$. Each subuniverse of $\mathcal{O}_A$ is a clone.

A set of operations is a clone iff it contains all projections and is closed with respect to composition.

For a positive integer n, an i-th hyperprojection on A of arity n, $1 \leq i \leq n$, is an n-ary partial hyperprojection $e_i^n : A^n \to P(A) \setminus \{\emptyset\}$, defined by $e_i^n(x_1, \ldots, x_n) = \{x_i\}$, $(x_1, \ldots, x_n) \in \rho$.

For positive integers n and m, we define the composition of hyperoperations $S_m^n : H_A^{(n)} \times (H_A^{(m)})^n \to H_A^{(m)}$, $(f, g_1, \ldots, g_n) \mapsto f(g_1, \ldots, g_n)$, where $f(g_1, \ldots, g_n) : A^m \to P(A) \setminus \{\emptyset\}$, $(x_1, \ldots, x_m) \mapsto \cup \{f(y_1, \ldots, y_n) :\ y_i \in g_i(x_1, \ldots, x_m), 1 \leq i \leq n\}$.

Let $f \in H_A^{(n)}$ and $g \in H_A^{(m)}$, then $\varsigma f \in H_A^{(n)}$, $\tau f \in H_A^{(n)}$, $\Delta f \in H_A^{(n-1)}$, $f \circ g \in H_A^{(m+n-1)}$ and $\nabla f \in H_A^{(n+1)}$ are defined by

$(\varsigma f)(x_1, x_2, \ldots, x_n) = f(x_2, \ldots, x_n, x_1),\ n \geq 2$
$(\tau f)(x_1, x_2, x_3, \ldots, x_n) = f(x_2, x_1, x_3, \ldots, x_n),\ n \geq 2$
$(\Delta f)(x_1, \ldots, x_{n-1}) = f(x_1, x_1, \ldots, x_{n-1}),\ n \geq 2,$
$(\varsigma f) = (\tau f) = (\Delta f) = f,\ n = 1,$
$(f \circ g)(x_1, \ldots, x_m, x_{m+1}, \ldots, x_{m+n-1}) = \cup\{f(y, x_{m+1}, \ldots, x_{m+n-1}) :\ y \in g(x_1, \ldots, x_m)\}$
$(\nabla f)(x_1, x_2, \ldots, x_{n+1}) = f(x_2, \ldots, x_{n+1}),$

where $x_1, \ldots, x_{m+n-1} \in A$. The full algebra of hyperoperations is $\mathcal{H}_A = (H_A, \circ, \varsigma, \tau, \Delta, e_1^2)$. Each subuniverse of $\mathcal{H}_A$ is a yperclone.

A set of hyperoperations is a hyperclone iff it contains all hyperprojections and is closed with respect to composition.

For a set C we denote the set of all n-ary elements from C by $C^{(n)}$. If C is a set of hyperoperations, we denote the set $\bigcup_{n \geq 1} \{\rho_f : f \in C^{(n)}\}$ by $D(C)$.

, Let us denote the set of all clones on A by L_{O_A} and the set of all hyperclones on A by L_{H_A}. Both sets form algebraic lattices. The atoms (dual atoms) are called minimal (maximal) elements. The least elements in both lattices, trivial clones, will be denoted by J_A. For a set F of hyperoperations, the least hyperclone containing F will be denoted by $\langle F \rangle$, and the least clone containing the set F of operations on A will be denoted by $\langle F \rangle_A$.

