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EXPLODING EIGENVALUES INVOLVING THE p-LAPLACIAN

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ABSTRACT. A review is given of recent work on eigenvalue problems involving

$$-\Delta_p u = (p-1)(\lambda r - q)|u|^{p-2}u$$

on a bounded subset Ω of $\mathbb{R}^N$, where $p>1$ and Δ_p is the p-Laplacian, from the viewpoint of two questions. One is whether eigenvalues can explode, i.e., generate arbitrarily large numbers of nearby eigenvalues under perturbation. The other is whether non-variational eigenvalues can exist.

It is shown that these two questions are related, and can be answered positively with small potential q and weight $r=1$, or with no potential and weight r close to one.

1. INTRODUCTION

We shall review recent work with Bryan Rynne on the equation

$$-\Delta_p u = (p-1)(\lambda r - q)E_p u \tag{1.1}$$

on a bounded subset Ω of $\mathbb{R}^N$, where $p>1$, $N\geq 1$, $\lambda\in\mathbb{R}$ and $q, r\in L_1(\Omega)$. The operator E_p satisfies

$$E_p u = |u|^{p-2}u,$$

where $|u|$ is the Euclidean norm of u, and Δ_p is the p-Laplacian operator, satisfying

$$\Delta_p u = \mathrm{div}(E_p \mathrm{grad} u).$$

The p-Laplacian operator has been associated with thousands of publications in the last few decades, and its popularity has much to do with applications in science and engineering - see, e.g., [11]. For example, fluid flow has been investigated with various velocity dependent viscosity laws. A notable one is the Ostwald-de Waele power law, leading to a classification of fluids into (i) pseudoplastic or shear thinning ($p<2$), (ii) Newtonian ($p=2$), and (iii) dilatant or shear thickening ($p>2$) types. Examples of the first category are blood plasma, latex paint and snow, while quicksand and automobile viscous coupling fluid belong to the third category.

It could be argued that theoretical work on the p-Laplacian operator dates back a long way (to equations involving power laws) but the case $N=1$, where $E_p u = |u|^{p-1}\mathrm{sgn}\, u$, shows that $\Delta_p u$ depends on $\mathrm{sgn}\, u'$ as well as a power of u'. Already in 1961, Beesack [2] examined equations with this effect in connection with an inequality of Hardy. More conventional formulations of $\Delta_p u$ were investigated by Dubinskii and Pohožaev, and also by Nečas, in the late 1960s, and by 1980 several methods of attack were in use, for example Elbert's modified Prüfer method for a (nonlinear Sturm-Liouville) case with $N=1$ and separated boundary conditions. In 1988, Guedda and Veron [18] showed that for certain equations of the form (1.1) under perturbations of a certain type, the (simple) eigenvalues were bifurcation points analogous to those of the linear case $p = 2$, and many publications have ensued on bifurcation theory.

For such eigenvalues, perturbations by terms of the form aE_pu (for example perturbations of the coefficients q, r) lead to nearby simple eigenvalues. The question of whether such perturbations can lead to more complicated behaviour is then of interest, and this is studied in Sections 2 and 3. It is shown that (nonsimple) eigenvalues can exist (even for $N = 1$) which "explode" under small perturbations of the coefficients into arbitrarily large numbers of nearby eigenvalues. This disproves a conjecture of Zhang [26]. The methods involve a detailed analysis of the inverse of Δ_p under periodic and antiperiodic boundary conditions, together with slightly nonstandard versions of tools used for bifurcation theory such as Lyapunov-Schmidt reduction, implicit function and degree theories.

Most of the early work on the p-Laplacian had a variational component. For example, Beesack used the classical calculus of variations, and Nečas and colleagues [15] employed Lyusternik-Šnirelman theory, which generalises the minimax principle from the case $p = 2$. A long-standing open question in the area is whether Lyusternik-Šnirelman theory generates all the eigenvalues, or, to put it another way, whether non-variational eigenvalues can exist. In Section 4 we shall show how to connect this question with that of explosion under perturbation, and we give examples with a positive answer (for each $N \geq 1$) for small potential q and weight $r = 1$, and also for no potential and weight r close to one. We conclude with some extensions and questions left open by our analysis.

2. Preliminaries for the case $N = 1$

2.1. **General concepts and notation.** Differentiability will be a key issue in our analysis and we start with our notations for derivatives. If f is a function between Banach spaces then $Df(u)$ denotes the Fréchet derivative of f at u. Partial derivatives will be indicated by subscripts, e.g., $D_ug(u,v)$, $D_vg(u,v)$ are the partial derivatives of a two argument function g. The special cases D_x and D_t will be denoted by the customary prime and dot.

The underlying Banach spaces that we will need are as follows. For $j = 0, 1$, we let $C^j[0, \pi_p]$ denote the space of j times continuously differentiable functions on $[0, \pi_p]$, with the usual sup-norm $|\cdot|_j$ (throughout, all function spaces will be real). $L^1(0, \pi_p)$, with norm denoted by $\|\cdot\|_1$, will be the usual space of integrable functions on $[0, \pi_p]$, and $W^{1,1}(0, \pi_p)$, with norm denoted by $\|\cdot\|_{1,1}$, will be the usual Sobolev space of absolutely continuous (AC) functions u on $[0, \pi_p]$, with derivative $u' \in L^1(0, \pi_p)$. It turns out that the ranges $p < 2$ and $p > 2$ will require different analysis in later sections, but a degree of unification will be achieved by writing

$$B_p := \begin{cases} C^1[0, \pi_p], & 1 < p \leq 2, \\ W^{1,1}(0, \pi_p) & p > 2. \end{cases} \tag{2.1}$$

