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Upper and Lower Bounds on the Number of Disjunctive Forms

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Abstract

In this paper we evaluate the upper and lower bounds on the number of disjunctive (normal) forms of an n-variable Boolean function (for our purpose we take the constant $\mathbf{1}$ function which always takes the value 1). The enumeration problem of the disjunctive forms is equivalent to enumerating elements of a distributive lattice, and it can be solved by enumerating antichains on the ternary n-cube which is isomorphic to the partially ordered set formed by all terms of the given function. For the upper bound we use a newly invented decomposition of the partially ordered set into chains (we introduce a tree structure which spans the cube). For the lower bounds, we evaluate the number of anticains on the cube by analyzing dependency among three consecutive layers instead of two. Put $|DF(\mathbf{1})|$ the number of different disjunctive forms for the constant $\mathbf{1}$ function. We obtain newly improved upper and lower bounds:

$$2^{2^r \cdot \binom{n}{r}} \cdot e^{\binom{n}{r} \cdot \left(1+e^{-r^2 2^{-r}}\right)} < |DF(\mathbf{1})| < \left(\frac{\sqrt[4]{3}}{3} n\right)^{2^r \cdot \binom{n}{r}},$$

where $r = 2n/3$, the Sperner rank for the ternary cube. This serves as a basis for the upper and lower bound on the number of disjunctive forms for all n-variable Boolean functions as well as a basis for the enumeration of many-valued logic functions.

Keywords: enumeration, disjunctive form, logic function, distributive lattice, antichain

1. Introduction

Logic function is usually represented by a *disjunctive form*, i.e., logical OR of logical product of its variables (terms), and so it is also called logical sum or disjunctive normal form. Given a logic function there are in general many finite number of disjunctive forms representing it. Finding the number of disjunctive forms for a function, or of the total number of them for the set of all n-variable logic function are fundamental for knowing the representing capability of disjunctive forms as well as for enumerating k-valued logic functions [1,2].

It is well-known that counting the number of disjunctive forms is equivalent to counting that of the elements of a distributive lattice, and generally it is considered to be a hard problem. Indeed, the exact numbers of them are known hitherto only up to $n = 4$ [3]. Among enumeration problems of logic functions the so-called Dedekind problem is a famous one with more than a hundred years history. It is a problem to count the number of monotone logical functions of Boolean-variables. This is equivalent to counting the number of elements of the free distributive lattice with n generating set [4]. This is a hard enumeration problem and the numbers are known only up to $n = 8$ [5]. As the set of disjunctive forms include as its proper subset the set of monotone logic functions, the numbers of disjunctive forms are larger than the Dededind numbers.

In this paper we present new upper and lower bounds on the number of disjunctive forms. It is known that the number of elements of a finite distributive lattice is equal to the number of antichains in the partial ordered set formed from its irreducible elements [6,7]. In our case terms (with some restrictions) coincides with the irreducible elements. So our task is to evaluate the number of antichains contained in the partial ordered set formed from terms. Our approach follows the one used to obtain the upper and lower bounds on the number of monotone logical functions [8,9], but applying technique used in the enumeration of the number of fuzzy logic functions [10,11]. To derive new bounds we introduce new ideas in both analyzing upper and lower bounds.

2. Definitions

Let n be a positive integer. Let 0 and 1 represent Boolean values as well (0 for False and 1 for True). Put $B = \{0,1\}$. An *n-variable logic* (or Boolean) function is a map $f : B^n \to B$. Let P_2^n

denote the set of all n-variable logic functions. Let us denote (Boolean) variables by x_i $(i = 1,\cdots,n)$ and let us denote logical operations (connectives) AND, OR and NOT by $\cdot$ (logical product operation), $\vee$ (logical sum) and $\sim$ (negation). A logical *formula* (of n variables) is a formula obtained by combining the variables x_i $(i = 1,\cdots,n)$ by the above logical operations. A formula represents an n-variable logical function. Call x_i and $\sim x_i$ a *literal*. A *term* is a product of literals where each index i appears at most once, i.e., no literals x_i and $\sim x_i$ appears in it simultaneously. Let T be the set of all terms. By definition null term (null string) is a term and we represent it by $\mathbf{1}$. Let α_i and α_j be distinct terms. Define a relation $\subset$ by:

$$\alpha_i \subset \alpha_j \Leftrightarrow \text{every literal of } \alpha_j \text{ appears in } \alpha_i . \qquad (1)$$

The relation $\subset$ naturally induces a partial order on the set T of all terms. For a set of terms $\{\alpha_1,\cdots,\alpha_s\}$ as the result does not depend on the application order of $\vee$, there corresponds a logical function defined by a formula $f = \alpha_1 \vee \cdots \vee \alpha_s$.

