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Construction of a single-peak solution of the Liouville-Gel'fand equation on a two-dimensional domain with a hole

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1 Introduction

We are concerned with the Liouville-Gel'fand equation

$$\begin{cases} \Delta u+\lambda e^{u}=0 & \text{in} \quad \Omega_{\varepsilon}, \\ u=0 & \text{on} \quad \partial\Omega_{\varepsilon}. \end{cases} \tag{LG}$$

Here $\lambda>0$ is a parameter and $\Omega_{\varepsilon}\subset \mathbb{R}^{2}$ is a planar domain with a hole whose size is $\varepsilon>0$. The precise definition of Ω_{ε} will be introduced later. What we discuss in this article is construction of a solution of (LG) caused by a hole in Ω_{ε}.

The equation (LG) has an interesting solution structure when a domain is non-simply connected. The case where Ω_{ε} is an annulus was investigated by S.-S. Lin [7] and Nagasaki and Suzuki [8]. They independently showed that radially symmetric solutions make a branch and it emanates from $(\lambda, u)=(0,0)$, bends back once and blows up at each point in Ω_{ε} as $\lambda\downarrow 0$. Moreover, S.-S. Lin found that the branch has infinitely many secondary bifurcation points from which non-radially symmetric solutions emanate. Nagasaki and Suzuki also obtained non-radially symmetric solutions which have rotational symmetry of order k ($k\in \mathbb{N}$) and is large in some sense. Additionally, Dancer [2] showed that the set of bifurcating non-radially symmetric solutions are unbounded in the bifurcation diagram. These results indicate that bifurcating non-radially symmetric solutions connect to the large solutions obtained by Nagasaki and Suzuki. In [5, 6], suggestive evidence of this expectation was given provided that the inside radius of Ω_{ε} is small.

For a general non-simply connected domain, Chen and C.-C. Lin [1] revealed the existence of a solution whose mass is not equal to $8\pi k$ ($k\in \mathbb{N}$). Furthermore, del Pino, Kowalczyk and Musso [3] proved that for each $k\in \mathbb{N}$, (LG) has a solution blowing up at k different points as $\lambda\to 0$.

Our motivation is to obtain more detailed information on the solution structure for general non-simply connected domains by extending the results in [5, 6]. What we consider in particular is a solution with one maximum point. In this article, only by a formal argument, we explain how such a solution can be constructed.

2 Construction of a formal solution

We begin with the definition of the domain Ω_ε. Let Ω and $D \subset \mathbb{R}^2$ be bounded domains including the origin. Then, for small $\varepsilon > 0$, we define Ω_ε by

$$\Omega_\varepsilon := \Omega \setminus \overline{(\varepsilon D)} = \{x \in \Omega\,; \varepsilon^{-1}x \notin \overline{D}\}.$$

The following figure is an example of Ω_ε.

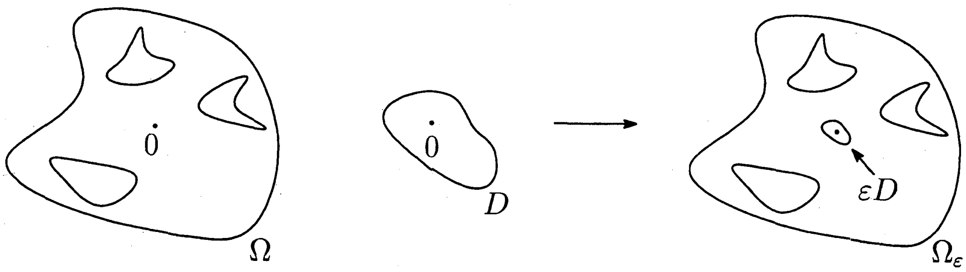

Figure : Domain Ω_ε

As will be seen below, an important factor to construct a formal solution is the regular part of a Green's function for Dirichlet Laplacian in Ω. We denote it by $H^\Omega = H^\Omega(x, y)$. Then, through this section, we assume that

$$\nabla_x H^\Omega(0,0) \neq 0. \tag{2.1}$$

This assumption leads to success of argument.

In what follows, we find a formal expansion of a solution $(\lambda.u) = (\lambda_\varepsilon, u_\varepsilon)$ by using the method of matched asymptotic expansions. To do this we separate Ω_ε into three parts. Two of them are regions near the boundary ($|x| \sim 1$ and $|x| \sim \varepsilon$) and the other is a region between them. The latter region is supposed to be $|x| \sim \delta_\varepsilon$, where δ_ε has a property $\varepsilon \ll \delta_\varepsilon \ll 1$ ($\varepsilon \to 0$) and is determined later. To obtain the expansion in this region, it is convenient to perform the change of variables $x = \delta_\varepsilon y$ and $v_\varepsilon(y) = u_\varepsilon(x) + \log(\delta_\varepsilon^2 \lambda_\varepsilon)$. Then we see that v_ε satisfies

$$\Delta v_\varepsilon + e^{v_\varepsilon} = 0 \quad \text{in} \quad (\delta_\varepsilon^{-1}\Omega) \setminus (\varepsilon\delta_\varepsilon^{-1}D).$$

