A Consideration on Functions Preserving Set Inclusion Relation

富山県立大学 工学部 高木昇（Noboru Takagi）
Department of Electronics and Informatics,
Toyama Prefectural University

Abstract—This paper discusses functions over the set of non-empty subsets of \{0, 1, \ldots, r - 1\} that are monotonic in the set inclusion relation. Min, Max and Literal operations play an important role in multiple-valued logic design/circuits because they can realize any function over \{0, 1, \ldots, r - 1\}. Operations over the set of non-empty subsets of \{0, 1, \ldots, r - 1\} that preserve the set inclusion relation are introduced from Min, Max and Literal operations over \{0, 1, \ldots, r - 1\}. Then, this paper proves some of mathematical properties of functions over the set of non-empty subsets of \{0, 1, \ldots, r - 1\} that are composed of the operations introduced.

Keywords: Multiple-Valued Logic Design/Circuits, Set Inclusion Relation, Clone Theory

1 Introduction

S. C. Kleene [1] first introduced regularity into ternary operations over the set of truth values \{0, 1, u\} in the following way.

A truth table for a ternary operation is regular if it satisfies the condition that "A given column (row) contains 1 in the u row (column), only if the column (row) consists entirely of 1's; and likewise for 0".

Kleene's regularity is one of the ways how binary operations can be expanded into ternary operations. Table 1 is the truth tables of regular ternary operations, which are given from the traditional binary operations AND, OR and NOT.

It is worth to notice that M. Goto [2] independently introduced ternary operations that are identical with the Kleene's ternary operations in Table 1. He showed that the ternary operations can be a model for analyzing undetermined behavior existing in binary systems, such as hazards in binary logic circuits. After Goto's work, M. Mukaidono studied mathematical properties of functions over \{0, 1, u\} that can be expressed by a formula composed of the three ternary operations (He called the ternary functions regular ternary logic functions). One of Mukaidono's main results[3] is that a function \(f\) over \{0, 1, u\} is a regular ternary logic function if and only if the function \(f\) is monotonic in the partial ordered relation, defined by Figure 1. I. G. Rosenberg [8] indicated that the set of regular ternary logic functions is this clone generated by the Kleene's ternary logic, i.e., the clone is identical with the clone over the 3-element universe \{\{0\}, \{1\}, \{0, 1\}\} that preserves the set inclusion relation \(\subseteq\).

This paper discusses functions over the set of non-empty subsets of \{0, 1, \ldots, r - 1\} when \(r\) is more than 2. In the following, \(E_r\) and \(P_r\) denote the \(r\)-valued set \{0, 1, \ldots, r - 1\} and the set of non-empty subsets of \(E_r\), respectively.

Table 1: Truth Tables of Regular Ternary Operations NOT, AND and OR

<table>
<thead>
<tr>
<th>NOT</th>
<th>AND</th>
<th>OR</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>u</td>
<td>u</td>
<td>u</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>u</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>u</td>
<td>u</td>
<td>u</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>u</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>u</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>u</td>
</tr>
</tbody>
</table>

\[ \begin{array}{ccc}
0 & 0 & 0 \\
1 & 1 & 1 \\
u & u & u \\
\end{array} \]
First, this paper shows a definition for expanding operations over $E_r$ into operations over $P_r$. This definition is identical with the Kleene's regularity when $r$ is equal to 2, and it has already been shown by M. Mukaidono [4] and I. G. Rosenberg [8]. Min, Max, and Literal operations play an important role in multiple-valued logic design/circuits, because they can realize any multiple-valued logic function over $E_r$. Therefore, Min, Max, and Literal operations are focused on in this paper. This paper then clarifies mathematical properties of functions over $P_r$, which are expressed by formulas composed of the operations given from Min, Max, and Literal operations over $E_r$.

This paper is organized below. Section 2 is for preliminaries. This section shows the definition for expanding operations over $E_r$ into operations over $P_r$, and then gives some of their mathematical properties. Section 3 focuses on Min, Max, and Delta Literal operations over $E_r$. They are expanded into operations over $P_r$, and then this section proves a necessary and sufficient condition for a function over $P_r$ to be expressed by a formula composed of these operations. Section 4 shows examples for the results obtained in Section 3. Section 5 discusses mathematical properties of functions over $P_r$ when we selected Min, Max, and Universal Literal operations over $E_r$. Then, Section 6 gives examples for the results appeared in Section 5 Section 7 concludes the paper.

2 Preliminaries

Let $E_r$ be the $r$-valued set $\{0, \ldots, r-1\}$, and let $P_r$ be the set of all non-empty subsets of $E_r$, i.e., $P_r = 2^{E_r} - \{\emptyset\}$, where $2^{E_r}$ is the power set of $E_r$. If a subset of $E_r$ consists of only one element, then it is called a singleton. The set of all singletons of $E_r$ is denoted by $S_r$, i.e., $S_r = \{\{0\}, \ldots, \{r-1\}\}$. It is evident that the set $P_r$ is a partial ordered set in the set inclusion $\subseteq$. In this paper, elements of the set $E_r$ are denoted by small letters such as $a$, $b$, $c$, $x$, $y$, etc., while elements of the set $P_r$ (i.e., non-empty subsets of $E_r$) are denoted by capital letters such as $A$, $B$, $C$, $X$, $Y$ etc.

**Definition 1** Let $o$ be an $n$-ary operation on $E_r$. Then, an $n$-ary operation $\hat{o}$ on $P_r$ with respect to $o$ is defined by setting

$$\hat{o}(A_1, \ldots, A_n) = \{o(a_1, \ldots, a_n) | a_1 \in A_1, \ldots, a_n \in A_n\}$$

for any element $(A_1, \ldots, A_n) \in P_r^n$. (End of Definition)

The following three operations play an important role in multiple-valued logic design because $r$-valued functions consisting of these operations and the constants $0, \ldots, r-1$ are
Table 3: Truth Table of $\sqcup$

<table>
<thead>
<tr>
<th>$X \setminus Y$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>01</th>
<th>02</th>
<th>12</th>
<th>012</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>01</td>
<td>01</td>
<td>01</td>
<td>01</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>01</td>
<td>02</td>
<td>12</td>
<td>012</td>
</tr>
<tr>
<td>01</td>
<td>0</td>
<td>01</td>
<td>01</td>
<td>01</td>
<td>01</td>
<td>01</td>
<td>01</td>
</tr>
<tr>
<td>02</td>
<td>0</td>
<td>01</td>
<td>02</td>
<td>01</td>
<td>02</td>
<td>12</td>
<td>012</td>
</tr>
<tr>
<td>12</td>
<td>0</td>
<td>1</td>
<td>12</td>
<td>01</td>
<td>012</td>
<td>012</td>
<td>012</td>
</tr>
<tr>
<td>012</td>
<td>0</td>
<td>01</td>
<td>012</td>
<td>012</td>
<td>012</td>
<td>012</td>
<td>012</td>
</tr>
</tbody>
</table>

Table 4: Truth Table of $X^S$

<table>
<thead>
<tr>
<th>$X$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>01</th>
<th>02</th>
<th>12</th>
<th>012</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X^0$</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>02</td>
<td>02</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$X^1$</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>02</td>
<td>0</td>
<td>02</td>
<td>02</td>
</tr>
<tr>
<td>$X^2$</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>02</td>
<td>0</td>
<td>02</td>
</tr>
<tr>
<td>$X^{01}$</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>2</td>
<td>02</td>
<td>02</td>
<td>02</td>
</tr>
<tr>
<td>$X^{02}$</td>
<td>2</td>
<td>0</td>
<td>2</td>
<td>02</td>
<td>2</td>
<td>02</td>
<td>02</td>
</tr>
<tr>
<td>$X^{12}$</td>
<td>0</td>
<td>2</td>
<td>2</td>
<td>02</td>
<td>02</td>
<td>2</td>
<td>02</td>
</tr>
<tr>
<td>$X^{012}$</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>

The unary operations $x^S$ are often called the universal literals. However, when $S$ is a singleton, $x^S$ is sometimes called a delta literal.