[Definition 1] A formula $f = \alpha_1 \vee \cdots \alpha_s$ is a *disjunctive form* if its every term is irreducible, that is $\alpha_i \not\subset \alpha_j$ for every $i \neq j$. □

It is well-known that every logical function can be represented by a disjunctive form. (A constant function $\mathbf{0}$ which takes the constant 0 is represented by null disjunctive form which has no term.)

For a function f there can be, in general, many disjunctive forms apart from the difference of orders of the terms. Put $V = \{0,1/2,1\}$. Define a partial order $\prec$ on V by

$$0 \prec 1/2 \text{ and } 1 \prec 1/2 . \qquad (2)$$

Put $\mathbf{V} = \{0,1/2,1\}^n$, the ternary n-cube which is the set of all ternary n-vectors. We extend the partial order $\prec$ on $\mathbf{V}$ coordinate-wise as follows.

[Definition 2] Let $\mathbf{a} = (a_1,\cdots,a_n)$ and $\mathbf{b} = (b_1,\cdots,b_n)$. Then

$$\mathbf{a} \prec \mathbf{b} \Leftrightarrow a_i \prec b_i \text{ for all } i \ (i = 1,\cdots,n). \qquad (3)$$

□

Define a 1:1 map between $\mathbf{V}$ and T as follows.

[Definition 3] For $\mathbf{a}=(a_1,\cdots,a_n)\in\mathbf{V}$ we map $\alpha(\mathbf{a})=x_1^{a_1}\cdot\cdots\cdot x_n^{a_n}$ where we put

$$\begin{cases} a_i=1 & \leftrightarrow & x_i^{a_i}=x_i \\ a_i=0 & \leftrightarrow & x_i^{a_i}=\sim x_i \\ a_i=1/2 & \leftrightarrow & x_i^{a_i}=1 \quad (x_i \text{ does not appear}) \end{cases} \tag{4}$$

□

It is easy to prove that the partial order $(T,\subset)$ and $(\mathbf{V},\prec)$ are isomorphic, i.e.,

$$\mathbf{a}\prec\mathbf{b} \;\Leftrightarrow\; \alpha(\mathbf{a})\subset\alpha(\mathbf{b}). \tag{5}$$

So the statements about $(T,\subset)$ can be interpreted as ones about $(\mathbf{V},\prec)$, and vice versa.

For $\mathbf{a}=(a_1,\cdots,a_n)\in\mathbf{V}$, put

$$I(\mathbf{a})=\{i\,|\,a_i=1 \text{ or } a_i=0\}. \tag{6}$$

The number $r(\mathbf{a})=|I(\mathbf{a})|$ is the *rank* of $\mathbf{a}$. Let us denote by V_k the set of vectors whose rank equals k:

$$V_k=\{\mathbf{a}\,|\,r(\mathbf{a})=k,\ \mathbf{a}\in\mathbf{V}\}. \tag{7}$$

We have $\mathbf{V}=\bigcup_{k=0}^{n}V_k$ and $|V_k|=2^k\cdot\binom{n}{k}$. The sole element $(1/2,\cdots,1/2)\in V_0$ is the maximal element of $\mathbf{V}$ (w.r.t. $\prec$) and every element of V_n $(=B^n=\{0,1\}^n)$ is a minimal element of $\mathbf{V}$ (no other element is minimal). We put $\mathbf{V}_{\bar{n}}\,(=\mathbf{V}\setminus V_n)$ which exactly corresponds to the set of terms T.

3. Preliminaries

Let $S=\{\mathbf{a}_1,\cdots,\mathbf{a}_s\}$ be a subset from the ternary n-cube $\mathbf{V}$.

[Definition 4] The set S is a *chain* if holds $\mathbf{a}_1\prec\cdots\prec\mathbf{a}_s$. It is an antichain if holds $\mathbf{a}_i\prec\mathbf{a}_j$ for no i,j $(i\neq j)$. □

The integer $|S|$ is the *length* (*size*) of the chain (antichain). We include the empty set as antichain.