If m, n are positive integers, $f \in H_A^{(n)}$ and $\alpha : \{1, \ldots, n\} \to \{1, \ldots, m\}$, we define $\delta_\alpha(f) \in H_A^{(m)}$ by $\delta_\alpha(f)(x_1, \ldots, x_m) = f(x_{\alpha(1)}, \ldots, x_{\alpha(n)})$, $(x_1, \ldots, x_m) \in A^m$. For a set F of hyperoperations δ-closure of F is the set $\delta(F) = \{\delta_\alpha(f) : f \in F^{(n)}, \alpha : \{1, \ldots, n\} \to \{1, \ldots, m\},\ m, n \in \mathbb{N}\}$. For a positive integer $n > 1$, the mapping $(\cdot)^\# : H_A^{(n)} \to O_{P(A)\setminus\{\emptyset\}}^{(n)}$, $f \mapsto f^\#$, where $f^\#$ is defined by $f^\#(X_1, \ldots, X_n) = \cup\{f(x_1, \ldots, x_n) | \ x_i \in X_i, 1 \leq i \leq n\}$, is an isomorphism from $(H_A^{(n)}, *, \varsigma, \tau, e_1^2)$ onto $(O_{P(A)\setminus\{\emptyset\}}^{(n)}, *, \varsigma, \tau, p_1^2)$ and there is a hyperoperation $h \in H_A$ such that $\lambda(\Delta h) \neq \Delta\lambda(h)$. ([14], [15]) We say that an operation $f^\#$ is extended from the hyperoperation f. If F is an arbitrary set of hyperoperations, then $F^\# = \{f^\# | f \in F\}$.

3 Hyperclone embeddings

3.1 From the hyperclone lattice on A to the clone lattice on $P(A) \setminus \{\emptyset\}$

The mapping $\lambda : L_{H_A} \to L_{P(A)\setminus\{\emptyset\}}$, $C \mapsto \langle C^\# \rangle_{P(A)\setminus\{\emptyset\}}$, is an order embedding, though not a full one, i.e. there are $F, G \in L_{H_A}$ such that $[\lambda(F), \lambda(G)] \setminus im\lambda \neq \emptyset$. ([5]).

Let A be $\{0, 1, 2, \ldots\}$, $|A| \geq 3$, $m \geq 2$ and $g_m \in H_A^{(m)}$ the hyperoperation defined by

$$g_m(x_1, \ldots, x_m) = \begin{cases} \{2\}, & (x_1, x_2, \ldots, x_m) \in J_m \\ \{0\}, & \text{otherwise} \end{cases},$$

where J_m is the set of all m-tuples with one coordinate equal 2 and all others equal 1.

Let us define the hyperoperation $f_{m+1} \in H_A^{(m+1)}$ by

$$f_{m+1}(x_1, x_2, \dots, x_{m+1}) = \begin{cases} A & , x_1 \neq x_2 \\ g_m(x_2, \dots, x_{m+1}) & , x_1 = x_2. \end{cases}$$

Thus, extended operation from f_{m+1} is operation $f^{\#}_{m+1} \in (P(A) \setminus \{\emptyset\})^{(m+1)}$ defined by

$$\begin{aligned} f^{\#}_{m+1}(X_1, X_2, \dots, X_{m+1}) &= \cup\{f_{m+1}(x_1, x_2, \dots, x_{m+1}) | x_i \in X_i\} \\ &= \begin{cases} g^{\#}_m(X_2, \dots, X_{m+1}) & , X_1 = X_2,\ |X_1| = 1 \\ A & , \text{otherwise} \end{cases} \end{aligned}$$

Lemma 3.1 *For every clone F of hyperoperations on A and for every $\emptyset \neq Q \subseteq \bigcup_{i \geq 2} \{f_i\}$ holds $\lambda(F) \neq \langle Q^{\#} \rangle_{P(A) \setminus \{\emptyset\}}$.*

Proof. Let Q be an arbitrary nonvoid subset of $\bigcup_{i \geq 2} \{f_i\}$. There is $m \geq 2$ such that $f_{m+1} \in Q$.

Suppose to the contrary, that there is a clone of hyperoperations F such that its λ image $\lambda(F)$ is the clone generated by $Q^{\#}$, i.e. $\lambda(F) = \langle Q^{\#} \rangle_{P(A) \setminus \{\emptyset\}}$. Then, there is a hyperoperation $h \in F$ with property $f^{\#}_{m+1} = \delta_\alpha h^{\#}$. From $f^{\#}_{m+1} \in H^{\#}_A$, it follows $f_{m+1} \in F$ (see [5]). Since F is a clone, the hyperoperation $g_m \in H_A^{(m)}$, defined by $g_m(x_1, \dots, x_m) = f_{m+1}(x_1, x_1, \dots, x_m)$, also belongs to F ($g_m = f_{m+1}(e^1_m, e^1_m, e^2_m, \dots, e^m_m)$). However, we shall prove that $g^{\#}_m \notin \langle Q^{\#} \rangle_{P(A) \setminus \{\emptyset\}}$.