For simplicity, in writing elements of $P_r$, we will remove brackets and put an underline if no confusion arises. That is, for example, $0$, $02$ and $012$ stand for $\{0\}$, $\{0, 2\}$ and $\{0, 1, 2\}$, respectively. Tables 2, 3 and 4 are truth tables of operations on $P_3$ with respect to $\cdot$, $+\mathbf{u}$ and $x^S$, respectively. Because this paper focuses on the operations on $P_r$ with respect to $\cdot$, $+$ and $x^S$, they are denoted by $\wedge$, $\sqcup$ and $X^S$, respectively.

This paper does not allow any kinds of compositions of the operations $\wedge$, $\sqcup$ and $X^S$ on $P_r$. Compositions are restricted by the form of the formulas defined below.

**Definition 2** Formulas are defined inductively in the following way.

1. Constants $\{0\}, \ldots, \{r - 1\}$ and literals $X_i^S$ ($i = 1, \ldots, n$ and $S \subseteq P_r$) are formulas.
2. If $G$ and $H$ are formulas, then $(G \wedge H)$ and $(G \sqcup H)$ are also formulas.
3. It is a formula if and only if we get it from (1) and (2) in a finite number of steps.

(End of Definition)
In writing formulas, we sometimes omit the operation $\land$ for simplicity.

It is evident that every formula expresses a function on $P_r$ when each variable $X_i$ takes an element of $P_r$. Furthermore, it is easy to verify that the formulas can not express all of the functions on $P_r$, i.e., the functions on $P_r$ expressed by the formulas are not functionally complete on $P_r$. Thus, one of the main subjects of the paper is to clear what functions on $P_r$ can be expressed by the formulas.

In the following, for any elements $(A_1, \ldots, A_n)$ and $(B_1, \ldots, B_n)$ of $P^n_r$, $(A_1, \ldots, A_n) \subseteq (B_1, \ldots, B_n)$ stands for $A_i \subseteq B_i$ for all $i$'s. Moreover, $(A_1, \ldots, A_n) \cap (B_1, \ldots, B_n) = \emptyset$ stands for $A_i \cap B_i = \emptyset$ for some $i$.

**Theorem 1** Suppose a function $f$ on $P_r$ can be expressed by a formula. Then, $f(A_1, \ldots, A_n) \in S_r$ holds for any element $(A_1, \ldots, A_n) \in S^n_r$.

**Theorem 2** Suppose a function $f$ on $P_r$ can be expressed by a formula. Then, $f(B_1, \ldots, B_n)$ holds for any elements $(A_1, \ldots, A_n)$ and $(B_1, \ldots, B_n)$ of $P^n_r$ such that $(A_1, \ldots, A_n) \subseteq (B_1, \ldots, B_n)$.

### 3 Functions Expressed by Formulas Composed of $\land$, $\lor$, and Delta Literals

This section shows a necessary and sufficient condition for functions on $P_r$ that can be expressed by formulas with the operations $\land$, $\lor$ and delta literals.

**Theorem 3** Let $A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_n$ be elements of $P_r$. If a function $f$ on $P_r$ is expressed by a formula, then the least element of $f(A_1, \ldots, A_{i-1}, A, A_{i+1}, \ldots, A_n)$ (which is a subset of $E_r$) is equal to the least element of $f(A_1, \ldots, A_{i-1}, B, A_{i+1}, \ldots, A_n)$ for any elements $A$ and $B$ of $P_r - S_r$, i.e.,

$$\min f(a_1, \ldots, A_{i-1}, A, A_{i+1}, \ldots, A_n) = \min f(A_1, \ldots, A_{i-1}, B, A_{i+1}, \ldots, A_n)$$

holds for any elements $A$ and $B$ of $P_r - S_r$.

From Theorems 1, 2 and 3, any function $f$ on $P_r$ expressed by a formula satisfies the following Condition A.

**Condition A:** Let $f$ be a function on $P_r$.

1. If $(A_1, \ldots, A_n) \in S^n_r$, then $f(A_1, \ldots, A_n) \in S_r$.
2. For any elements $(A_1, \ldots, A_n)$ and $(B_1, \ldots, B_n)$ of $P^n_r$, $(A_1, \ldots, A_n) \subseteq (B_1, \ldots, B_n)$ implies $f(A_1, \ldots, A_n) \subseteq f(B_1, \ldots, B_n)$.
3. Let $A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_n$ be elements of $P_r$. Then, the least element of $f(A_1, \ldots, A_{i-1}, A, A_{i+1}, \ldots, A_n)$ is equal to the least element of $f(A_1, \ldots, A_{i-1}, B, A_{i+1}, \ldots, A_n)$ for any elements $A$ and $B$ of $P_r - S_r$, i.e.,

$$\min f(A_1, \ldots, A_{i-1}, A, A_{i+1}, \ldots, A_n) = \min f(A_1, \ldots, A_{i-1}, B, A_{i+1}, \ldots, A_n)$$

holds for any elements $A$ and $B$ of $P_r - S_r$.

---

2All of the proofs in this paper are omitted because of the limitation of the space.
In the remainder of this section, it is proven that Condition A is a necessary and sufficient condition for a function on $P_r$ to be expressed by a formula with the operations $\wedge$, $\sqcup$, and delta literals.

**Definition 3** Let $f$ be a function on $P_r$, and let $A = (A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_n)$ be an element of $S_r^{n-1}$. Then, we define one-variable functions $\check{f}_A^i(X)$ and $\hat{f}_A^i(X)$ ($i = 1, \ldots, n$) expressed by the following formulas.

$$\check{f}_A^i(X) = \bigcup_{s \in E_r} \left( \{s\} \wedge \bigcup_{B \in P^i_A(s)} \sqcup X^B \right),$$  

(1)

where $P^i_A(s)$ is the set of all maximal elements of the set

$$P^i_A(s) = \{B \in P_r | \min f(A_1, \ldots, A_{i-1}, B, A_{i+1}, \ldots, A_n) = s\},$$

(2)

and

$$\hat{f}_A^i(X) = \bigcup_{s \in P_r - S_r} \left( \bigcup_{t \in S} \left\{ \{t\} \wedge \bigcup_{B \in Q^i_A(s)} \left( \bigwedge_{e \in B} X^{\{e\}} \right) \right\} \right),$$

(3)

where $Q^i_A(s)$ is the set of all minimal elements of

$$Q^i_A(s) = \{B \in P_r - S_r | f(A_1, \ldots, A_{i-1}, B, A_{i+1}, \ldots, A_n) = S\}.$$  

(4)

In the formulas (1) and (3), if $P^i_A(s)$ and $Q^i_A(s)$ are the empty set, then

$$\bigcup_{B \in P^i_A(s)} X^B$$

and

$$\bigcup_{B \in Q^i_A(s)} \left( \bigwedge_{e \in B} X^{\{e\}} \right)$$

are defined as $\{0\}$, respectively. Moreover, in the formula (1), when $B = E_r$, then $X^B$ is the constant $\{r-1\}$.

In the formula (1), if $f$ is a function satisfying Condition A, then any subset $P^i_A(s)$ is a subset of $S_r$, or it is equal to $\{E_r\}$. Now, let us show this property. Suppose an element $B$ of $P_r - S_r$ is a member of $P^i_A(s)$. Then, by Condition A(3), it follows that $E_r$ is also a member of $P^i_A(s)$. Therefore, when an element of $P_r - S_r$ is a member of $P^i_A(s)$, then $E_r$ is also a member of $P^i_A(s)$. This fact implies that $P^i_A(s)$ is a subset of $S_r$, or it is equal to $\{E_r\}$. So, the formula (1) is well-defined when $f$ is a function satisfying Condition A.