[Theorem 1] For an antichain of $\mathbf{V}$ there exists a disjunctive form (uniquely up to the order of its terms) and vice versa. □

Thus there is a 1:1 correspondence between the sets of antichains of $\mathbf{V}$ and those of disjunctive forms of T. So, counting the disjunctive forms is reduced to that of antichains in $\mathbf{V}$. As $|\mathbf{V}| = 3^n$ the number of disjunctive form is bounded from above by 2^{3^n}. It is well-known:

[Theorem 2] The set of disjunctive forms of n-variables forms a distributive lattice with respect to the operations $\vee$ (OR) and $\wedge$ (AND). □

So our problem is also to count the number of elements of a finite distributive lattice. It is clear that the Dedekind problem (counting the monotone functions) is a special case of our problem as disjunctive forms can be of terms with only positive literals as a special subset of disjunctive forms.

First we consider the number of disjunctive forms for a given logic function f. Assume that an n-variable logic function $f(x_1,\cdots,x_n)$ is given. We evaluate the number of disjunctive forms which represent f.

Let $\alpha = x_1^{a_1} \cdot\cdots\cdot x_n^{a_n}$ be a term. A term α belongs to f if holds

$$f([a_1],\cdots,[a_n]) = 1, \tag{8}$$

where $[a_i]$ denotes largest integer not exceeding a_i, e.g., $[1/2] = 0$. The set $V(f)$ of terms of f is

$$V(f) = \{\alpha = x_1^{a_1} \cdot\cdots\cdot x_n^{a_n} \mid f([a_1],\cdots,[a_n]) = 1,\ a_1,\cdots,a_n \in V\} \tag{9}$$

To $\mathbf{a} = (a_1,\cdots,a_n)$ assign a subset of B^n.

$$\mathbf{a}^* = \{(b_1,\cdots,b_n) \in B^n \mid b_i = 0\, or\, 1 \ \ if\ a_i = 1/2,\ \ b_i = a_i \ \ otherwise\} \tag{10}$$

[Theorem 3] Let an n-ary logic function f be represented by a disjunctive form $f = \alpha_1 \vee \cdots \vee \alpha_s$ and let $\alpha_i = x_1^{a_{i_1}} \cdot\cdots\cdot x_n^{a_{i_n}}$ and let $\mathbf{a}_i = (a_{i_1},\cdots,a_{i_n})$ for i $(i = 1,\cdots,n)$. Then the set $\{\mathbf{a}_1,\cdots,\mathbf{a}_n\}$ is an antichain of $\mathbf{V}$ and holds

$$\mathbf{a}_1^* \cup \cdots \cup \mathbf{a}_s^* = V_n(f) \tag{11}$$

Conversely, if the last sentence is true, then the set of terms $\{\alpha_1,\cdots,\alpha_s\}$ defined by $\mathbf{a}_i$ $(i=1,\cdots,n)$ is a disjunctive form of f.

(Proof) This follows from Theorem 1. □

We have the following.

[Theorem 4] The set of disjunctive forms of f forms a distributive lattice with respect to the operations $\vee$ (OR) and $\cdot$ (AND). □

Thus the problem to find all disjunctive forms for f is to find all antichains $\{\mathbf{a}_1,\cdots,\mathbf{a}_n\}$ in $V(f)$ such that the equation (11) holds.

The next statement is a slight improvement of this statement (we omit the proof). Put $\mathbf{V}_{\bar{n}}(f)=\mathbf{V}(f)\backslash V_n(f)$.

[Theorem 5] The number of disjunctive forms of f is equal to the number of antichains in $\mathbf{V}_{\bar{n}}(f)$.

□

Let us denote the set of disjunctive forms of f by $DF(f)$. Let **DF** denote the set of all disjunctive forms for the n-variable function. Then

$$\mathbf{DF}=\bigcup_{f\in P_2^n} DF(f). \tag{12}$$

As $DF(f)$'s are disjoint we have

$$|\mathbf{DF}|=\sum_{f\in P_2^n}|DF(f)|. \tag{13}$$

[Theorem 6] The maximal number of $|DF(f)|$ is attained when $f=\mathbf{1}$ (a constant function which always takes the value 1). □

Thus we have

$$|DF(\mathbf{1})|<|\mathbf{DF}|<2^{2^n}\cdot|DF(\mathbf{1})| \tag{14}$$

We use this formula for evaluating the bounds for $|\mathbf{DF}|$. So we concentrate on the evaluation of $DF(\mathbf{1})$.