Assuming that v_ε can be expanded as $v_\varepsilon(y) = v_0(y) + \delta_\varepsilon v_1(y) + \cdots$, we have

$$\begin{aligned} \Delta v_0 + e^{v_0} &= 0, \\ \Delta v_1 + e^{v_0} v_1 &= 0 \end{aligned} \quad \text{in} \quad \mathbb{R}^2 \setminus \{0\}.$$

Since a solution which we find has one peak, it is appropriate to choose v_0 as

$$v_0(y) = \log \frac{8(1-\rho^2)}{(1-\rho^2 + |y - \rho\omega|)^2},$$

or, in polar coordinates $y = (r\cos\theta, r\sin\theta)$,

$$v_0(y) = \log\frac{8(1-\rho^2)}{r^2\{r + r^{-1} - 2\rho\cos(\theta-\gamma)\}^2}. \tag{2.2}$$

Here $\rho \in (0,1)$ and $\gamma \in \mathbb{R}/2\pi\mathbb{Z}$ are parameters and $\omega = (\cos\gamma, \sin\gamma)$. Substituting this into the equation for v_1, we have

$$\Delta v_1 + \frac{8(1-\rho^2)}{r^2\{r + r^{-1} - 2\rho\cos(\theta-\gamma)\}^2} v_1 = 0 \quad \text{in} \quad \mathbb{R}^2 \setminus \{0\}. \tag{2.3}$$

To determine v_1, boundary conditions at the origin and infinity is needed. They are obtained as matching conditions, and therefore we consider the expansion near the boundary. First we treat the region $|x| \sim 1$. We formally expand $u_\varepsilon(x) = u_0(x) + \delta_\varepsilon u_1(x) + \cdots$ as $\varepsilon \to 0$. Then, for $j = 0, 1$, we have

$$\begin{cases} \Delta u_j = 0 & \text{in} \quad \Omega \setminus \{0\}, \\ u_j = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{2.4}$$

Since the maximum principle implies that u_ε is positive, u_0 must be nonnegative. Hence u_0 is given by

$$u_0(x) = c_0 G_0^\Omega(x). \tag{2.5}$$

Here c_0 is a nonnegative constant and G_0^Ω is a Green's function for the Dirichlet Laplacian in Ω with a singularity at the origin.

We substitute $x = \delta_\varepsilon^{\frac{1}{2}}\tilde{x}$ in (2.5) and $y = \delta_\varepsilon^{-\frac{1}{2}}\tilde{x}$ in (2.2), and compare the asymptotic behavior as $\varepsilon \to 0$. As $\varepsilon \to 0$,

$$\begin{aligned}
u_0(\delta_\varepsilon^{\frac{1}{2}}\tilde{x}) &= c_0\left(\frac{1}{2\pi}\log\frac{1}{|\delta_\varepsilon^{\frac{1}{2}}\tilde{x}|} - H_0^\Omega(\delta_\varepsilon^{\frac{1}{2}}\tilde{x})\right) \\
&\sim c_0\left(\frac{1}{4\pi}\log\frac{1}{\delta_\varepsilon} + \frac{1}{2\pi}\log\frac{1}{|\tilde{x}|} - H_0^\Omega(0) - \delta_\varepsilon^{\frac{1}{2}}\nabla H_0^\Omega(0)\cdot\tilde{x}\right) \\
&= c_0\left(\frac{1}{4\pi}\log\frac{1}{\delta_\varepsilon} + \frac{1}{2\pi}\log\frac{1}{\tilde{r}} - H_0^\Omega(0) - \delta_\varepsilon^{\frac{1}{2}}\mu\tilde{r}\cos(\tilde{\theta}-\tau)\right), \\
v_0(\delta_\varepsilon^{-\frac{1}{2}}\tilde{x}) &= \log\frac{8(1-\rho^2)}{(\delta_\varepsilon^{-\frac{1}{2}}\tilde{r})^2\{(\delta_\varepsilon^{-\frac{1}{2}}\tilde{r}) + (\delta_\varepsilon^{-\frac{1}{2}}\tilde{r})^{-1} - 2\rho\cos(\tilde{\theta}-\gamma)\}^2} \\
&\sim \log\{8(1-\rho^2)\delta_\varepsilon^2\} + 4\log\frac{1}{\tilde{r}} + \delta_\varepsilon^{\frac{1}{2}}\frac{4\rho\cos(\tilde{\theta}-\gamma)}{\tilde{r}} \\
&= \log\{8(1-\rho^2)\delta_\varepsilon^2\} + 4\log\frac{1}{\tilde{r}} + \delta_\varepsilon^{\frac{1}{2}}\frac{4\rho\tilde{x}\cdot\tilde{\omega}}{|\tilde{x}|^2},
\end{aligned}$$

where $H_0^\Omega(x) = H^\Omega(x,0)$, $\nabla H_0^\Omega(0) = (\mu\cos\tau, \mu\sin\tau)$, $\tilde{x} = (\tilde{r}\cos\tilde{\theta}, \tilde{r}\sin\tilde{\theta})$ and $\tilde{\omega} = (\cos\gamma, \sin\gamma)$. By matching two expansions $\log 1/(\delta_\varepsilon^2\lambda_\varepsilon) + v_0(z) + \delta_\varepsilon v_1(z) + \cdots$ and $u_0(x)+$