**Lemma 1** Let $A = (A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_n)$ be an element of $S_r^{n-1}$. Then, for a function $f$ satisfying Condition A,

$$\check{f}_A^i(B) = \begin{cases} f(A_1, \ldots, A_{i-1}, B, A_{i+1}, \ldots, A_n) & \text{if } f(A_1, \ldots, A_{i-1}, B, A_{i+1}, \ldots, A_n) \in S_r \\ K & \text{otherwise} \end{cases}$$

holds for any element $B$ of $P_r$, where $K$ is an element of $P_r$ such that

$$\{f_0\} \subseteq K \subseteq f(A_1, \ldots, A_{i-1}, B, A_{i+1}, \ldots, A_n)$$

and $f_0$ is the least element of $f(A_1, \ldots, A_{i-1}, B, A_{i+1}, \ldots, A_n)$.

(End of Lemma)
Lemma 2 Let $A = (A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_n)$ be an element of $S_r^{n-1}$. Then, for a function $f$ satisfying Condition A,

$$\hat{f}_A^i(B) = \begin{cases} \{0\} & \text{if } f(A_1, \ldots, A_{i-1}, B, A_{i+1}, \ldots, A_n) \in S_r \\ \{0\} \cup f(A_1, \ldots, A_{i-1}, B, A_{i+1}, \ldots, A_n) & \text{otherwise} \end{cases}$$

holds for any element $B \in P_r$.

Lemma 3 Let $A = (A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_n)$ be an element of $S_r^{n-1}$. Then, for a function $f$ satisfying Condition A,

$$\check{f}_A^i(B) \cup \hat{f}_A^i(B) = f(A_1, \ldots, A_{i-1}, B, A_{i+1}, \ldots, A_n)$$

holds for any element $B \in P_r$.

In the following, this section proves that any function satisfying Condition A can be expressed by a formula, and also shows a method how a formula can be formulated by a function satisfying Condition A.

Definition 4 Let $f$ be a function on $P_r$. Then, $f_1$ is defined as a function on $P_r$ expressed by the following formula.

$$f_1(X_1, \ldots, X_n) = \bigcup_{i=1}^{n} f^i(X_1, \ldots, X_n), \quad (5)$$

where

$$f^i(X_1, \ldots, X_n) = \bigcup_{A=(A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_n) \in S_r^{n-1}} \left( \bigwedge_{j=1(j \neq i)}^{n} X_j^{A_j} \wedge \left( \check{f}_A^i(X_i) \cup \hat{f}_A^i(X_i) \right) \right).$$

(End of Definition)

Here, let us introduce a subset of $P_r^n$, which will be denoted by $I(r, n)$, below.

$$I(r, n) = \bigcup_{i=1}^{n} \{(A_1, \ldots, A_n) \in P_r^n \mid A_i \in P_r - S_r \text{ and } A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_n \in S_r \}$$

That is, each element $(A_1, \ldots, A_n)$ of $I(r, n)$ consists of elements of $S_r$, but except for one.

Lemma 4 Let $F$ be a function satisfying Condition A. Then,

$$f_1(A_1, \ldots, A_n) = \begin{cases} f(A_1, \ldots, A_n) & \text{if } f(A_1, \ldots, A_n) \in S_r \cup I(r, n) \\ K & \text{otherwise} \end{cases}$$

holds for any element $(A_1, \ldots, A_n) \in P_r^n$, where $K$ is an element of $P_r$ such that $\{0\} \subseteq K \subseteq \{0\} \cup f(A_1, \ldots, A_n)$.

(End of Lemma)
Definition 5 Let \( f \) be a function on \( P_r \), let \( S \) be an element of \( P_r - S_r \), and let \( \hat{T}(f, S) \) is the set of all minimal elements of the following subset of \( P^n_r \).

\[
T(f, S) = \{(A_1, \ldots, A_n) \in P_r^n \mid f(A_1, \ldots, A_n) = S \text{ and } (A_1, \ldots, A_n) \notin S^n_r \cup I(r, n)\}
\]  

Then, \( f_2 \) is defined as a function on \( P_r \) expressed by the following formula.

\[
f_2(X_1, \ldots, X_n) = \{s_0\} \cup \bigcup_{S \subseteq P_r - S_r} \left\{ \bigwedge_{b \in A_1} X_1^{(b)} \land \cdots \land \bigwedge_{b \in A_n} X_n^{(b)} \right\}
\]

where

\[
f_S(X_1, \ldots, X_n) = \begin{cases} 
\bigcup_{(A_1, \ldots, A_n) \in \hat{T}(f, S)} \left\{ \bigwedge_{b \in A_1} X_1^{(b)} \land \cdots \land \bigwedge_{b \in A_n} X_n^{(b)} \right\} & \text{if } \hat{T}(f, S) \neq \emptyset \\
\{0\} & \text{otherwise}
\end{cases}
\]

and \( s_0 \) is the least element of \( \bigcup_{(A_1, \ldots, A_n) \in P_r^n} f(A_1, \ldots, A_n) \).  

(End of Definition)

Lemma 5 Let \( f \) be a function on \( P_r \) satisfying Condition A. Then,

\[
f_2(A_1, \ldots, A_n) = \begin{cases} 
\{s_0\} & \text{if } (A_1, \ldots, A_n) \in S^n_r \cup I(r, n) \\
 f(A_1, \ldots, A_n) & \text{otherwise}
\end{cases}
\]

holds for any element \((A_1, \ldots, A_n) \in P^n_r \), where \( s_0 \) is the least element of the union \( \bigcup_{(A_1, \ldots, A_n) \in P^n_r} f(A_1, \ldots, A_n) \).  

(End of Lemma)

Theorem 4 Let \( f \) be a function on \( P_r \) satisfying Condition A. Then,

\[
f(A_1, \ldots, A_n) = f_1(A_1, \ldots, A_n) \cup f_2(A_1, \ldots, A_n)
\]

holds for any element \((A_1, \ldots, A_n) \in P^n_r \), where \( f_1 \) and \( f_2 \) are the formulas (5) and (7), respectively.  

(End of Theorem)

4 Examples of Functions Satisfying Condition A

Consider the function \( f \) on \( P_3 \) whose truth table is given in Table 5. It is not difficult to verify that \( f \) satisfies Condition A. Then, this section illustrates how we can form the formula that expresses the function \( f \).

Example 1 Let us first consider the formulas (1) and (3). It follows by Eq. (2) that we have the following three subsets of \( P_3 \).

\[
\begin{align*}
P_2^0(0) &= \{B \in P_3 \mid \min f(B, \emptyset) = 0\} = \{0, 2, 01, 02, 12, 012\} \\
P_2^1(1) &= \{B \in P_3 \mid \min f(B, \emptyset) = 1\} = \{1\} \\
P_2^2(2) &= \{B \in P_3 \mid \min f(B, \emptyset) = 2\} = \emptyset
\end{align*}
\]
Table 5: Example of Function $f$ Satisfying Condition A

<table>
<thead>
<tr>
<th>$X \setminus Y$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>01</th>
<th>02</th>
<th>12</th>
<th>012</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>02</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>12</td>
<td>12</td>
<td>12</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>02</td>
<td>0</td>
<td>02</td>
<td>02</td>
</tr>
<tr>
<td>01</td>
<td>01</td>
<td>01</td>
<td>02</td>
<td>012</td>
<td>012</td>
<td>012</td>
<td>012</td>
</tr>
<tr>
<td>02</td>
<td>0</td>
<td>02</td>
<td>0</td>
<td>02</td>
<td>0</td>
<td>02</td>
<td>02</td>
</tr>
<tr>
<td>12</td>
<td>012</td>
<td>012</td>
<td>02</td>
<td>012</td>
<td>012</td>
<td>012</td>
<td>012</td>
</tr>
<tr>
<td>012</td>
<td>012</td>
<td>012</td>
<td>02</td>
<td>012</td>
<td>012</td>
<td>012</td>
<td>012</td>
</tr>
</tbody>
</table>

Thus, since $\tilde{P}_0^1(0) = \{012\}$, $\tilde{P}_0^1(1) = \{1\}$, and $\tilde{P}_0^1(2) = \emptyset$, we have the formula $\tilde{f}_0^1(X)$ by Eq.