$\langle Q^{\#} \rangle \subset Pol_{P(A) \setminus \{\emptyset\}}(A)$, because for every $i \geq 2$ $f^{\#}_i(A, A, \dots, A) = A$, and $g^{\#}_m \notin Pol_{P(A) \setminus \{\emptyset\}}(A)$, because $g^{\#}_m(A, A, \dots, A) = \{0, 2\} \neq A$.

So, $g^{\#}_m \in F^{\#}$ and $g_m \notin \langle Q^{\#} \rangle_{P(A) \setminus \{\emptyset\}}$.

(Note that Q is not a clone of hyperoperations on A.) □

Lemma 3.2 *For every $i \geq 3$ holds $f^{\#}_i \notin \langle \bigcup_{j \geq 3, j \neq i} \{f^{\#}_j\} \rangle_{P(A) \setminus \{\emptyset\}}$.*

Proof. Let us define for every $m \geq 2$ relation $\rho_m \in P^m_A$ by $\rho_m = A_m \cup B_m$, where A_m is the set of all m-tuples with exactly one coordinate equal $\{2\}$ and all others equal $\{1\}$ and $B_m = \{\{0\}, \{2\}, \{0, 2\}, A\}^m \setminus (\{2\}, \{2\}, \dots, \{2\})$. We are going to show that $f^{\#}_{m+1} \notin Pol_{P(A) \setminus \{\emptyset\}} \rho_m$ and $f^{\#}_{i+1} \in Pol_{P(A) \setminus \{\emptyset\}} \rho_m, i \neq m$.

For m-tuples $(\{2\}, \{1\}, \dots, \{1\}), (\{2\}, \{1\}, \dots, \{1\}), (\{1\}, \{2\}, \dots, \{1\}), \dots, (\{1\}, \{1\}, \dots, \{2\}) \in A_m$ (the first one and the second one are equal), holds $(f^{\#}_{m+1}(\{2\}, \{2\}, \{1\}, \dots, \{1\}), f^{\#}_{m+1}(\{1\}, \{1\}, \{2\}, \dots, \{1\}), \dots, f^{\#}_{m+1}(\{1\}, \{1\}, \{1\}, \dots, \{2\})) = (\{2\}, \{2\}, \dots, \{2\}) \notin \rho_m$. So, $f^{\#}_{m+1} \notin Pol_{P(A) \setminus \{\emptyset\}} \rho_m$.

Suppose that there is $i \neq m$ such that $f^{\#}_{i+1} \notin Pol_{P(A) \setminus \{\emptyset\}} \rho_m$. Then, there are tuples $\mathbf{X}_1 := (X_{11}, \dots, X_{m1}), \dots, \mathbf{X}_{i+1} := (X_{1(i+1)}, \dots, X_{m(i+1)}) \in \rho_m$, such that $(Y_1, Y_2, \dots, Y_m) := (f^{\#}_{i+1}(X_{11}, \dots, X_{1(i+1)}), \dots, f^{\#}_{i+1}(X_{m1}, \dots, X_{m(i+1)})) \notin \rho_m$. Since $im f^{\#}_{i+1} = \{\{0\}, \{2\}, \{0, 2\}, A\}$, it follows that $(Y_1, \dots, Y_m) = (\{2\}, \{2\}, \dots, \{2\})$ and it is possible only for $\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_{i+1} \in A_m$, $\mathbf{X}_1 = \mathbf{X}_2$ and $i = m$. This is a contradiction. □

Theorem 3.1 *There are continuumly many pairwise distinct clones of operations on $P(A) \setminus \{\emptyset\}$ in the interval $[\lambda(Jh_A), \lambda(H_A)]$ that are not in the set of all images $im\lambda$ of the operation λ.*

Proof. Let $R = \bigcup_{i \geq 3} \{f_i\}$.