4. Upper bound

First, we introduce the upper bound described in [10]. For m the size of a largest antichain in the poset $\mathbf{V}_n^-$, Dilworth's theorem[7,12] says that $\mathbf{V}_n^-$ is the disjoint (except the maximal element of $\mathbf{V}_n^-$) union of m chains. Each chain has at most n elements from $\mathbf{V}_n^-$ and shares the maximal element (that is, element in V_0). Thus, if S is an antichain in $\mathbf{V}_n^-$, then S is uniquely determined by its intersection with each of m chains; the intersection contains at most one element. In the case that S has the maximal element, all elements in S are included by this element. As a result, we can safely ignore the maximal element of $\mathbf{V}_n^-$, and so assume that each chain has at most $n-1$ elements. Since there are at most n possibilities (including the case that we select no element) for the intersection of S with any chain, $|DF(\mathbf{1})|$ is bounded as follows:

$$|DF(\mathbf{1})| < n^m \tag{15}$$

Here, since $\mathbf{V}_n^-$ enjoys the Sperner property [7,14] from [13], we assume that the largest-sized antichain in $\mathbf{V}_n^-$ has cardinality m, i.e., we put $m := \max(|V_k|)$. Let r be the rank of the layer in $\mathbf{V}_n^-$ having the cardinality m (let us call it Sperner rank). Then, $r \approx \lceil 2n/3 \rceil$ holds [10]. So, the above upper bound becomes as follows:

$$|DF(\mathbf{1})| < n^{2^r \binom{n}{r}}. \tag{16}$$

This argument is similar to one used in [8] for estimating the upper bound of monotone logic functions with n-variables.

Now, we proceed to improve the upper bound by incorporating a new idea. The poset $\mathbf{V}_n^-$ we divide at the r – th (the Sperner) rank into two parts, and we focus our attention on the part of 2^r ternary n-vecters derived by expanding ($a_{i,j} = 0,1 : 1 \le j \le r$) the ternary n-vector $(a_{i_1}, 1/2, \cdots, 1/2, a_{i_r})$, where 1/2 appears at $n-r$ fixed (but any) coordinates; three are $\binom{n}{r}$ such copies (see Figure 1, shaded part).

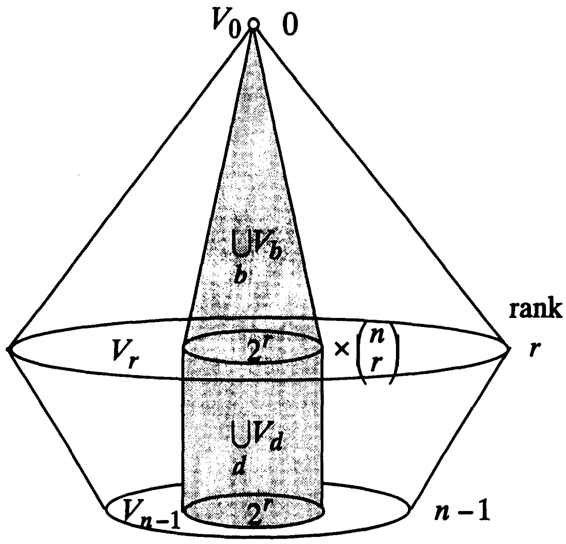

Figure 1. Dividing the poset V_n^- into two parts at the Sperner rank r.

First part is a poset $\bigcup_{0\le b\le r} V_b$, which is composed of $r+1$ layers in V_n^- whose rank b is equal to or smaller than r. Second part is a poset $\bigcup_{r+1\le d\le n-1} V_d$, which consists of $n-1-r$ layers in V_n^- with the rank d larger than r. Here, we can observe that $\bigcup_{0\le b\le r} V_b$ can be divided into disjoint except the maximal element $(1/2,\cdots,1/2)$ $\binom{n}{r}$ binary trees, each of which has 2^r minimum elements in V_r (where, the $1/2$'s positions of each minimum element is the same) (see Figure 2).