$$\tilde{f}_0^1(X) = (0 \land 012) \cup (1 \land X^1) \cup (2 \land 0)$$

$$= 1 \land X^1$$

In a similar way, we have the formulas $\tilde{f}_1^1(X), \tilde{f}_2^1(X), \tilde{f}_0^2(Y), \tilde{f}_1^2(Y), \tilde{f}_2^2(Y)$, below.

$$\tilde{f}_1^1(X) = 1X^1 \cup X^2$$
$$\tilde{f}_2^1(X) = X^1$$
$$\tilde{f}_0^2(Y) = \emptyset$$
$$\tilde{f}_1^2(Y) = 1 \cup Y^2$$
$$\tilde{f}_2^2(Y) = Y^1$$

Moreover, it follows by Eq. (4) that we have

$$Q_0^1(01) = \{B \in P_3 - S_3 \mid f(B, 0) = 01\} = \{01\},$$
$$Q_0^1(02) = \{B \in P_3 - S_3 \mid f(B, 0) = 02\} = \emptyset,$$
$$Q_0^1(12) = \{B \in P_3 - S_3 \mid f(B, 0) = 12\} = \emptyset,$$ and
$$Q_0^1(012) = \{B \in P_1 - S_3 \mid f(B, 0) = 012\} = \{12, 012\}.$$ 

Thus, since $Q_0^1(01) = \{01\}$, $Q_0^1(02) = Q_0^1(12) = \emptyset$ and $Q_0^1(012) = \{12\}$, we have the formula $\tilde{f}_0^1(X)$ by Eq. (3).

$$\tilde{f}_0^1(X) = (0X^0X^1 \cup 1X^0X^1) \cup 0 \cup 0 \cup (0X^2X^2 \cup 1X^2X^2 \cup 2X^2X^2)$$

$$= 1X^0X^1 \cup 1X^1X^2 \cup X^1X^2.$$ 

In a similar way, we have the formulas $\tilde{f}_1^1(X), \tilde{f}_1^1(Y), \tilde{f}_2^1(Y), \tilde{f}_1^2(Y), \tilde{f}_2^2(Y)$, below.

$$\tilde{f}_1^1(X) = 1X^0X^1 \cup X^0Y^2 \cup 1X^1X^2 \cup 1X^1X^2,$$
$$\tilde{f}_2^1(X) = 1X^0X^1 \cup 1X^1X^2,$$
$$\tilde{f}_0^2(Y) = Y^1Y^2,$$
$$\tilde{f}_1^2(Y) = 1Y^2Y^2 \cup X^2Y^2 \cup 1Y^1Y^2 \cup Y^1Y^2,$$
$$\tilde{f}_2^2(Y) = Y^2Y^1 \cup Y^1Y^2.$$
Table 6: Truth Tables of $\check{f}_{A}^{1}(X)$ and $\hat{f}_{A}^{1}(X)$

<table>
<thead>
<tr>
<th>$X$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>01</th>
<th>02</th>
<th>12</th>
<th>012</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_{0}^{1}(X)$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>01</td>
<td>01</td>
<td>01</td>
<td>01</td>
</tr>
<tr>
<td>$f_{1}^{1}(X)$</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>01</td>
<td>02</td>
<td>02</td>
<td>12</td>
</tr>
<tr>
<td>$f_{2}^{1}(X)$</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>01</td>
<td>02</td>
<td>02</td>
<td>02</td>
</tr>
<tr>
<td>$f_{1}^{1}(X)$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>01</td>
<td>02</td>
<td>02</td>
<td>02</td>
</tr>
<tr>
<td>$f_{2}^{1}(X)$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>01</td>
<td>02</td>
<td>02</td>
<td>02</td>
</tr>
</tbody>
</table>

Table 7: Truth Tables of $\check{f}_{A}^{2}(Y)$ and $\hat{f}_{A}^{2}(Y)$

<table>
<thead>
<tr>
<th>$Y$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>01</th>
<th>02</th>
<th>12</th>
<th>012</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_{0}^{2}(Y)$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>01</td>
<td>02</td>
<td>02</td>
<td>02</td>
</tr>
<tr>
<td>$f_{1}^{2}(Y)$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>01</td>
<td>02</td>
<td>02</td>
<td>02</td>
</tr>
<tr>
<td>$f_{2}^{2}(Y)$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>01</td>
<td>02</td>
<td>02</td>
<td>02</td>
</tr>
<tr>
<td>$f_{1}^{2}(Y)$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>01</td>
<td>02</td>
<td>02</td>
<td>02</td>
</tr>
<tr>
<td>$f_{2}^{2}(Y)$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>01</td>
<td>02</td>
<td>02</td>
<td>02</td>
</tr>
</tbody>
</table>

Table 8: Truth Table of $f_{A}^{i} = \check{f}_{A}^{i} \cup \hat{f}_{A}^{i}$

<table>
<thead>
<tr>
<th>$X$ or $Y$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>01</th>
<th>02</th>
<th>12</th>
<th>012</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_{0}^{1}(X)$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>01</td>
<td>02</td>
<td>02</td>
<td>02</td>
</tr>
<tr>
<td>$f_{1}^{1}(X)$</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>01</td>
<td>02</td>
<td>02</td>
<td>02</td>
</tr>
<tr>
<td>$f_{2}^{1}(X)$</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>02</td>
<td>02</td>
<td>02</td>
<td>02</td>
</tr>
<tr>
<td>$f_{0}^{2}(Y)$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>02</td>
<td>02</td>
<td>02</td>
</tr>
<tr>
<td>$f_{1}^{2}(Y)$</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>12</td>
<td>12</td>
<td>12</td>
</tr>
<tr>
<td>$f_{2}^{2}(Y)$</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>02</td>
<td>02</td>
<td>02</td>
<td>02</td>
</tr>
</tbody>
</table>

Tables 6 and 7 show the truth tables of $\check{f}_{A}$ and $\hat{f}_{A}$ for which $i = 1, 2$ and $A \in \{0, 1, 2\}$.

(End of Example)

It follows by Lemma 3 that

$$f(X, B) = \check{f}_{A}^{1}(X) \cup \hat{f}_{A}^{1}(X) \quad \text{and} \quad f(B, Y) = \check{f}_{A}^{2}(Y) \cup \hat{f}_{A}^{2}(Y)$$

hold for every $A \in \{0, 1, 2\}$ and every $B \in P_{3}$, where $\check{f}_{A}(X)$, $\hat{f}_{A}(X)$, $\check{f}_{A}(Y)$ and $\hat{f}_{A}(Y)$ have been obtained in Eqs. (9), (10), (11) and (12). Table 8 shows the truth tables of $\check{f}_{A} \cup \hat{f}_{A}$, where $i = 1, 2$ and $A \in \{0, 1, 2\}$.

Example 2 Let us next consider the formula (5) in Definition 4. It follows by Eqs. (9) and (11) that we have the formula $\check{f}_{A}^{i}(X) \cup \hat{f}_{A}^{i}(X)$ below.

$$\check{f}_{0}^{1}(X) \cup \hat{f}_{0}^{1}(X) = \underline{1}X^{1} \sqcup \underline{1}X^{2} \sqcup \underline{1}X^{2} \sqcup X^{1}X^{2}$$

$$= \underline{1}X^{1} \sqcup X^{1}X^{2}$$

In a similar way, by Eqs. (10), (11) and (12), we have the formulas

$$\check{f}_{0}^{2}(Y) \cup \hat{f}_{0}^{2}(Y) = Y^{1}Y^{2},$$

$$\check{f}_{1}^{2}(Y) \cup \hat{f}_{1}^{2}(Y) = \underline{1} \sqcup Y^{2},$$

$$\check{f}_{2}^{2}(Y) \cup \hat{f}_{2}^{2}(Y) = Y^{1}. $$

Therefore, the formula $f_{1}(X, Y)$ of (5) in Definition 4 is given as

$$f_{1}(X, Y) = f^{1}(X, Y) \cup f^{2}(X, Y),$$

(13)
Table 9: Truth Table of $f_1(X, Y)$

<table>
<thead>
<tr>
<th>$X \setminus Y$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>01</th>
<th>02</th>
<th>12</th>
<th>012</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>12</td>
<td>12</td>
<td>12</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>01</td>
<td>01</td>
<td>01</td>
<td>02</td>
<td>01</td>
<td>012</td>
<td>12</td>
<td>12</td>
</tr>
<tr>
<td>02</td>
<td>02</td>
<td>02</td>
<td>02</td>
<td>02</td>
<td>02</td>
<td>02</td>
<td>02</td>
</tr>
<tr>
<td>12</td>
<td>012</td>
<td>012</td>
<td>012</td>
<td>012</td>
<td>012</td>
<td>012</td>
<td>012</td>
</tr>
<tr>
<td>012</td>
<td>012</td>
<td>012</td>
<td>012</td>
<td>012</td>
<td>012</td>
<td>012</td>
<td>012</td>
</tr>
</tbody>
</table>