(a) Since λ is an order embedding, λ is injective and for $F, G \in L_{H_A}$ $F \leq G$ is equivalent to $\lambda(F) \leq \lambda(G)$. So, for $\langle Q \rangle \leq H_A$ it follows $\langle Q^{\#} \rangle_{P(A) \setminus \{\emptyset\}} \leq \lambda(\langle Q \rangle) \leq \lambda(H_A), Q \subseteq R$.

On the other hand, $\lambda(Jh_A) \leq \langle Q^{\#} \rangle_{P(A) \setminus \{\emptyset\}}$, because $\lambda(Jh_A) = J_{P(A) \setminus \{\emptyset\}}$. This being the result of the following:

$(e^n_i)^{\#}(X_1, \dots, X_n) = \bigcup_{x_i \in X_i} e^n_i(x_1, \dots, x_n) = \bigcup_{x_i \in X_i} \{x_i\} = X_i = p_i^{n, P(A) \setminus \{\emptyset\}}(X_1, \dots, X_n)$.

(see [14],[15])

(b) It follows from Lemma3.1 that for every $Q \subseteq R$ holds $\langle Q^{\#}\rangle_{P(A)\setminus\{\emptyset\}} \notin im\lambda$.

(c) From Lemma3.2 follows that for all $Q_1, Q_2 \subseteq R$ if $Q_1^{\#} \neq Q_2^{\#}$ then $\langle Q_1^{\#}\rangle_{P(A)\setminus\{\emptyset\}} \neq \langle Q_2^{\#}\rangle_{P(A)\setminus\{\emptyset\}}$.

□

3.2 From the clone lattice on A to the hyperclone lattice on $P(A)\setminus\{\emptyset\}$

Let us define a map $\alpha : L_A \to L_{H_A}$ by $\alpha(C) = \bigcup_{n\geq 1} \{f \in H_A^{(n)} : \exists f' \in C \quad \forall(x_1,\dots,x_n) \in A^n \ \ f(x_1,\dots,x_n) = \{f'(x_1,\dots,x_n)\}\}$. It is easy to show that $\alpha(C)$ is a hyperclone.

Lemma 3.3 *The map α is full order embedding.*

Proof. Obviously, α is 1-1 map, and holds $C_1 \leq C_2 \Leftrightarrow \alpha(C_1) \leq \alpha(C_2)$.

Let H be an arbitrary hyperclone with the property $\alpha(J_A) \subseteq H \subseteq \alpha(O_A)$. From the definition of α immediately follows that there is clone C such that $\alpha(C) = H$. □

Without loss of generality, we will sometimes identify the hyperclone $\alpha(C)$ and the clone C, in order to simplify the presentation.

Lemma 3.4 *Let A be a finite set with $|A| \geq 2$. Every hyperclone generated by constant hyperoperation on A is minimal.*

Corollary 3.1 *Let A be a finite set with $|A| \geq 2$. There are least $2^{|A|} - 1$ minimal clones in the lattice L_{H_A}.*

Theorem 3.2 *On any finite set A, with $|A| \geq 2$, there are three minimal hyperclones such that their join contains all hyperoperations.*

Proof. It is proved in [4] that there are two minimal clones such that their join contains all operations on any finite set A. Romov proved in [13] that O_A is maximal hyperclone. Hence, it is enough to choose the third minimal hyperclone from the set of minimal hyperclones that are not minimal in the clone lattice on A. From previous lemma follows that such a set is not empty. □

Theorem 3.3 *The interval $[\alpha(\langle O_A^{(1)}\rangle), \alpha(O_A)]$ is a chain.*

Proof. It is the chain obtained by Burle in 1967 [11]. He has shown that interval $[\langle O_A^{(1)}\rangle, O_A]$ is $(|A|+1)$-element chain

$$\langle O_A^{(1)}\rangle = U_1 \subset L \subset U_2 \subset \ldots \subset U_k = O_A.$$

U_i is the set of all operations depending on at most one variable and operations taking at most j values and L is the set of operations depending of one variable and operations $f(x_1,\dots,x_n) = \alpha(\psi_1(x_{i_1}) + \ldots \psi_t(x_{i_t}))$, where $\alpha : \{0,1\} \to A$ and $\psi_j : A \to \{0,1\}$, $j \in \{i_1,\dots,i_t\}$, $1 \leq i_1 < \ldots < i_t \leq n$, are arbitrary maps and $+$ is addition modulo 2. □

Corollary 3.2 *There are finite maximal chains in the hyperclone lattice.*

Proof. It is known that there are finite maximal chains in the interval $[J_A, \langle O_A^{(1)}\rangle]$. With the maximal chain from previous theorem and the clone H_A (since O_A is maximal in the hyperclone lattice), we get the finite maximal chain in the hyperclone lattice. □

4 Minimal hyperclones

A hyperclone is minimal if it is not trivial and its only subclone is trivial.