We turn now to notation for (1.1). We start with the signed power function in the form $[x]^\alpha := |x|^\alpha \mathrm{sgn}\, x$, for α, $x \in \mathbb{R}$. We first note that this function satisfies the simple identities $[x]^\alpha = x|x|^{\alpha-1}$ and $[[x]^\alpha]^\beta = [x]^{\alpha\beta}$, for $\alpha, \beta > 0$, $x \in \mathbb{R}$, and, for a differentiable function f, $([f]^\alpha)'(x) = \alpha|f(x)|^{\alpha-1}f'(x)$, when $f(x) \neq 0$. Now (1.1) can be written in the form

$$-\left([u']^{p-1}\right)' = (p-1)(\lambda r - q)[u]^{p-1}, \quad \text{on } (0, \pi_p). \tag{2.2}$$

The above notation clarifies the various detailed power estimates underlying our perturbation analysis. In particular, periodic boundary conditions

$$u(0) = u(\pi_p) \quad \text{and} \quad u'(0) = u'(\pi_p) \tag{2.3}$$

make sense for (1.1).

In the operator notation used at the outset (which indicates powers more appropriate for variational analysis),

$$E_p : x \mapsto [x]^{p-1}, \quad \Delta_p : u \mapsto (E_p(u'))'.$$

In general, we will simplify our notation by keeping the same symbols for operators and their restrictions. For example, the operator of differentiation (denoted by D as above) can map AC to L^1, C^1 to C^0, etc. Similarly for the operator $\mathcal{I}$ of integration in Section 2.3, Δ_p and its inverse, and so on.

2.2. **The constant coefficient case.** The constant coefficient case will play an essential part in our analysis, both as an unperturbed state, and to provide the definition of certain generalised sine functions which will be used frequently. When the coefficients are constant, we may translate the eigenparameter so as to ensure that $q = 0$. Then (2.2) takes the form

$$-([u']^{p-1})' = (p-1)\lambda[u]^{p-1}. \tag{2.4}$$

We denote the (unique) maximal solution of the initial value problem for (2.4) with $\lambda = 1$, $u(0) = 0$, $u'(0) = 1$, by $\sin_p$. A construction of this function is described in [14] and shows that $\sin_p$ is a C^1 function on $\mathbb{R}$, and is $2\pi_p$-periodic, where $\pi_p := 2(\pi/p)/\sin(\pi/p)$. Moreover

$$\sin_p(x + \pi_p) = -\sin_p(x), \quad x \in \mathbb{R}, \tag{2.5}$$

$$|\sin_p|^p + |\sin_p'|^p \equiv 1. \tag{2.6}$$

and $\sin_p(m\pi_p) = 0$, $\sin_p'((m + \frac{1}{2})\pi_p) = 0$, $m \in \mathbb{Z}$. Thus the graph of $\sin_p$ resembles a sine wave, and indeed, $\sin_2$ reduces to the usual sin function, and $\pi_2 = \pi$.

Remark 2.1. The notation $\sin_p$ (and π_p) has also been used for different functions (and their zeros) in several works. See [5] for further details.

To determine the periodic eigenvalues and eigenfunctions of (1.1), we introduce the functions $e_k(t) \in B_p$, for integer $k \geq 0$ and $t \in \mathbb{R}$, defined by

$$e_0(t)(x) = 1, \quad e_k(t)(x) = \sin_p(2k(x+t)), \quad x \in [0, \pi_p]. \tag{2.7}$$

It is clear that the mappings $t \to e_k(t) : \mathbb{R} \to B_p$ are π_p-periodic.

Lemma 2.2. *For $q = 0$ and $k \geq 0$, the kth periodic eigenvalue λ_k^0 equals $(2k)^p$, with corresponding eigenfunctions $e_k(t)$, $t \in \mathbb{R}$. There are no other periodic eigenvalues, and (up to scaling) no other eigenfunctions. Each eigenfunction has a finite number of zeros, all simple, in $[0, 2\pi_p)$.*

This is a straightforward calculation (cf. [20, pp. 442-3], where other boundary conditions are also considered). We remark that the eigenvalues in Lemma 2.2 are to be understood in our standing sense of classical solutions, and are numbered without attempting to count any "multiplicity".

Lemma 2.2 also shows that for any $k \geq 1$, the eigenvalue λ_k is not simple. Let us consider the mapping $e_k : t \to e_k(t) : \mathbb{R} \to B_p$ in more detail. It will be shown in Lemma 2.3 that this mapping is C^1, and by periodicity, $e_k(t)$ parametrizes a non-trivial closed loop of eigenfunctions in B_p. Also, denoting the set of all eigenfunctions corresponding to λ_k by E_k, we see from the homogeneity of the problem that E_k is parametrised by the mapping $(s,t) \to se_k(t) : \mathbb{R}\setminus\{0\}\times\mathbb{R} \to B_p$. Thus E_k is a two-dimensional, C^1 manifold in B_p, and the tangent space of E_k at the point $e_k(t)$ has a basis given by $e_k(t)$ and the t derivative $\dot{e}_k(t)$. This tangent space will play an important rôle for us as the nullspace of an appropriate linearisation of (1.1), (2.3).

2.3. **Domains, ranges and differentiability.** When we need to be specific about periodic boundary conditions, we will denote the periodic p-Laplacian, with (maximal) domain consisting of u such that

$$u,\ E_p(u') \text{ are } AC \text{ and satisfy (2.3)}, \tag{2.8}$$

by Δ_{pp}. As indicated earlier, we will also use Δ_{pp} to denote restrictions as needed. We consider the problem

$$\Delta_{pp}u = h, \quad h \in L^1(0,\pi_p). \tag{2.9}$$

Since we allow $h \in L^1(0,\pi_p)$ in (2.9), this equation is taken to hold a.e. on $(0,\pi_p)$, in the Carathéodory sense.