Table 10: Truth Table of $f_2(X, Y)$

<table>
<thead>
<tr>
<th>$X \setminus Y$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>01</th>
<th>02</th>
<th>12</th>
<th>012</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>01</td>
<td>01</td>
<td>01</td>
<td>02</td>
<td>01</td>
<td>012</td>
<td>12</td>
<td>12</td>
</tr>
<tr>
<td>02</td>
<td>02</td>
<td>02</td>
<td>02</td>
<td>02</td>
<td>02</td>
<td>02</td>
<td>02</td>
</tr>
<tr>
<td>12</td>
<td>012</td>
<td>012</td>
<td>012</td>
<td>012</td>
<td>012</td>
<td>012</td>
<td>012</td>
</tr>
<tr>
<td>012</td>
<td>012</td>
<td>012</td>
<td>012</td>
<td>012</td>
<td>012</td>
<td>012</td>
<td>012</td>
</tr>
</tbody>
</table>

where

\[
f^1(X, Y) = Y^2 \left( \tilde{f}^2_0(X) \cup \tilde{f}^2_0(X) \right) \cup Y^1 \left( \tilde{f}^1_0(X) \cup \tilde{f}^1_0(X) \right) \cup Y^2 \left( \tilde{f}^1_0(X) \cup \tilde{f}^1_0(X) \right) \quad \text{and}
\]

\[
f^2(X, Y) = X^2 \left( \tilde{f}^2_0(Y) \cup \tilde{f}^2_0(Y) \right) \cup X^1 \left( \tilde{f}^1_0(Y) \cup \tilde{f}^1_0(Y) \right) \cup X^2 \left( \tilde{f}^1_0(Y) \cup \tilde{f}^1_0(Y) \right).
\]

Table 9 is the truth table of $f_1(X, Y)$.  

(End of Example)

**Example 3**  
In this example, let us consider the formula (7) in Definition 5. It follows by Eq. (6) that we have the following subsets of $P_3^2$.

\[
T(f, 01) = \emptyset,
\]

\[
T(f, 02) = \{(02, 01), (02, 12), (02, 012)\},
\]

\[
T(f, 12) = \emptyset,
\]

\[
T(f, 012) = \{(A, B) | A \in \{01, 12, 012\} \text{ and } B \in P_3 - S_3\}
\]

Therefore, since we have

\[
\hat{T}(f, 01) = \emptyset,
\]

\[
\hat{T}(f, 02) = \{(02, 01), (02, 12)\},
\]

\[
\hat{T}(f, 12) = \emptyset, \text{ and}
\]

\[
\hat{T}(f, 012) = \{(01, 01), (01, 02), (01, 12), (12, 01), (12, 02), (12, 12)\},
\]

it follows by Eq. (8) that we have the following formulas.

\[
f_{01}(X, Y) = \emptyset
\]

\[
f_{02}(X, Y) = X^0 X^2 Y^2 Y^1 \cup X^0 X^2 Y^2 Y^2
\]

\[
f_{12}(X, Y) = \emptyset
\]

\[
f_{012}(X, Y) = X^0 X^1 Y^2 Y^1 \cup X^0 X^1 Y^2 Y^2 \cup X^0 X^1 Y^2 Y^1 \cup X^1 X^2 Y^2 Y^1 \cup X^1 X^2 Y^2 Y^2 \cup X^1 X^2 Y^2 Y^2
\]

Thus, the formula $f_2(X, Y)$ of (7) in Definition 5 is obtained as the formula below.

\[
f_2(X, Y) = \emptyset \cup \{f_{02}(X, Y) \cup 2 f_{02}(X, Y)\} \cup \{0 f_{012}(X, Y) \cup f_{012}(X, Y) \cup 2 f_{012}(X, Y)\}
\]

\[
= f_{02}(X, Y) \cup f_{012}(X, Y) \cup f_{012}(X, Y)
\]

(14)

Table 10 is the truth table of $f_2(X, Y)$.  

(End of Example)

It follows by Theorem 4 that the function $f$ of Table 5 can be expressed by the formula $f_1(X, Y) \cup f_2(X, Y)$, where $f_1(X, Y)$ and $f_2(X, Y)$ are the formulas given in (13) and (14), respectively.
5 Functions Expressed by Formulas Composed of $\land$, $\lor$ and Universal Literals

This section discusses functions on $P_r$ expressed by formulas, which are composed of the operations $\land$, $\lor$ and universal literals. Then, a necessary and sufficient condition for a function on $P_r$ to be expressed by a formula when $r$ is equal to 3.

**Theorem 5** Let $f$ be a function on $P_r$. If $f$ can be expressed by a formula, then

$$\bigcap_{A\in P_r-S_r} f(A_1, \ldots, A_{i-1}, A, A_{i+1}, \ldots, A_n) \neq \emptyset$$

holds for any elements $A_1, \ldots, A_{i-1}, A, A_{i+1}, \ldots, A_n$ of $P_r$. (End of Theorem)

By Theorems 1, 2 and 5, any function $f$ on $P_r$ expressed by a formula satisfies the following Condition B.

**Condition B:** Let $f$ be a function on $P_r$.

1. If $(A_1, \ldots, A_n) \in S_r^n$, then $f(A_1, \ldots, A_n) \in S_r$.
2. For any elements $(A_1, \ldots, A_n)$ and $(B_1, \ldots, B_n)$ of $P_r^n$, $(A_1, \ldots, A_n) \subseteq (B_1, \ldots, B_n)$ implies $f(A_1, \ldots, A_n) \subseteq f(B_1, \ldots, B_n)$.
3. $$\bigcap_{A\in P_r-S_r} f(A_1, \ldots, A_{i-1}, A, A_{i+1}, \ldots, A_n) \neq \emptyset$$ holds for any elements $A_1, \ldots, A_{i-1}, A, A_{i+1}, \ldots, A_n$ of $P_r$.

In the following, this section proves that Condition B is a necessary and sufficient condition for a function on $P_3$ to be expressed by a formula with the operations $\land$, $\lor$, and universal literals.

**Definition 6** Let $(A_1, \ldots, A_n)$ be any element of $P_r^n$. Then, $\alpha = X_1^{A_1} \land \cdots \land X_n^{A_n}$ is said to be the type-1 term corresponding to $(A_1, \ldots, A_n)$. Next, let $(B_1, \ldots, B_n)$ be any element of $P_r^n - S_r^n$. Then, $\beta = \bigwedge_{e\in B_1} X_1^{\{e\}} \land \cdots \land \bigwedge_{e\in B_n} X_n^{\{e\}}$ is said to be the type-2 term corresponding to $(B_1, \ldots, B_n)$. (End of Definition)

Let $S$ be an element of $P_r$, and let $T$ be an element of $P_r - S_r$. Then, it is easy to verify that the following two equations are valid.

$$X^S = \begin{cases} \{r-1\} & \text{if } X \subseteq S \\ \{0\} & \text{if } X \cap S = \emptyset \\ \{0, r-1\} & \text{otherwise} \end{cases}$$ (15)

$$\bigwedge_{e\in T} X^{\{e\}} = \begin{cases} \{0, r-1\} & \text{if } T \subseteq X \\ \{0\} & \text{otherwise} \end{cases}$$ (16)

Therefore, for any type-1 term $\alpha$ and any type-2 term $\beta$, $\alpha(A_1, \ldots, A_n) = \{r-1\}, \{0, r-1\}, \text{or } \{0\}$, and $\beta(A_1, \ldots, A_n) = \{0, r-1\} \text{ or } \{0\}$ hold for any element $(A_1, \ldots, A_n) \in P_r^n$.