A hyperoperation of minimal arity in a minimal hyperclone, that is not a hyperprojection, is called minimal hyperoperation.

A ternary majority hyperoperation on A, $ma \in H_A^{(3)}$, is a ternary hyperoperation on A defined by $ma(x,x,y) = ma(x,y,x) = ma(y,x,x) = \{x\}$ for all $x, y \in A$.

A ternary minority hyperoperation on A, $mi \in H_A^{(3)}$, is a ternary hyperoperation on A defined by $mi(x,x,y) = mi(x,y,x) = mi(y,x,x) = \{y\}$ for all $x, y \in A$.

For $n > 2$ and $1 \le i \le n$, every n-ary hyperoperation s with $s(x_1, \ldots, x_n) = \{x_i\}$, $|\{x_1, \ldots, x_n\}| < n$ is called semi-hyperprojection.

It is easy to show that the theorem analogous to Rosenberg's classification theorem ([16]) holds for minimal hyperoperations.

Theorem 4.1 *Every minimal hyperoperation is one of the following types:*

(1) a unary hyperoperation,

(2) a binary idempotent hyperoperation,

(3) a ternary majority hyperoperation,

(4) a ternary minority hyperoperation,

(5) an n-ary semi-hyperprojection, $n > 2$.

Lemma 4.1 *Let $\langle g\rangle_{P(A)\setminus\{\emptyset\}}$ be a minimal clone on $P(A) \setminus \{\emptyset\}$. If there is a hyperoperation f, such that $g = f^{\#}$, then $\langle f\rangle$ is a minimal hyperclone on A.*

Proof. Since $\langle\{f^{\#}\}\rangle_{P(A)\setminus\{\emptyset\}}$ is a minimal clone in $L_{P(A)\setminus\{\emptyset\}}$, and the mapping $\alpha : L_{H_A} \to L_{P(A)\setminus\{\emptyset\}}$, $\alpha(C) = \langle C^{\#}\rangle_{P(A)\setminus\{\emptyset\}}$, studied in [5] is an order embedding we prove the claim.

4.1 Minimal hyperclones on a two-element set

Let $A = \{0, 1\}$.

Lemma 4.2 *Let f be a hyperoperation on A. $\langle\{f\}\rangle$ is a minimal hyperclone on A iff $\langle\{f^{\#}\}\rangle_{P(A)\setminus\{\emptyset\}}$ is a minimal clone on $P(A) \setminus \{\emptyset\}$.*

Lemma 4.3 *There are 13 minimal hyperclones on A.*

Lemma 4.4 *There is no pair of minimal hyperclones whose join is the clone of all hyperoperations.*

Lemma 4.5 *There are 12 three-element sets of minimal hyperoperations whose union generate the clone of all hyperoperations on A.*

Proof. Since we know all the pairs of minimal clones whose join is O_A ([4],[17]) and O_A is maximal in the hyperclone lattice, it follows that $\langle\{\neg, min, f_i\}\rangle = \langle\{\neg, max, f_i\}\rangle = H_A$, $1 \le i \le 6$. □

Lemma 4.6 *There are 36 four-element sets of minimal hyperoperations whose union is the clone of all hyperoperations on A.*

4.2 Unary minimal hyperclones

Lemma 4.7 *Let $f, g \in H_A^{(1)}$. Then,*

(a) $(f \circ g)^{\#} = f^{\#} \circ g^{\#}$.

(b) $(\Delta f)^{\#} = \Delta f^{\#}$.

Proof.