We next define

$$Mu(x) := \frac{1}{\pi_p}\int_0^{\pi_p} u, \quad u \in L^1(0,\pi_p),\ x \in [0,\pi_p],$$

so M maps $L^1(0,\pi_p)$ to constant functions. By integrating (2.9) over $[0,\pi_p]$ and using (2.3) we obtain $Mh = 0$, so

$$M\Delta_{pp}u = 0, \tag{2.10}$$

for all u in the domain of Δ_{pp}. In view of this we define

$$E := \{v \in L^1(0,\pi_p) : Mv = 0\}, \quad E^j := E \cap C^j[0,\pi_p],\ j = 0,1, \tag{2.11}$$

and so $R(\Delta_{pp}) \subset E$.

We continue with some additional properties of the functions e_k, $k \geq 1$, defined in (2.7).

Lemma 2.3. *For any $p > 1$ $(p \neq 2)$ and $k \geq 1$, the mapping $e_k : \mathbb{R} \to B_p$ is C^1. For any $t \in \mathbb{R}$,*

$$e_k(t) = -\Delta_{pp}^{-1}(\lambda_k[e_k(t)]^{p-1}) \tag{2.12}$$

and

$$M(e_k(t)) = M([e_k(t)]^{p-1}) = M(\dot{e}_k(t)) = M(|e_k(t)|^{p-2}\dot{e}_k(t)) = 0. \tag{2.13}$$

The proofs of this and the remaining results in this section (some of which are quite technical) can be found in [4].

We note that M and $I-M$ are projections on $L^1(0,\pi_p)$, and are $\langle\cdot,\cdot\rangle$-symmetric, in the sense that

$$\langle Mu_1, u_2\rangle = (\pi_p)^{-1}\int_0^{\pi_p} u_1 \int_0^{\pi_p} u_2 = \langle u_1, Mu_2\rangle, \quad u_1, u_2 \in L^1(0,\pi_p). \tag{2.14}$$

Moreover Δ_{pp} commutes with M and with $I-M$ — these are separate statements since Δ_{pp} is nonlinear. More precisely, we have the following

Lemma 2.4. *M is C^1 from $L^1(0,\pi_p)$ to $C^1[0,\pi_p]$, and for any u in the domain of Δ_{pp} (given by (2.8)),*

$$M\Delta_{pp}u = \Delta_{pp}Mu = 0, \quad (I-M)\Delta_{pp}u = \Delta_{pp}(I-M)u. \tag{2.15}$$

In particular, Δ_{pp}^{-1} commutes with M and with $I-M$ on $R(\Delta_{pp}) = E = R(I-M)$.

Combining these results with more complicated ones on domains, ranges and differentiability of Δ_{pp}^{-1} for different ranges of p, we have the following conclusion, which will be needed in the next section.

Theorem 2.5. *The operator $\Phi_p(u) := \Delta_{pp}^{-1} \circ (I-M) \circ E_p$ maps $C^1[0,\pi_p]$ to B_p if $1<p<2$ (resp. $C^0[0,\pi_p]$ to B_p if $p>2$), and is C^1 on a neighbourhood of $e_k(t)$, $t \in \mathbb{R}$. In each case, the derivative $D\Phi_p(u)$ is compact on the specified spaces.*

3. Exploding eigenvalues for $N = 1$

First we recall λ_k^0 from Lemma 2.2. The main result of this section is

Theorem 3.1. *Suppose that $N = 1, p > 1, p \neq 2$ and $r = 1$. For any integers k, $n \geq 1$ and any $\epsilon > 0$, there exists $q = q_{k,n} \in C^1[0,\pi_p]$ with norm $< \epsilon$ such that there are at least n periodic eigenvalues of (2.2) in $(\lambda_k^0 - \epsilon, \lambda_k^0 + \epsilon) \cap \sigma_{2k}$.*

The proof is rather involved, but we shall give some of the ideas. Full details can be found in [4].

To construct a suitable $q_{k,n}$ we consider the equation

$$-\Delta_{pp}(u) + \epsilon q \phi_p(u) = (\lambda_k^0 + \epsilon\mu)E_p(u), \tag{3.1}$$

where $q \in C^1[0,\pi_p]$ and $\epsilon \in \mathbb{R}$. By Lemma 2.3, when $\epsilon = 0$, the mapping $t \to e_k(t)$ gives a closed, C^1 curve of solutions of (3.1) in B_p. We will find $q \in C^1[0,\pi_p]$ such that solutions "bifurcate" from this curve when $\epsilon \neq 0$.

From now on we simplify our notation by suppressing the subscripts from λ_k^0 and e_k.

We first reformulate (3.1) as a functional equation. Defining

$$f(\mu,u,\epsilon) := (\epsilon(q-\mu) - \lambda^0)E_p(u),$$

for $(\mu,u,\epsilon) \in \mathbb{R} \times B_p \times \mathbb{R}$, we can rewrite (3.1) as

$$\Delta_{pp}u = f(\mu,u,\epsilon). \tag{3.2}$$

Now define $F : \mathbb{R} \times B_p \times \mathbb{R} \to B_p$ by

$$F(\mu,u,\epsilon) := u - \Delta_{pp}^{-1}(I-M)f(\mu,u,\epsilon) - M(u + f(\mu,u,\epsilon)). \tag{3.3}$$

Lemma 3.2. *Equation (3.1) is equivalent to the equation*

$$F(\mu,u,\epsilon) = 0. \tag{3.4}$$

Moreover

$$F(\mu,e(t),0) = 0, \quad (\mu,t) \in \mathbb{R}^2. \tag{3.5}$$

3.1. **Linearisation and projection.** It can be shown that

$$L(t) := D_v F(\mu, e(t), 0) : B_p \to B_p,$$

and the mapping $t \to L(t)$ is C^0 on $\mathbb{R}$. Moreover, there is an alternative characterization of the operator $L(t)$, more in keeping with the original operator Δ_p, as follows.