**Lemma 6** For any type-1 term $\alpha$ corresponding to $(A_1, \ldots, A_n) \in P_r^n$,
(1) \((B_1, \ldots, B_n) \subseteq (A_1, \ldots, A_n)\) iff \(\alpha(B_1, \ldots, B_n) = \{r - 1\}\),
(2) \((A_1, \ldots, A_n) \cap (B_1, \ldots, B_n) = \emptyset\) iff \(\alpha(B_1, \ldots, B_n) = \{0\}\),
(3) \((B_1, \ldots, B_n) \not\in (A_1, \ldots, A_n)\) and \((A_1, \ldots, A_n) \cap (B_1, \ldots, B_n) = \emptyset\) iff \(\alpha(B_1, \ldots, B_n) = \{0, r - 1\}\)

hold for any \((B_1, \ldots, B_n) \in P_r^n\).

(End of Lemma)

**Lemma 7** For any type-2 term \(\alpha\) corresponding to \((A_1, \ldots, A_n) \in P_r^n - S_r^n\),
(1) \((A_1, \ldots, A_n) \subseteq (B_1, \ldots, B_n)\) iff \(\alpha(B_1, \ldots, B_n) = \{0, r - 1\}\),
(2) \((A_1, \ldots, A_n) \not\subset (B_1, \ldots, B_n)\) iff \(\alpha(B_1, \ldots, B_n) = \{0\}\)

hold for any \((B_1, \ldots, B_n) \in P_r^n\).

(End of Lemma)

Let \(f\) be a function satisfying Condition B, and let \(S\) be an element of \(P_r\). Then, define two subsets of \(P_r^n\), denoted by \(L(f, S)\) and \(U(f, S)\), below.

\[
L(f, S) = \{(A_1, \ldots, A_n) \in P_r^n \mid f(A_1, \ldots, A_n) \subseteq S\}
\]

\[
U(f, S) = \{(A_1, \ldots, A_n) \in P_r^n \mid f(A_1, \ldots, A_n) \cap S \neq \emptyset\}.
\]

Let \(\bar{L}(f, S)\) and \(U'(f, S)\) be the sets of all maximal elements of \(L(f, S)\) and of all minimal elements of \(U(f, S)\), respectively. Further, let \(\bar{U}(f, S) = U'(f, S) - S_r^n\).

**Lemma 8** Let \(f\) be a function satisfying Condition B, and let \(S\) be an element of \(P_r\). Then, \((f)^S\) can be expressed by the following formula.

\[
(f)^S = \begin{cases} \bigcup_{A \in \bar{L}(f, S)} \alpha_A \sqcup \bigcup_{A \in U'(f, S)} \beta_A & \text{if } \bar{L}(f, S) \neq \emptyset \text{ or } \bar{U}(f, S) \neq \emptyset \\ \{0\} & \text{otherwise} \end{cases}
\]

(17)

where \(\alpha_A\) and \(\beta_A\) are the type-1 and type-2 terms corresponding to \(A\), respectively.

(End of Lemma)

Now, let us consider formulas of one-variable functions satisfying Condition B. Any one-variable function \(f\) satisfying Condition B is in at least one of the following three cases 3.

(B-1) \(f(A) \neq 0\) holds for any element \(A \in P_3 - S_3\).
(B-2) \(f(A) \neq 02\) holds for any element \(A \in P_3 - S_3\).
(B-3) \(f(A) \neq 12\) holds for any element \(A \in P_3 - S_3\).

---

3If \(f\) is in neither one of the cases (B-1), (B-2), (B-3), then it implies that we have three distinct elements \(A, B, C\) in \(P_3 - S_3\) such that \(f(A) = 01\), \(f(B) = 02\), and \(f(C) = 12\). However, this contradicts to the fact that \(f\) satisfies Condition B(3).
Table 11: Example of (B-4)

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>01</th>
<th>02</th>
<th>12</th>
<th>012</th>
<th>012</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>01</td>
<td>02</td>
<td>12</td>
<td>012</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>2</td>
<td>1</td>
<td>02</td>
<td>012</td>
<td>12</td>
<td>012</td>
<td></td>
<td></td>
</tr>
<tr>
<td>01</td>
<td>02</td>
<td>12</td>
<td>2</td>
<td>012</td>
<td>12</td>
<td>012</td>
<td>012</td>
<td></td>
<td></td>
</tr>
<tr>
<td>02</td>
<td>0</td>
<td>12</td>
<td>12</td>
<td>012</td>
<td>12</td>
<td>012</td>
<td>012</td>
<td></td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>0</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>012</td>
<td>12</td>
<td>12</td>
<td></td>
<td></td>
</tr>
<tr>
<td>012</td>
<td>0</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 12: Example of (B-5)

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>01</th>
<th>02</th>
<th>12</th>
<th>012</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>01</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>2</td>
<td>2</td>
<td>02</td>
<td>02</td>
<td>2</td>
<td>2</td>
<td>02</td>
</tr>
<tr>
<td>01</td>
<td>01</td>
<td>01</td>
<td>12</td>
<td>012</td>
<td>012</td>
<td>12</td>
<td>012</td>
<td></td>
</tr>
<tr>
<td>02</td>
<td>02</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>012</td>
<td>012</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>0</td>
<td>12</td>
<td>12</td>
<td>012</td>
<td>012</td>
<td>12</td>
<td>012</td>
<td></td>
</tr>
<tr>
<td>012</td>
<td>012</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td></td>
</tr>
</tbody>
</table>

Table 13: Example of (B-6)

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>01</th>
<th>02</th>
<th>12</th>
<th>012</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>2</td>
<td>2</td>
<td>02</td>
<td>02</td>
<td>2</td>
<td>2</td>
<td>02</td>
</tr>
<tr>
<td>01</td>
<td>0</td>
<td>12</td>
<td>02</td>
<td>012</td>
<td>12</td>
<td>012</td>
<td>12</td>
<td></td>
</tr>
<tr>
<td>02</td>
<td>0</td>
<td>12</td>
<td>2</td>
<td>012</td>
<td>12</td>
<td>012</td>
<td>12</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>0</td>
<td>12</td>
<td>12</td>
<td>012</td>
<td>12</td>
<td>012</td>
<td>12</td>
<td></td>
</tr>
<tr>
<td>012</td>
<td>0</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td></td>
</tr>
</tbody>
</table>

**Property 1** Any one-variable function $f$ satisfying Condition (B) can be expressed by the following formula.

\[
f(X) = \begin{cases} 
  f^{12}(X) \wedge (1 \cup f^{00}(X)) & \text{if } f \text{ is in the case (B-1)} \\
  (1 \wedge f^1(X)) \cup f^2(X) \cup (1 \wedge f^{12}(X)) & \text{if } f \text{ is in the case (B-2)} \\
  (1 \wedge f^1(X)) \cup f^2(X) & \text{if } f \text{ is in the case (B-3)} 
\end{cases}
\]  

(18) (End of Property)

By Property 1, every one-variable function satisfying Condition (B) can be expressed by a formula.

Next, let us consider the case where functions satisfying Condition (B) depend more than one variable. Then, any function $f$ satisfying Condition (B) is in at least one of the three cases below.

**(B-4)** $f(A_1, \ldots, A_n) \neq 01$ holds for any element $(A_1, \ldots, A_n) \in (P_3 - S_3)^n$.

**(B-5)** $f(A_1, \ldots, A_n) \neq 12$ holds for any element $(A_1, \ldots, A_n) \in (P_3 - S_3)^n$.

**(B-6)** \( \bigcap_{(A_1, \ldots, A_n) \in (P_3 - S_3)^n} f(A_1, \ldots, A_n) = 1 \).

Tables 11, 12 and 13 are examples of two-variable functions being in the cases (B-4), (B-5), and (B-6), respectively.