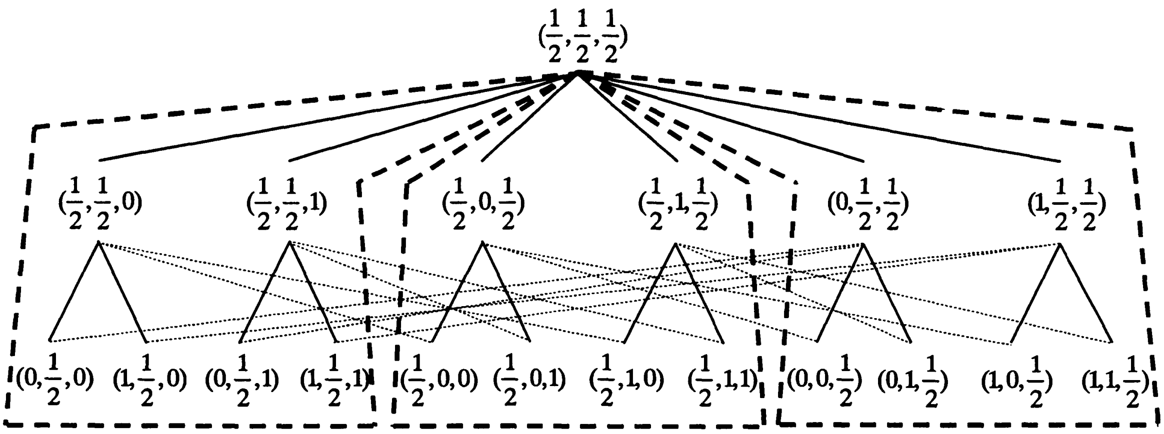

Figure 2. Partition of the upper cube into $\binom{n}{r}$ binary trees ($n=3$, $r=2$).

Now we divide each tree into disjoint chains in the following way: For a given tree we select a largest path (chain) from the root of the tree (if there are many we take the leftmost path). Then we remove the chain from the tree breaking the tree into a forest (a set of trees). We repeat the same procedure for each tree of the forest until there remains a tree which consists of a single node (see Figure 3). Now from the largest chain we remove the largest element of the chain; i.e. the element of V_0. In this way, each binary tree is divided into 2 chains of length r and 2^k chains of length $r-k$, $1 \le k \le r-1$ (see Figure 4). The sum of these chains is $2 + \sum_k 2^k \; (= 2^r)$.

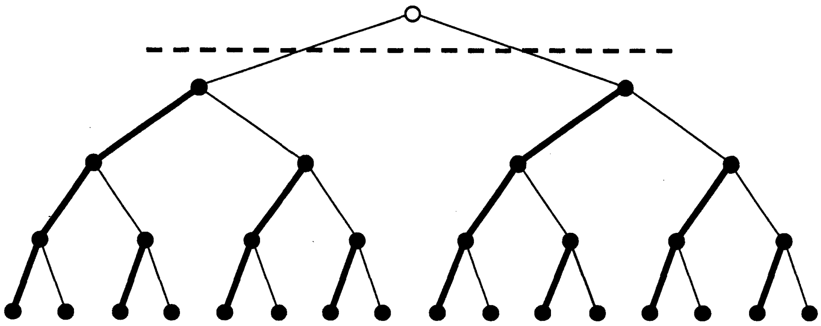

Figure 3. Decomposition of a binary tree into disjoint chains.

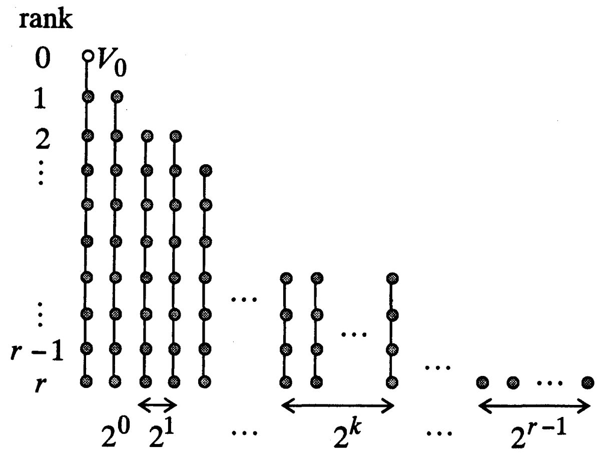

Figure 4. 2^r chains (in $\bigcup_b V_b$) obtained from a binary tree.

In addition, the second poset $\bigcup_d V_d$ can be divided into disjoint $2^r\binom{n}{r}$ chains with a length of at most $n-1-r$ by Dilworth's theorem [7,12] (see Figure 5).

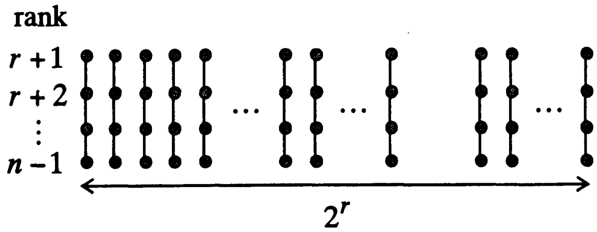

Figure 5. 2^r **chains (in** $\bigcup_d V_d$ **) obtained by Dilworth's theorem.**