Lemma 3.3. *For any $t \in \mathbb{R}$ and $v \in B_p$, if $w = L(t)v$ then*

$$-(|e(t)'|^{p-2}(v - w)')' = \lambda(I - M)(|e(t)|^{p-2}v). \tag{3.6}$$

The operator $L(t)$ is not one-to-one. In fact we have the following result.

Lemma 3.4. *For each $t \in \mathbb{R}$,*

$$N(L(t)) = \operatorname{span}\{e(t), \dot{e}(t)\}, \tag{3.7}$$

and $R(L(t))$ is closed, with $\operatorname{codim} R(L(t)) = 2$.

The operator $L(t)$ is not $\langle \cdot , \cdot \rangle$-symmetric, but by introducing some new inner products we can define a type of orthogonal projection onto $N(L)$. For each $t \in \mathbb{R}$ let

$$\langle v_1, v_2 \rangle_t := \langle v_1, v_2 |e(t)|^{p-2} \rangle, \quad v_1, v_2 \in B_p.$$

Now, for any $t \in \mathbb{R}$ we define $P(t) : B_p \to N(L(t))$ by

$$P(t)v := \frac{\langle v, e(t)\rangle_t}{\langle e(t), e(t)\rangle_t} e(t) + \frac{\langle v, \dot{e}(t)\rangle_t}{\langle \dot{e}(t), \dot{e}(t)\rangle_t} \dot{e}(t), \quad v \in B_p, \tag{3.8}$$

and we let $Q(t) := I - P(t)$. By the above results, the operator functions P, Q are C^0 on $\mathbb{R}$.

Lemma 3.5. *For each $t \in \mathbb{R}$,*

$$\langle e(t), \dot{e}(t)\rangle_t = 0, \tag{3.9}$$

and hence $P(t)$, $Q(t)$ are $\langle \cdot , \cdot \rangle_t$-symmetric projections from B_p to $N(L(t))$ and $R(L(t))$, respectively. Moreover

$$Q(t)e(t) = 0, \quad Q(t)\dot{e}(t) = 0, \quad P(t)L(t) = 0. \tag{3.10}$$

3.2. **A bifurcation equation.** We now use the projections P, Q to reformulate (3.4) as a bifurcation-type equation on the null-spaces $N(L(t))$, $t \in \mathbb{R}$.

We look for solutions (μ, u, ϵ) of (3.4) near to $(\mu_0, e(t_0), 0)$, with u having the form $u = e(t) + w$, where $w \in W_0$ is small. Equation (3.4) is equivalent to the pair of equations

$$Q(t)F(\mu, e(t) + w, \epsilon) = 0, \tag{3.11}$$

$$P(t)F(\mu, e(t) + w, \epsilon) = 0, \tag{3.12}$$

and it is clear by (3.5) that $(w, \epsilon) = (0,0)$ satisfies (3.11)-(3.12) for all $(\mu, t) \in \mathbb{R}^2$. The function F is C^1 (when w, ϵ are small), but P, Q are only C^0, so the functions on the left hand sides of (3.11) and (3.12) are C^1 with respect to (μ, w, ϵ) and C^0 with respect to t. Also, denoting the left hand side of (3.11) by $F_Q(\mu, t, w, \epsilon)$, we see from (3.5) that

$$F_Q(\mu, t, 0, 0) \equiv 0, \quad D_w F_Q(\mu_0, t_0, 0, 0)\bar{w} = L(t_0)\bar{w}, \quad \bar{w} \in W_0.$$

By construction and Lemma 3.5, the mapping $L(t_0) : W_0 \to W_0$ is linear and bijective, so is non-singular. By slightly nonstandard implicit function theory, equation

(3.11) has a solution $w(\mu, t, \epsilon)$, which is defined and continuous on a neighbourhood of $(\mu_0, t_0, 0)$, the derivative $D_{(\mu,\epsilon)}w(\mu, t, \epsilon)$ exists and is continuous on this neighbourhood, and

$$w(\mu, t, 0) \equiv 0. \tag{3.13}$$

Substituting the solution w into (3.12), we see that (3.1) is locally equivalent to the equation

$$F_P(\mu, t, \epsilon) := P(t)F(\mu, e(t) + w(\mu, t, \epsilon), \epsilon) = 0.$$

By developing the apppropriate smoothness properties of these constructions, we are led to the following bifurcation-type equation in the two parameters w, μ for each small enough ϵ.

Lemma 3.6. *For $\epsilon \neq 0$, equation* (3.1) *is locally equivalent to the equation*

$$H(\mu, t, \epsilon) := \begin{pmatrix} \langle G(\mu, t, \epsilon), e(t)\rangle_t \\ \langle G(\mu, t, \epsilon), \dot{e}(t)\rangle_t \end{pmatrix} = 0 \tag{3.14}$$

where

$$G(\mu, t, \epsilon) := \begin{cases} \epsilon^{-1}\lambda(p-1)F_P(\mu, t, \epsilon), & \epsilon \neq 0, \\ P(t)\big((I - M)(q - \mu)e(t)\big), & \epsilon = 0. \end{cases}$$

In order to analyse (3.14), we introduce the function J given by

$$J(t, q) := \int_0^{\pi_p} q|e(t)|^p\, dx, \quad t \in \mathbb{R}. \tag{3.15}$$

Although later the q dependence of $J(t, q)$ will be important, for now we regard $q \in C^1[0, \pi_p]$ as fixed and we simply write $J(t)$.