Then, we can prove Properties 1 \sim 6, which show a way for constructing formulas of $n$-variable functions satisfying Condition (B).

Let $A$ be an element $(A_1, \ldots, A_{n-1}, A_{n+1}, \ldots, A_n)$ of $P_r^{r-1}$. Then, denote the one-variable function $f(A_1, \ldots, A_{n-1}, X, A_{n+1}, \ldots, A_n)$ by $f_A^X(X)$.
Property 2 Suppose a function $f$ satisfying Condition B is in the case (B-5). Let $f'$ be a function expressed by the formula

$$f' = p^1 \sqcup \cdots \sqcup p^n,$$

(19)

where

$$p^i(X_1, \ldots, X_n) = \bigcup_{A=(A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_n) \in S_{3}^{n-1}} (f_A^i(X_i) \wedge X_1^{A_1} \wedge \cdots \wedge X_{i-1}^{A_{i-1}} \wedge X_{i+1}^{A_{i+1}} \wedge \cdots \wedge X_n^{A_n}).$$

(20)

Then, for any element $(A_1, \ldots, A_n) \in P_3^n$,

$$f'(A_1, \ldots, A_n) = \begin{cases} f(A_1, \ldots, A_n) & \text{if } (A_1, \ldots, A_n) \not\in (P_3 - S_3)^n \\ K & \text{otherwise} \end{cases}$$

where $K$ is an element of $P_r$ such that $\{0\} \subseteq K \subseteq \{0\} \cup F(A_1, \ldots, A_n).$ (End of Property)

Property 3 Suppose a function $f$ satisfying Condition B is in the case (B-5). Let $f''$ be a function expressed by the formula

$$f''(X_1, \ldots, X_n) = \bigcup_{S \in P_3 - S_3} \left( \bigcup_{t \in \hat{T}(S)} \left( \bigwedge_{e(A_1, \ldots, A_n) \in \hat{T}(S)} X^e \wedge \cdots \wedge \bigwedge_{e \in A_n} X^e \right) \right),$$

(21)

where $\hat{T}(S)$ is the set of all minimal elements of the set

$$T(S) = \{(A_1, \ldots, A_n) \in (P_3 - S_3)^n \mid f(A_1, \ldots, A_n) = S\}$$

and

$$\hat{T}(S) = \bigcup_{(A_1, \ldots, A_n) \in \hat{T}(S)} \left( \bigwedge_{e(A_1, \ldots, A_n) \in \hat{T}(S)} X^e \wedge \cdots \wedge \bigwedge_{e \in A_n} X^e \right).$$

(22)

Then, for any element $(A_1, \ldots, A_n) \in P_3^n$,

$$f''(A_1, \ldots, A_n) = \begin{cases} \{0\} & \text{if } (A_1, \ldots, A_n) \not\in (P_3 - S_3)^n \\ f(A_1, \ldots, A_n) & \text{otherwise} \end{cases}$$

(End of Property)

Property 4 Any function $f$ satisfying Condition B can be expressed by $f = f' \cup f''$, if $f$ is in the case (B-5).

(End of Property)

Property 5 Suppose a function $f$ satisfying Condition B is in the case (B-4). Let $A = (A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_n)$ be an element of $S_3^{n-1}$. Then, define $g_A^i$ and $h$ as functions on $P_r$ expressed by the following formulas.

$$g_A^i(X_1, \ldots, X_n) = f_A^i(X_i) \sqcup \cdots \sqcup f_A^{i-1}(X_{i-1}) \wedge X_{i+1}^{A_{i+1}} \wedge \cdots \wedge X_n^{A_n},$$

(23)

where $A_j = E_3 - A_j (j = 1, \ldots, i - 1, i + 1, \ldots, n)$.

$$h(X_1, \ldots, X_n) = \left( \left( \bigcup_{i=1}^n X_i \right) \sqcup f^{12}(X_1, \ldots, X_n) \right) \wedge \left( \left( \bigcup_{i=1}^n X_i \right) \sqcup f^{02}(X_1, \ldots, X_n) \right) \cup \_$$

(24)
Then, $f$ can be expressed by the following formula.

$$f(X_1, \ldots, X_n) = G(X_1, \ldots, X_n) \land h(X_1, \ldots, X_n),$$  \hspace{1cm} (25)

where $G$ is $\land$-ing of all the $g_A^i$'s of Eq. (23), i.e.,

$$G(X_1, \ldots, X_n) = \bigwedge_{i=1}^{n} \left( \bigwedge_{A \in S_3^{n-1}} g_A^i(X_1, \ldots, X_n) \right),$$  \hspace{1cm} (26)

(End of Property)

**Property 6** Suppose a function $f$ satisfying Condition B is in the case (B-6). Then, $f$ is in either one of the following two cases.

1. $f(A) \neq \underline{02}$ holds for any element $A \in P_3^n$, or
2. $f(A) = \underline{02}$ holds for some element $A \in P_3^n$.

If $f$ is in the case (1), then $f$ can be expressed by

$$f(X_1, \ldots, X_n) = (\bot \land f^1(X_1, \ldots, X_n)) \cup f^2(X_1, \ldots, X_n) \cup (\bot \land f^{12}(X_1, \ldots, X_n)).$$  \hspace{1cm} (27)

Let $w$ be a function expressed by the following formula.

$$w(X_1, \ldots, X_n) = \bigcup_{(A_1, \ldots, A_n) \in Q_{\underline{02}}} \xi_1(A_1) \cup \cdots \cup \xi_n(A_n),$$  \hspace{1cm} (28)

where $Q_{\underline{02}} = \{(A_1, \ldots, A_n) \in P_3^n \mid f(A_1, \ldots, A_n) = \underline{02}\}$, and

$$\xi_i(A) = \begin{cases} X^A_i & \text{if } A \in S_3 \\ 0 & \text{otherwise.} \end{cases}$$

Then, $f$ can be expressed by the following formula, if $f$ is in the case (2).

$$f(X_1, \ldots, X_n) = G(X_1, \ldots, X_n) \land w(X_1, \ldots, X_n),$$  \hspace{1cm} (29)

where $G(X_1, \ldots, X_n)$ is given by Eq. (26).  \hspace{1cm} (End of Property)

## 6 Examples of Function Satisfying Condition B

This section shows examples of 2-variable functions satisfying Condition B, and illustrates how they can be expressed by formulas.

**Example 4** Consider the function $f$ defined by Table 12, which is in the case (B-5). The formula expressing $f$ is given by Properties 2, 3, and 4. First, consider the formulas $f^1_A(X)$ and $f^2_A(Y)$, which appear in Eq. (20). Table 14 shows the truth tables of the six one-variable functions $f^1_1(X)$, $f^2_1(X)$, $f^1_2(X)$, $f^2_2(Y)$, $f^1_2(Y)$ and $f^2_2(Y)$. Since $f^1_1(X)$ and $f^1_2(Y)$ are in (B-2) (or (B-3)), $f^2_1(X)$ is in (B-3), and $f^2_2(Y)$ is in (B-1), it follows by Eq. (17) and (18) that these one-variable functions are expressed by the following formulas.