Then, for 2^r elements in V_r, there exist 2 chains with a length of at most $r+(n-1-r)$ and 2^k chains with a length of at most $(r-k)+(n-1-r)$, $1 \le k \le r-1$ in the poset $\bigcup_b V_b$ and $\bigcup_d V_d$. If S is any antichain in $\bigcup_b V_b$ and $\bigcup_d V_d$ (namely, in $\mathbf{V}_{\overline{n}}$), then, for 2^r elements in V_r, S is uniquely determined by its intersection with each of the above chains. Considering all possibilities, including the case of no select, the intersections of S with any chain give the upper bound of $|DF(\mathbf{1})|$ as follows:

$$\left((r+(n-1-r)+1)^2 \cdot \prod_{k=1}^{r-1}\left((r-k)+(n-1-r)+1\right)^{2^k}\right)^{\binom{n}{r}}$$

$$=\left(n \cdot \prod_{k=0}^{r-1}(n-k)^{2^k}\right)^{\binom{n}{r}} = \left(n^{2^{r-1}} \cdot (n-r+1)^{2^{r-1}} \cdot \prod_{k=0}^{r-2}\left(1-\frac{k}{n}\right)^{2^k}\right)^{\binom{n}{r}}. \tag{17}$$

Put $r = 2n/3$, the Sperner rank, then the equation (17) becomes as follows:

$$(17) < \left(n^{2^{r-1}} \cdot \left(\frac{n}{3}\right)^{2^{r-1}} \cdot \left(\frac{1}{3}\right)^{2^{r-2}}\right)^{\binom{n}{r}} = \left(\frac{\sqrt[4]{3}}{3}n\right)^{2^r\binom{n}{r}}. \tag{18}$$

From the above result, using the following Stirling's approximation [15] for factorials:

$$2^r\binom{n}{r} = \frac{3}{2\sqrt{\pi}} \cdot \frac{3^n}{\sqrt{n}} \cdot \left(1+O\left(\frac{1}{n}\right)\right), \tag{19}$$

the new upper bound of $|DF(\mathbf{1})|$ is rewritten as follows:

$$|DF(\mathbf{1})| < 2^{\alpha}, \tag{20}$$

$$\text{where } \alpha = 3^n \cdot \frac{3}{2\sqrt{n\pi}}\left(1+\frac{c}{n}\right)\cdot \log_2 \frac{\sqrt[4]{3}n}{3}.$$

5. Lower bound

We first introduce the lower bound described in [10]. For any $0 < k < n$, in two adjacent ranks k and $k-1$ in a poset $\mathbf{V}_{\bar{n}}$, each element in V_k is covered by k elements in V_{k-1} (see Figure 6). So, s elements in V_k are covered by at most ks elements in V_{k-1}. Since the remaining $|V_{k-1}|-ks$ elements in V_{k-1} do not cover the s elements in V_k, any subset of the remaining elements in V_{k-1} forms an antichain. Therefore, in the case of choosing s elements (where, $0 \le s \le |V_k|$) in V_k, the number of antichain obtained from the remaining elements in V_{k-1} is at least as follows, and this gives the lower bound of $|DF(\mathbf{1})|$.

$$\sum_s \binom{|V_k|}{s} 2^{|V_{k-1}|-ks} = 2^{|V_{k-1}|}\cdot \sum_s \binom{|V_k|}{s} 2^{-ks} \tag{21}$$

By applying the binomial theorem [15], the equation (21) becomes as follows:

$$(22) = 2^{|V_{k-1}|}\cdot \left(1+2^{-k}\right)^{|V_k|} = 2^{|V_{k-1}|}\cdot e_u^{\binom{n}{k}} \tag{22}$$

$$\text{where, } e_u = \left(1+2^{-k}\right)^{2^k}.$$

In equation (22), take k as r, where $r = 2n/3$ is the Sperner rank as described in the previous section. If $n \to \infty$, then

$$e_u \to e \text{ (the base of natural logarithm) and } \binom{n}{r-1} \to 2\cdot\binom{n}{r}.$$

So

$$|V_{r-1}| \approx |V_r| \tag{23}$$

holds. Consequently, we obtain the lower bound of $|DF(\mathbf{1})|$ in [10] as follows:

$$|DF(\mathbf{1})| > 2^{|V_r|}\cdot e^{\binom{n}{r}} \left(= 2^{2^r\binom{n}{r}}\cdot e^{\binom{n}{r}} \right) \tag{24}$$

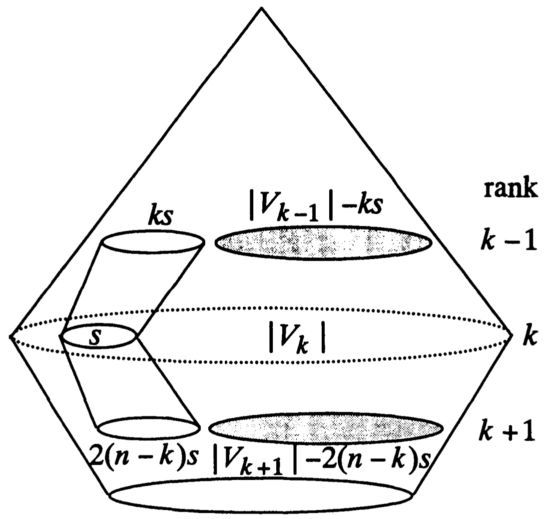

Figure 6. Counting the antichains in two adjacent layers.