If $\dot{J}(t) = 0$ then t is a *critical point* of J, with *critical value* $J(t)$; a critical point t is *non-degenerate* if $\ddot{J}(t) \neq 0$. Using

$$H(\mu, t, 0) = \begin{pmatrix} J(t) - \mu\gamma \\ \dot{J}(t)/p \end{pmatrix} = 0 \tag{3.16}$$

where

$$\gamma = \int_0^{\pi_p} |e(t)|^p\, dx, \tag{3.17}$$

and

$$D_{(\mu,t)}H(\mu, t, 0) = \begin{pmatrix} \dot{J}(t) & -\gamma \\ \ddot{J}(t)/p & 0 \end{pmatrix}, \tag{3.18}$$

we can use arguments based on the implicit function theorem and degree theeory to establish existence of solutions to (3.1) as follows.

Theorem 3.7. *Suppose that t_0 is a non-degenerate critical point of J. Then there is an $\epsilon_0 > 0$ such that if $|\epsilon| < \epsilon_0$ then* (3.1) *has an eigenvalue $\lambda(\epsilon) \in \sigma_k(\epsilon q)$ of the form $\lambda(\epsilon) = \lambda + \epsilon\mu(\epsilon)$, where $\mu(\epsilon) \to J(t_0)/\gamma$ as $\epsilon \to 0$, where γ satifies* (3.17).

3.3. **Multiplicities of higher eigenvalues.** Fix $k \geq 1$ and $p \neq 2$, and let $E_k^0 \subset W_P^{1,1}$ denote the set of eigenfunctions corresponding to the periodic, constant coefficient eigenvalue λ_k^0. As noted earlier, the elements of E_k^0 are C^1, but it is well known that they lack some higher derivatives. The following result will suffice for our purposes. Let $O_p = \mathbb{R} \setminus \{j\pi_p/2 : j \in \mathbb{Z}\}$.

Lemma 3.8. *The function* $\sin_p$ *is analytic on O_p. If $p < 2$ (respectively $p > 2$) then* $\sin_p$ *is not C^3 at* 0 *(respectively at $\pi_p/2$).*

Proof. The analyticity of $\sin_p$ on O_p follows from the analyticity of the system (4.4) except where $u = 0$ or $u' = 0$ (see [8, Theorem 8.1, Ch. 1], recalling that $q = 0$, $r = 1$). Restricting our attention to $(0, \pi_p/2)$, where $\sin_p$ and $\sin_p' > 0$, we see from (2.4) that

$$\begin{aligned}\sin_p'' &= -(\sin_p)^{p-1}(\sin_p')^{2-p},\\ \sin_p''' &= -(p-1)(\sin_p)^{p-2}(\sin_p')^{3-p} - (p-2)(\sin_p)^{2p-2}(\sin_p')^{3-2p}.\end{aligned}$$

The proof now follows from $\sin_p(0) = 0 = \sin_p'(\pi_p/2)$ and (2.6). □

We now use this result to show that the (linear) dimension of E_k^0 is infinite.

Proposition 3.9. *For $k \geq 1$, the (linear) span of E_k^0 has infinite dimension.*

Proof. Choose an arbitrary integer $m \geq 1$, and let $\psi_j = e_{2k}(\frac{j\pi_p}{8m})$, $j = 1, \ldots, m$. By Lemma 3.8, ψ_j is analytic on $\mathbb{R}$, except for a discrete set of points Ψ_j. Since $\Psi_i \cap \Psi_j = \emptyset$, if $i \neq j$, the set of functions $\{\psi_j : j = 1, \ldots, m\}$ is linearly independent on $\mathbb{R}$. Since these functions are anti-symmetric and $2\pi_p$-periodic, they are also linearly independent on the interval $[0, \pi_p]$. Hence, $\dim(\mathrm{span} E_k^0) \geq m$, and since m was arbitrary this completes the proof. □

Our final lemma shows that we can choose a function q in Theorem 3.7 for which the corresponding functional $J(\cdot, q)$ has sufficiently many non-degenerate critical points. A proof, which depends on Lemma 3.8, Proposition 3.9 and a genericity argument, can be found in [4].

Lemma 3.10. *For each k, $n \geq 1$, there exists a function $q_{k,n} \in C^1[0, \pi_p]$, such that the functional $J(\cdot, q_{k,n})$ has at least n non-degenerate critical points in $(0, \pi_p)$, with distinct critical values, and no degenerate critical points.*

We can now substitute $q = q_{k,n}$ from Lemma 3.10 into Theorem 3.7 to complete the proof of Theorem 3.1.

Let us make the following informal

Definition 3.11. The perturbation multiplicity of an eigenvalue λ of (1.1) is the supremum of the number of eigenvalues near λ which can be produced by small perturbations of q.

According to Theorem 3.1, the perturbation multiplicity of the constant coefficient, periodic eigenvalue λ_k^0 is infinite for $k \geq 1$, and one of the key ingredients for this result is the infinite dimension in Proposition 3.9.

4. Variational and non-variational eigenvalues for $N = 1$

In this section we consider the equation

$$-\left([u']^{p-1}\right)' = (\lambda r - q)[u]^{p-1}, \quad \text{a.e. on } (0, \pi_p), \tag{4.1}$$

mainly for periodic boundary conditions

$$u(0) = u(\pi_p), \tag{4.2}$$
$$u'(0) = u'(\pi_p). \tag{4.3}$$

4.1. **Carathéodory and variational eigenvalues.** We define λ to be a (Carathéodory) eigenvalue of (4.1)–(4.3) if the system

$$\begin{aligned} u' &= [v]^{1/(p-1)}, \\ v' &= -(\lambda r - q)[u]^{p-1}, \end{aligned} \tag{4.4}$$

equivalent to (4.1), admits a nonzero periodic solution in the sense of Carathéodory. In particular, u and $v = [u']^{p-1}$ must be absolutely continuous, so both sides of (4.1) are L^1 functions, and the boundary conditions make sense.