$\overline{Q_{\underline{02}} \cap (P_3 - S_3)^n = \emptyset}$ holds, since $f$ is in the case (B-6).
Table 14: One-Variable Functions \( f_A^1 \) and \( f_A^2 \) of Example 4

<table>
<thead>
<tr>
<th>( X ) or ( Y )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>01</th>
<th>02</th>
<th>12</th>
<th>012</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f_A^1(X) )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( f_A^1(Y) )</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>01</td>
<td>02</td>
<td>12</td>
<td>012</td>
</tr>
<tr>
<td>( f_A^2(X) )</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( f_A^2(Y) )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 15: Truth Table of \( f' \) of Example 4

<table>
<thead>
<tr>
<th>( X ) ( \times ) ( Y )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>01</th>
<th>02</th>
<th>12</th>
<th>012</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f'_0 \langle X \rangle )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( f'_1 \langle X \rangle )</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>01</td>
<td>02</td>
<td>01</td>
<td>012</td>
</tr>
<tr>
<td>( f'_2 \langle X \rangle )</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( f'_0 \langle Y \rangle )</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( f'_1 \langle Y \rangle )</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( f'_2 \langle Y \rangle )</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 16: Truth Table of \( f'' \) of Example 4

<table>
<thead>
<tr>
<th>( X ) ( \times ) ( Y )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>01</th>
<th>02</th>
<th>12</th>
<th>012</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f''_0 \langle X \rangle )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( f''_1 \langle X \rangle )</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>01</td>
<td>02</td>
<td>01</td>
<td>012</td>
</tr>
<tr>
<td>( f''_2 \langle X \rangle )</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( f''_0 \langle Y \rangle )</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( f''_1 \langle Y \rangle )</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( f''_2 \langle Y \rangle )</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

\[
f_A^1(X) = 0,
\]
\[
f_A^1(Y) = 1X^1 \sqcup X^2,
\]
\[
f_A^2(X) = 1X^2
\]
\[
f_A^2(Y) = 0.
\]

Therefore, it follows by Eq. (19) that \( f' \) is expressed by the following formula:

\[
f' = X^0 f_A^2(Y) \sqcup X^1 f_A^2(Y) \sqcup X^2 f_A^2(Y) \sqcup f_A^2(X) Y^1 \sqcup f_A^1(X) Y^2 \tag{30}\]

Table 15 is the truth table of \( f' \). Next, consider \( f'' \) in Eq. (21). Since

\[
T'(01) = \{(01, 01), (01, 12)\},
\]
\[
T'(02) = \hat{T}'(12) = \emptyset, \text{ and}
\]
\[
T'(012) = \{(02, 01), (02, 02), (02, 12), (12, 01), (12, 02), (12, 12)\},
\]

it follows by Eq. (22) that we have the following formulas.

\[
f_{T'(01)}(X, Y) = X^0 X^1 Y^2 Y^1 \sqcup X^0 X^1 Y^2 Y^2
\]
\[
f_{T'(012)}(X, Y) = X^0 X^2 Y^1 Y^1 \sqcup X^0 X^2 Y^2 Y^2 \sqcup X^0 X^2 Y^2 Y^1 \sqcup X^1 X^2 Y^2 Y^2
\]

We then have \( f''(X, Y) \) below by Eq. (21).

\[
f''(X, Y) = \hat{1} f_{T'(01)}(X, Y) \sqcup \hat{1} f_{T'(012)}(X, Y) \sqcup f_{T'(012)}(X, Y) \tag{31}\]

Table 16 is the truth table of \( f'' \). It follows by Property 4 that \( f(X, Y) = f'(X, Y) \sqcup f''(X, Y) \).

(End of Example)
Example 5 Consider the function $f$ defined by Table 11, which is in $(B-4)$. The formula expressing $f$ is given by Property 5. It follows by Eq. (17) and (18) that the one-variable functions $f^i_A(X)$ and $f^i_A(Y)$ are obtained below.

$$f^0_A(X) = X^1, \quad f^1_A(X) = 1 \cup X^{12}, \quad f^2_A(X) = 1 \cup X^{01},$$
$$f^0_A(Y) = 1 \cup Y^{12}, \quad f^1_A(Y) = 2, \quad f^2_A(Y) = Y^{01} \cup 1 \cup Y^{01}.$$

Then, by Eq. (23), we have

$$g^0_A(X, Y) = X^1 \cup Y^{12}, \quad g^1_A(X, Y) = 1 \cup X^{12} \cup Y^{02}, \quad g^2_A(X, Y) = X^{0} \cup Y^{0} \cup Y^{02},$$
$$g^0_B(X, Y) = X^{12} \cup 1 \cup Y^{2}, \quad g^1_B(X, Y) = 2, \quad g^2_B(X, Y) = X^{0} \cup Y^{0} \cup Y^{02}.$$

By Eq. (26), we have the function $G(X, Y)$ expressed by and-ing of all the above $g_A^i(X, Y)$'s. Table 17 is the truth table of $G(X, Y)$. Next, consider $h$ in Eq. (24). It follows by Eq. (17) that the functions $f^{12}(X, Y)$ and $f^{02}(X, Y)$ are expressed by the following formulas.

$$f^{12}(X, Y) = X^1 \cup X^{12} \cup Y^{1} \cup Y^{2} \cup X^{0} \cup X^{12} \cup Y^{0} \cup Y^{2} \cup X^{1} \cup X^{2} \cup Y^{0} \cup Y^{2},$$
$$f^{02}(X, Y) = X^{0} \cup X^{0} \cup X^{0} \cup X^{0} \cup X^{1} \cup X^{2} \cup X^{0} \cup X^{1} \cup X^{2} \cup Y^{0} \cup Y^{2}.$$

Thus, by Eq. (24), we have

$$h(X, Y) = \left( v(X, Y) \cup f^{12}(X, Y) \right) \land \left( v(X, Y) \cup f^{02}(X, Y) \cup 1 \right),$$

where $v(X, Y) = X^{0} \cup X^{0} \cup X^{0} \cup X^{0} \cup Y^{0} \cup Y^{2} \cup Y^{0} \cup Y^{2}$. Table 18 is the truth table of $h(X, Y)$. Lastly, by Eq. (25), $f(X, Y)$ are expressed by the following formula.

$$f(X, Y) = G(X, Y) \land h(X, Y)$$
$$= g^0_A(X, Y), g^1_A(X, Y) \land g^2_A(X, Y) \land g^0_B(X, Y) \land g^1_B(X, Y) \land g^2_B(X, Y) \land h(X, Y)$$

(End of Example)

Example 6 Consider the function $f$ defined by Table 13, which is in $(B-6)$. The formula expressing $f$ is given by Property 6. Since $f$ is in the case (2) of Property 6, $f$ is expressed by the formula given in Eq. (29).

First, consider the formula $G(X, Y)$. It follows by Eq. (17) and (18) that one-variable functions $f^i_A(X)$ and $f^i_A(Y)$ are obtained below.
Thus, we obtain $g_\lambda^1(X, Y)$ and $g_\lambda^2(X, Y)$ of Eq. (23) below.

$$
\begin{align*}
g_0^1(X, Y) &= Y^{12} \\
g_0^2(X, Y) &= X^{12} \\
g_1^1(X, Y) &= (1 \land X^{12}) \cup X^2 \\
g_1^2(X, Y) &= X^{12} \land (1 \lor X^{02}) \\
g_2^1(X, Y) &= X^{12} \wedge (1 \lor X^{02}) \cup Y^{01} \\
g_2^2(X, Y) &= X^{01} \cup Y^{12}
\end{align*}
$$

By Eq. (26), we have the function $G(X, Y)$ expressed by $\land$-ing of all the above $g_\lambda^i(X, Y)$'s. Table 19 is the truth table of $G$.

Next, consider $w(X, Y)$ of Eq. (28). Because $Q_{02} = \{(2, 01), (2, 02), (2, 012), (02, 02)\}$, it follow by Eq. (28) that the formula $w$ is obtained below.

$$
w(X, Y) = 1 \lor X^2 \lor Y^2
$$

Table 20 is the truth table of $w$. Lastly, it follows by Eq. (29) that the following formula expresses the function $f$.

$$
f(X, Y) = G(X, Y) \land w(X, Y)
$$

7 Conclusions

This paper discussed functions over $P_r$ that preserves the set inclusion relation $\subseteq$. We referred the three kinds of operations $\text{Min}$, $\text{Max}$, and $\text{Literals}$ over $E_r$, because they are functionally complete on the $r$-valued set $E_r$. This paper then proved some of the mathematical properties of functions over $P_r$ that can be expressed by formulas. It is one of the open problems that which set of operations $\hat{o}_1, \hat{o}_2, \ldots, \hat{o}_m$ over $P_r$ can realize any function over $P_r$ preserving the set inclusion relation $\subseteq$. 
References