This argument follows in a similar way one used in [9] for counting antichains in two adjacent ranks between k and $k-1$ in a poset $\mathbf{V}_{\bar{n}}$ for estimating the lower bound of monotone logic functions with n-variables.

Here, we can obtain the same lower bound as the equation (24) by counting antichains in two adjacent ranks k and $k+1$ in a poset $\mathbf{V}_{\bar{n}}$ (see Figure 6). That is, each element in V_k covers $2(n-k)$ elements in V_{k+1}. So, s elements in V_k cover at most $2(n-k)s$ elements in V_{k+1}. Since the remaining $|V_{k+1}|-2(n-k)s$ elements in V_{k+1} are not covered by s elements in V_k, any subset of the remaining elements in V_{k+1} forms an antichain. Therefore, in the case of choosing s elements ($0 \le s \le |V_k|$) in V_k, the number of antichains obtained from the remaining elements in V_{k+1} is at least as follows, and gives the lower bound of $|DF(1)|$.

$$\sum_s \binom{|V_k|}{s} 2^{|V_{k+1}|-2(n-k)s} = 2^{|V_{k+1}|} \cdot \sum_s \binom{|V_k|}{s} 2^{-2(n-k)s} \tag{25}$$

By applying the binomial theorem [15], the equation (25) is as follows.

$$(25) = 2^{|V_{k+1}|} \cdot \left(1 + 2^{-2(n-k)}\right)^{|V_k|} = 2^{|V_{k+1}|} \cdot e_d^{2^{3k-2n}\binom{n}{k}}, \tag{26}$$

where $e_d = \left(1+2^{-2(n-k)}\right)^{2^{2(n-k)}}$.

In the equation (26), take k as r, where $r = 2n/3$, Sperner rank. Then $2^{3r-2n} = 1$ holds. If $n \to \infty$, then

$$e_d \to e \text{ (the base of natural logarithm) and } \binom{n}{r-1} \to 2 \cdot \binom{n}{r}.$$

So

$$|V_{r+1}| \sim |V_r| \tag{27}$$

holds. Consequently, we obtain the same lower bound as the equation (24), as follows:

$$|DF(\mathbf{1})| > 2^{|V_r|} \cdot e^{\binom{n}{r}} \tag{28}$$

In two equations (24) and (28), antichains we have counted are independent each other (because, the former is derived from the relation between V_r and V_{r-1}, and the latter from one between V_r and V_{r+1}). By adding both results, we can improve the lower bound only slightly as follows:

$$|DF(\mathbf{1})| > 2 \cdot 2^{|V_r|} \cdot e^{\binom{n}{r}}. \tag{29}$$

Now, we proceed to improve the lower bound by counting antichains in three adjacent-ranked posets (V_{r-1}, V_r and V_{r+1}) of $\mathbf{V}_{\overline{n}}$, where we take $r = 2n/3$, the Sperner rank (see Figure 7).

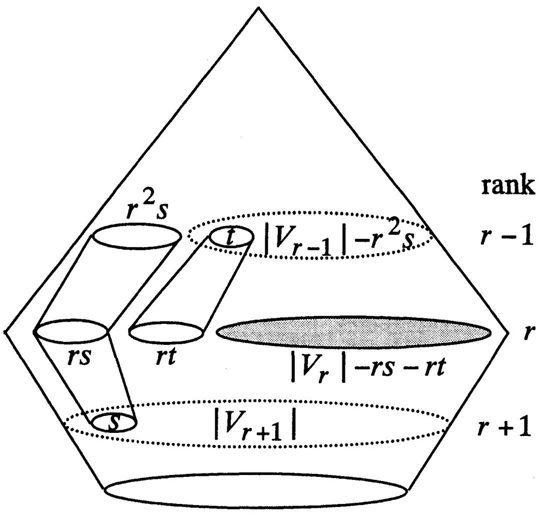

Figure 7. Counting the antichains in the three adjacent layers, where $r = 2n/3$, the Sparner rank.