We now briefly sketch the Ljusternik-Šnirelman construction of the variational eigenvalues. Further details can be found in [17, Chapter 3] or [25]. Let

$$W_P^{1,1} := \{w \in W^{1,p}(0, \pi_p) : w(0) = w(\pi_p)\},$$

and let

$$G(u) := \int_0^{\pi_p} (|u'|^p + q|u|^p), \quad H(u) := \int_0^{\pi_p} r|u|^p, \quad u \in W_P^{1,1}. \tag{4.5}$$

We next recall a standard definition of Lyusternik-Šnirelmann theory. Setting

$$\mathcal{M} := \left\{u \in W_P^{1,1} : H(u) = 1\right\},$$

and

$$\mathcal{A} := \{A \subset \mathcal{M} : A \text{ is non-empty, compact and symmetric } (A = -A)\}, \tag{4.6}$$

we define the Krasnoselskij genus of $A \in \mathcal{A}$ by

$$\gamma(A) := \inf\{m \in \mathbb{N} : \exists \text{ a continuous, odd } f : A \to \mathbb{R}^m \setminus \{0\}\},$$

where $\gamma(A) = \infty$ if no such m exists. Now, for any integer $k \geq 0$, let

$$\mathcal{F}_k := \{A \in \mathcal{A} : \gamma(A) \geq k\},$$

and

$$\mu_k := \inf_{A \in \mathcal{F}_{k+1}} \sup_{u \in A} G(u). \tag{4.7}$$

It is clear from this definition that $\mu_{k+1} \geq \mu_k$ for all $k \geq 0$.

Theorem 4.1. *For each $k \geq 0$, μ_k is a (Carathéodory) eigenvalue of* (4.1)–(4.3).

Proof. Standard arguments (cf. [3, Section 5], [17, Chapter 3] or [25]) show that to each $\lambda = \mu_k$ there corresponds a nonzero $u = u_k \in W_P^{1,1}$ satisfying the weak form of (4.1)–(4.3), viz.,

$$\int_0^{\pi_p} \{[u']^{p-1} w' - (\lambda r - q)[u]^{p-1} w\} = 0, \quad \forall w \in W_P^{1,1}. \tag{4.8}$$

Writing

$$v(t) = \int_0^t (\lambda r - q)[u]^{p-1}, \quad t \in [0, \pi_p],$$

we see that v is absolutely continuous and $[u']^{p-1} = v$, and hence u satisfies (4.1) in the Carathéodory sense. Furthermore, u automatically satisfies (4.2), and (4.3) then follows from (4.8) in a standard way by appropriate choices of $w \in W_P^{1,1}$. □

In view of Theorem 4.1, we call μ_k the kth *variational* periodic eigenvalue of (4.1)–(4.3). The case $k = 0$ is somewhat special, so from now on, we restrict our attention to $k \geq 1$. We next consider the relationship between these eigenvalues and the variational periodic eigenvalues μ_k^0, constructed in (4.7).

Theorem 4.2. *All the eigenvalues* λ_k^0, $k \geq 1$, *are variational, with* $\mu_{2k-1}^0 = \mu_{2k}^0 = \lambda_k^0 = (2k)^p$, $k \geq 1$.

A proof can be found in [5].

4.2. **Non-variational eigenvalues.** In the constant coefficient case it is easily seen from the construction of the periodic eigenvalues and eigenfunctions in Lemma 2.2 that the corresponding set σ_{2k}^0 consists of the singleton $\{\lambda_k^0\}$. By contrast, in the general case we have the following result.

Theorem 4.3. *Suppose that* $p \neq 2$ *and* $r = 1$. *For any integers* $k, n \geq 1$ *and any* $\epsilon > 0$, *there exists* $q \in C^1[0, \pi_p]$ *with norm* $< \epsilon$ *such that there are at least* n *non-variational periodic eigenvalues of* (4.1) *in* $(\lambda_k^0 - \epsilon, \lambda_k^0 + \epsilon) \cap \sigma_{2k}$.

Proof. Choose $\epsilon_1 \in (0, \epsilon)$ such that $\lambda_{k-1}^0 < \lambda_k^0 - \epsilon_1$ and $\lambda_k^0 + \epsilon_1 < \lambda_{k+1}^0$. Then, by Theorem 3.1, there exist $\tilde{q} \in C^1$ and $\eta > 0$ with the following property: if $q = \alpha\tilde{q}$, with $|\alpha| < \eta$, then (4.1) has at least $n + 2$ distinct periodic eigenvalues in $(\lambda_k^0 - \epsilon_1, \lambda_k^0 + \epsilon_1) \cap \sigma_{2k}$ (so the constant coefficient eigenvalue λ_k^0, corresponding to $q = 0$, splits into at least $n + 2$ nearby distinct eigenvalues, when $q = \alpha\tilde{q}$).

For the remainder of the proof, we shall exhibit the dependence of the eigenvalues on q explicitly, so we label the variational periodic eigenvalues of (4.1) by $\mu_k(q)$. From the variational construction (4.7) we see that each $\mu_m(\alpha\tilde{q})$, $m \geq 1$, depends continuously on α. Hence, by Theorem 4.2, there exists $\zeta > 0$ such that, if $|\alpha| < \zeta$, then $\mu_{2k-2}(\alpha\tilde{q}) < \lambda_k^0 - \epsilon_1$ and $\lambda_k^0 + \epsilon_1 < \mu_{2k+1}(\alpha\tilde{q})$. It now suffices to take $q = \alpha\tilde{q}$ for $|\alpha| < \min\{\zeta, \eta, \epsilon/\|\tilde{q}\|\}$. □

It is natural to ask which of the Carathéodory eigenvalues of this problem are variational and which are not. We shall give an explicit answer to this question, in terms of the set σ_{2k}. As remarked above, in the constant coefficient case $\sigma_{2k}^0 = \{\lambda_k^0\}$, so by Theorem 4.2 this set is realised variationally. On the other hand, Theorem 4.3 shows that in general σ_{2k} may contain a large number of non-variational eigenvalues. The following theorem shows that σ_{2k} contains its minimal and maximal elements, and that these are precisely the variational eigenvalues in σ_{2k}.