(a) Let X be an arbitrary subset of A. Then $(f \circ g)^{\#}(X) = \cup\{(f \circ g)(x) : x \in X\} = \cup\{\cup\{f(y) : y \in g(x)\} : x \in X\} = \cup\{f(y) : y \in \cup\{g(x) \in X\}\} = \cup\{f(y) : y \in g^{\#}(X)\} = f^{\#}(g^{\#}(X))$.

(b) It follows immediately from the definition of Δ,when f and $f^{\#}$ are unary, that $(\Delta f)^{\#} = f^{\#} = \Delta f^{\#}$.

$\square$

Corollary 4.1 *The mapping $f \mapsto f^{\#}$ is isomorphism from $(H_A^{(1)}; \Delta)$ onto $(\lambda(H_A^{(1)}), \Delta)$.*

Corollary 4.2 *Let $f \in H_A^{(1)}$. Then, $\langle\{f^{\#}\}\rangle_{P(A)\setminus\{\emptyset\}} = \langle\{f\}\rangle^{\#}$.*

Corollary 4.3 *The restriction of the mapping $\lambda : L_{H_A} \to L_{P(A)\setminus\{\emptyset\}}$, $C \mapsto \langle C^{\#}\rangle_{P(A)\setminus\{\emptyset\}}$ to the interval $[J_A, \langle H_A^{(1)}\rangle]$ is a full order embedding.*

Corollary 4.4 *Let $f \in H_A^{(1)}$. Then, $\langle\{f\}\rangle$ is a minimal hyperclone iff $\langle\{f^{\#}\}\rangle_{P(A)\setminus\{\emptyset\}}$ is a minimal clone.*

Lemma 4.8 *Let $f \in H_A^{(1)}$. Then, it holds*

(a) $f^2 = f \Leftrightarrow (f^{\#})^2 = f^{\#}$.

(b) $f^p = e_1^1 \Leftrightarrow (f^{\#})^p = \pi_1^1$, for some prime p.

Proof.

(a) $(\to)$ If $f^2 = f$ then $(f^{\#})^2 = (f^2)^{\#} = f^{\#}$. $(\leftarrow)$ If $f^{\#}(f^{\#}(X)) = f^{\#}(X)$ holds for every $X \subseteq A$, then it also holds for $|X| = 1$. It means that for every $x \in A$, $f^{\#}(f^{\#}(\{x\})) = f^{\#}(\{x\})$, i.e. $f(f(x)) = f(x)$.

(b) $(\to)$ If $f^p = e_1^1$ then $(f^{\#})^p = (f^p)^{\#} = (e_1^1)^{\#} = \pi_1^1$. $(\leftarrow)$ For every $X \subseteq A$ $(f^{\#})^p(X) = X$ implies that for every $x \in A$ holds $(f^{\#})^p(\{x\}) = \{x\}$, i.e. for every $x \in A$ holds $f^p(x) = x$.

$\square$

Theorem 4.2 *Let $f \in H_A^{(1)}$. Then, $\langle\{f\}\rangle$ is minimal iff $f^2 = f$ or $f^p = id_A$, for some prime p.*

Proof. Let $f^2 = f$ or $f^p = id_A$, for some prime p. From Lemma4.8 ($f^2 = f$ iff $(f^{\#})^2 = f^{\#}$) and ($f^p = e_1^1$, for some prime p iff $(f^{\#})^p = \pi_1^1$, for some prime p). It is known ([16], [3],[18]) that ($(f^{\#})^2 = f^{\#}$) or $(f^{\#})^p = \pi_1^1$), for some prime p) iff $\langle\{f^{\#}\}\rangle_{P(A)\setminus\emptyset}$ is minimal clone in $L_{O_{P(A)\setminus\{\emptyset\}}}$. From Lemma4.4, $\langle\{f^{\#}\}\rangle_{P(A)\setminus\{\emptyset\}}$ is minimal clone in $L_{O_{P(A)\setminus\{\emptyset\}}}$ iff $\langle\{f\}\rangle$ is minimal hyperclone in L_{H_A}. $\square$

Example 1 *There are* 6 *unary minimal hyperclones on $A = \{0, 1\}$.*

Example 2 *There are* 64 *unary minimal hyperclones on the set $A = \{0, 1, 2\}$.*

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