Here, to simplify calculation, we regard that an element in one layer covers (or, is covered by) r elements in the other layer between two adjacent ranked-layers. That is, s elements in V_{r+1} are covered by at most rs elements in V_r, and are covered by at most $r^2 s$ elements in V_{r-1} (similarly, t elements in V_{r-1} cover at most rt elements in V_r, and cover at most $r^2 t$ elements in V_{r+1}).

In Figure 7, we consider the case that, firstly s elements are chosen in V_{r+1}, and secondly t elements are chosen from the elements which have remained after removing the covering $r^2 s$ elements in V_{r-1} (since the number of remaining elements is at most $|V_{r-1}|-r^2 s$ elements). Then, as for inclusion relations, the remaining elements (namely, $|V_r|-rs-rt$ elements) in V_r are included neither in the shadow of the s elements in V_{r+1} nor in the anti-shadow of the t elements in V_{r-1}, respectively. So, any subset of the remaining elements in V_r forms an antichain. Therefore, in the case of choosing s elements ($0 \le s \le |V_{r+1}|$) in V_{r+1} and choosing t elements ($0 \le t \le |V_{r-1}|-r^2 s$) in V_{r-1}, the number of antichains obtained from the remaining elements in V_r can be at least as follows, and this gives a new lower bound of $|DF(1)|$:

$$\sum_s \binom{|V_{r+1}|}{s} \sum_t \binom{|V_{r-1}|-r^2 s}{t} 2^{|V_r|-rs-rt}$$

$$= 2^{|V_r|} \cdot \sum_s \binom{|V_{r+1}|}{s} 2^{-rs} \sum_t \binom{|V_{r-1}|-r^2 s}{t} 2^{-rt} . \tag{30}$$

Here, in the above equation, by applying the binomial theorem we have:

$$\sum_t \binom{|V_{r-1}|-r^2 s}{t} 2^{-rt} = \left(1+2^{-r}\right)^{|V_{r-1}|-r^2 s} = e_1^{\frac{|V_{r-1}|}{2^r}} \cdot e_1^{-\frac{r^2 s}{2^r}} , \tag{31}$$

where $e_1 = \left(1+2^{-r}\right)^{2^r}$.

So the equation (30) becomes as follows:

$$2^{|V_r|} \cdot e_1^{\frac{|V_{r-1}|}{2^r}} \cdot \sum_s \binom{|V_{r+1}|}{s} 2^{-r\left(1+\frac{r}{2^r}\log_2 e_1\right)s} . \tag{32}$$

Furthermore, in the above equation (32), put $A = 1 + \frac{r}{2^r}\log_2 e_1$. By repeated application of the binomial theorem we have:

$$\sum_s \binom{|V_{r+1}|}{s} 2^{-rAs} = \left(1 + 2^{-rA}\right)^{|V_{r+1}|} = e_2^{\frac{|V_{r+1}|}{2^{rA}}}, \tag{33}$$

where $e_2 = \left(1 + 2^{-rA}\right)^{2^{rA}}$.

So the equation (32) becomes as follows:

$$2^{|V_r|} \cdot e_1^{\frac{|V_{r-1}|}{2^r}} \cdot e_2^{\frac{|V_{r+1}|}{2^{rA}}}. \tag{34}$$

In equation (34), if $n \to \infty$, then $e_1 \to e$ and $e_2 \to e$. Then, $|V_{r-1}| \sim |V_r|$ and $|V_{r+1}| \sim |V_r|$ hold. Consequently, the new lower bound of $|DF(\mathbf{1})|$ we obtain as follows:

$$|DF(\mathbf{1})| > 2^{|V_r|} \cdot e^{\binom{n}{r}} \cdot e^{\binom{n}{r} \cdot e^{-r^2 2^{-r}}} = 2^{|V_r|} \cdot e^{\binom{n}{r} \cdot \left(1 + e^{-r^2 2^{-r}}\right)} \tag{35}$$

Combining (14), (18), (35) we can obtain bounds for the number of disjunctive forms for all n-variable logic functions.

6. Conclusions

In this paper we have presented new upper and lower bounds on the number of disjunctive forms of an n-variable binary logic function (we took the constant function $\mathbf{1}$ for our purpose). We have followed the methods described in [10] (they originate from [8,9]) incorporating new ideas. It is interesting to note that the lower bound by counting antichains contained in the adjacent three layers (the Sperner rank, $r = 2n/3$ at the center) can be simplified as shown herein. If we apply Gilbert's method [8] and Shapiro's method [9] to upper and lower bound, respectively, for the poset $\mathbf{V}_n^-$ satisfying Sperner's lemma, we would not expect to obtain an essential improvement over the upper and lower bounds obtained here.

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