Theorem 4.4. *Assume the conditions of Theorem 4.3. For any* $k \geq 1$, *the set* σ_{2k} *is non-empty and compact, and the periodic variational eigenvalues* μ_{2k-1} *and* μ_{2k} *are the minimal and maximal elements, respectively, in* σ_{2k}.

See [5] for a proof. We remark that the extremal elements of σ_k are periodic eigenvalues if k is even, and are antiperiodic eigenvalues if k is odd (see [7]).

To conclude this section, we note that each of unperturbed eigenvalues λ_k^0, $k \geq 1$, equals exactly two of the μ_j^0 in Theorem 4.2. Moreover it is shown in [5] that the corresponding set of "normalised" eigenfunctions in $W_P^{1,1}$ is homeomorphic to the unit circle $S^1 \subset \mathbb{R}^2$, and hence has genus two. It is natural to define this as the "variational" multiplicity (compare Definition 3.11). Thus Theorem 4.4 is consistent with Theorem 4.2, and the fact that even under perturbation there are only two variational eigenvalues $\mu_k(q)$ near to μ_k^0. Of course, in the linear case $p = 2$, all these eigenvalues have (algebraic=geometric) multiplicity two.

5. Further results in one and higher dimensions

This section is devoted to analogues of Theorem 4.3, in one and higher dimensions, for the case where $q = 0$.

5.1. $N = 1$. We start with an alternative variational formulation as follows — cf. Szulkin [25]. First we translate the λ origin so that all eigenvalues are positive, and then we replace the pair (G, H) in (4.5) by $(-H, G)$. This leads to a characterization of the negative reciprocals of the eigenvalues, but the important point for us is that they are now continuous in r (in a sense we shall make precise below) for fixed q — in fact we shall take $q = 0$. We then have the following analogue of Theorem 4.3 in one dimension.

Theorem 5.1. *Suppose that $p \neq 2$ and $q = 0$. For any integers k, $n \geq 1$ and any $\epsilon > 0$, there exist $\beta > 0$ and $r : (0, \beta) \to C^1[0, \pi_p]$ such that for each $\alpha \in (0, \beta)$, there are at least n non-variational periodic eigenvalues in $(\lambda_k^0 - \epsilon, \lambda_k^0 + \epsilon) \cap \sigma_{2k}$ for (4.1) with $r = r(\alpha)$. Moreover $r(\alpha)$ converges to 1 in the $C^1[0, \pi_p]$ norm as $\epsilon \to 0$.*

Proof. Starting again with the unperturbed problem $q = 0 = r - 1$, we use [7, Theorem 4.3] instead to give $\tilde{r} \in C^1$ so that the constant coefficient eigenvalue λ_k^0 splits into at least $n + 2$ nearby distinct eigenvalues, when $q = 0$ and $r = r(\alpha)$, where

$$r(\alpha) = 1 + \alpha\tilde{r}, \tag{5.1}$$

for sufficiently small α. As indicated above, the variational periodic eigenvalues of (4.1), which we now denote by $\mu_k(\alpha)$, depend continuously on α. We then conclude the proof as for Theorem 4.3, replacing the one parameter family αq by $r(\alpha)$. $\square$

Remark 5.2. In what follows, we will scale the interval $[0, \pi_p]$ to $[0, 2\pi]$, and denote the corresponding procedure (which scales the eigenvalues, eigenfunctions and weight function r) by carets. For example, $r(\alpha)$ from (5.1) scales to $\hat{r}(\alpha)$ defined on $[0, 2\pi]$, and $\hat{\lambda}_k^0$ is an unperturbed eigenvalue corresponding to $\hat{r}(0)$.

5.2. $N > 1$. We turn now to an analogue of Theorem 4.3 in higher dimensions, and we consider the Neumann problem for $q = 0$ in a bounded domain $\Omega \subset \mathbb{R}^N$, with $N \geq 2$, $p \neq 2$. We note that the p-Laplacian operator in $\mathbb{R}^N$ has the form

$$\Delta_p u := \text{div}\left(|\text{grad } u|^{p-2}\text{grad } u\right),$$

where $|\cdot|$ denotes the usual Euclidean norm in $\mathbb{R}^N$. For the purposes here it will suffice to consider weak solutions in $W^{1,p}(\Omega)$, although more regularity can be ensured — cf. [12]. We construct variational solutions as for Theorem 5.1, but with $W_P^{1,1}$ replaced by $W^{1,p}(\Omega)$. For a given $r \in C^1(\overline{\Omega})$, the Lyusternik-Šnirelman theory (as in [25]) yields an increasing sequence of variational eigenvalues μ_j, accumulating at $+\infty$.

Theorem 5.3. *Suppose that $1 < p \neq 2$, $q = 0$, and $N \geq 2$. For any integers k, $n \geq 1$ and any $\epsilon > 0$, there exist $\beta > 0$, $\Omega \subset \mathbb{R}^N$ and $r : (0, \beta) \to C^1(\overline{\Omega})$ such that for each $\alpha \in (0, \beta(\epsilon))$, there are at least n non-variational Neumann eigenvalues, within ϵ of $\hat{\lambda}_k^0$ from Remark 5.2, of (1.1) with $r = r(\alpha)$ in Ω. Moreover $r(\alpha)$ converges to 1 in the $C^1(\overline{\Omega})$ norm as $\epsilon \to 0$.*

Proof. We first consider the case $N = 2$. Let Ω be the annulus $\Omega := \{x \in \mathbb{R}^2 : 1 < |x| < 1 + 2\epsilon\}$, and let (ρ, θ) denote standard polar coordinates in $\mathbb{R}^2$ given by $x = \rho\cos\theta$, $y = \rho\sin\theta$.

Let $\hat{r}$ be a real valued C^1 function on $[0, 2\pi]$, and let $\hat{u}$ be an eigenfunction corresponding to an eigenvalue $\hat{\lambda}$ of (1.1) with $r = \hat{r}$ on $[0, 2\pi]$. Define $u(\rho, \theta) = \hat{u}(\theta)$

on Ω. Using the standard polar formulae for grad and div we see that

$$\Delta_p u = \rho^{-1}([\rho^{-1}u_\theta]^{p-1})_\theta = \rho^{-p}([u_\theta]^{p-1})_\theta,$$

suffix denoting partial differentiation.

It follows that u is a (nonzero, weak) solution of (1.1) on Ω, with $\lambda = \hat{\lambda}$ and r defined by

$$r(\rho,\theta) = \rho^p \hat{r}(\theta). \tag{5.2}$$

Moreover u obviously satisfies Neumann boundary conditions on $\partial\Omega$, so $\hat{\lambda}$ is also an eigenvalue of (1.1) on Ω with r as in (5.2).

We shall apply this below to $\hat{r} = \hat{r}(\alpha)$ of Remark 5.2, denoting r from (5.2) by $r(\alpha)$, and the corresponding variational eigenvalues by $\mu_j(\alpha)$. When $\alpha = 0$ this process is independent of the function $\tilde{r}$ used in the proof of Theorem 5.1, so we can write $r(0)$ and $\mu_j(0)$ unambiguously. Moreover, for fixed $k \geq 1$, $\hat{\lambda}_k^0$ is an eigenvalue of (1.1) on Ω with $r = r(0)$, and we write $m \geq 0$ for the (finite) variational multiplicity of this eigenvalue. More precisely, we find $l \geq 1$ and m such that

$$\mu_{l-1}(0) < \lambda_k^0 = \mu_l(0) = \cdots = \mu_{l+m-1}(0) < \mu_{l+m}(0). \tag{5.3}$$

Now we can use Theorem 5.1 and Remark 5.2, with n there replaced by $m+n$, to obtain $r(\alpha)$ as indicated above via (5.2). For sufficiently small $\alpha > 0$, there are at least $m+n$ eigenvalues of (1.1) on $[0, 2\pi]$ with $r = \hat{r}(\alpha)$, and hence of (1.1) on Ω with $r = r(\alpha)$, within ϵ of $\hat{\lambda}_k^0$. Since each $\mu_j(\alpha)$ is continuous in α, (5.3) shows that at least n of these eigenvalues must be non-variational.

For $N > 2$, we use cylindrical polar coordinates $(\rho, \theta, x_3, \ldots, x_N)$ for a similar construction. Instead of rotating the line segment $|\rho - 1 - \epsilon| < \epsilon$ through $\theta \in [0, 2\pi)$ to obtain an annulus for Ω, this time we rotate the ball with centre $\rho = 1 + \epsilon$, $x_3 = \cdots = x_N = 0$ and radius ϵ, to obtain a torus for the domain. Details will be left to the reader. □

5.3. **Conclusion and open problems.** We have shown that exploding eigenvalues and non-variational eigenvalues both exist near the constant coefficient case. In fact, since the variational and perturbation multiplicities are respectively finite and infinite, the "non-variational" multiplicity is also infinite, so the non-variational eigenvalues are also exploding. Theorems 5.1 and 5.3 extend corresponding results in [5] by requiring not only $q = 0$ but also r close to 1. One could require $r = 1$ and q close to 0 instead.

There are various related questions that remain open. One concerns the infinite multiplicities above. Our examples exhibit explosion into (arbitrarily large) finite numbers of eigenvalues, but can there be infinitely many? Also the constructions (with $q = 0$) in Theorem 5.3 involves a simple (annular/toroidal) domain and complicated r. Can one have $r = 1$ with a complicated domain?

Further questions stem from extensions of the basic theory based on Berestycki's "half-eigenvalue" problem. This involves the equation

$$-\Delta_p(u) + q[u]^{p-1} = \alpha[u^+]^{p-1} - \beta[u^-]^{p-1} + \lambda[u]^{p-1}. \tag{5.4}$$

We assume periodic boundary conditions with α, β and $\lambda \in \mathbb{R}$ although other possibilities exist – see [7]. Clearly, (5.4) is of the form considered in previous subsections (with $r = 1$) when $\alpha = \beta = 0$. Also it is known as the Fučík eigenvalue problem when $\lambda = 0$. Indeed, under certain conditions, the latter problem leads to a set of "Fučík " curves in the (α, β) plane, and any half-eigenvalue λ of (5.4) corresponds to a point of intersection of these curves with the line parametrized by $\{(\alpha+\lambda, \beta+\lambda) \in \mathbb{R}^2 : \lambda \in \mathbb{R}\}$.

It turns out that our perturbation results in λ extend to (5.4), so the intersection points of the Fučík curves with the line $\alpha = \beta$ explode into nearby intersection points as above. It is an interesting question, however, whether these points remain on (exploded) curves, i.e., whether there really are curves any more under the kind of perturbation of q and/or r that we have been discussing.

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