$\delta_\varepsilon u_1(x) + \cdots$ in the region $|x| \sim \delta_\varepsilon^{\frac{1}{2}}$, we have $c_0 = 8\pi$ and

$$u_1(x) = 4\rho \frac{x \cdot \tilde{\omega}}{|x|^2} + o\left(\frac{1}{|x|}\right) \quad \text{as} \quad x \to 0. \tag{2.6}$$

(2.4) and (2.6) give

$$u_1(x) = 4\rho\left(\frac{x \cdot \tilde{\omega}}{|x|^2} - 2\pi \nabla_y H(x,0) \cdot \tilde{\omega}\right) + c_1 G_0^\Omega(x),$$

where $c_1 \in \mathbb{R}$ is an undetermined constant. From this,

$$\begin{aligned} u_1(\delta_\varepsilon^{\frac{1}{2}} \tilde{x}) &\sim 4\rho\left(\delta_\varepsilon^{-\frac{1}{2}} \frac{\tilde{x} \cdot \omega}{|\tilde{x}|^2} - 2\pi\mu\omega \cdot \tilde{\omega}\right) + c_1\left(\frac{1}{2\pi} \log \frac{1}{|\delta_\varepsilon^{\frac{1}{2}} \tilde{x}|} - H_0^\Omega(0)\right) \\ &= \delta_\varepsilon^{-\frac{1}{2}} \frac{4\rho \cos(\tilde{\theta} - \tau)}{\tilde{r}} + \frac{c_1}{4\pi} \log \frac{1}{\delta_\varepsilon} + \frac{c_1}{2\pi} \log \frac{1}{\tilde{r}} - 8\pi\rho\mu\cos(\gamma - \tau) - c_1 H_0^\Omega(0). \end{aligned}$$

Thus it is appropriate to impose the condition

$$v_1(y) = -c_0 \mu r \cos(\theta - \tau) + \frac{c_1}{2\pi} \log \frac{1}{r} + a_1 + o(1) \quad \text{as} \quad r \to \infty. \tag{2.7}$$

Here a_1 is a constant determined later. Moreover,

$$\begin{aligned} &\frac{c_0}{4\pi} \log \frac{1}{\delta_\varepsilon} - c_0 H_0^\Omega(0) + \frac{c_1}{2\pi} \delta_\varepsilon \log \frac{1}{\delta_\varepsilon} - \delta_\varepsilon\left(8\pi\rho\mu\cos(\gamma - \tau) + c_1 H_0^\Omega(0)\right) \\ &\quad = \log \frac{1}{\delta_\varepsilon^2 \lambda_\varepsilon} + \log\{8(1 - \rho^2)\delta_\varepsilon^2\} - \delta_\varepsilon a_1, \end{aligned}$$

which gives

$$\begin{aligned} \lambda_\varepsilon = 8(1 - \rho^2)\delta_\varepsilon^2 \exp &\left[c_0 H_0^\Omega(0) + \frac{c_1}{2\pi} \delta_\varepsilon \log \frac{1}{\delta_\varepsilon}\right. \\ &\left. + \delta_\varepsilon\left\{a_1 - 8\pi\rho\mu\cos(\gamma - \tau) - c_1 H_0^\Omega(0)\right\}\right]. \end{aligned} \tag{2.8}$$

Next we consider the expansion in $|x| \sim \varepsilon$. Performing the change of variables $x = \varepsilon z$ and putting $w_\varepsilon(z) = u_\varepsilon(x)$, we have

$$\begin{cases} \Delta w_\varepsilon + \varepsilon^2 \lambda_\varepsilon e^{w_\varepsilon} = 0 & \text{in} \quad \varepsilon^{-1}\Omega_\varepsilon = (\varepsilon^{-1}\Omega) \setminus \overline{D}, \\ w_\varepsilon = 0 & \text{on} \quad \partial(\varepsilon^{-1}\Omega_\varepsilon). \end{cases}$$

Hence the formal expansion $w_\varepsilon(z) = w_0(z) + \delta_\varepsilon w_1(z) + \cdots$ gives

$$\begin{cases} \Delta w_j = 0 & \text{in} \quad \mathbb{R}^2 \setminus D, \\ w_j = 0 & \text{on} \quad \partial D \end{cases}$$

for $j = 0, 1$. To find a solution of the above equation, we perform the Kelvin transform

$$w_j^*(z^*) = w_j(z), \qquad z^* = \frac{z}{|z|^2}.$$

Then w_j^* satisfies

$$\begin{cases} \Delta w_j^* = 0 & \text{in} \quad D^* \setminus \{0\}, \\ \quad w_j^* = 0 & \text{on} \quad \partial D^*, \end{cases}$$

where $D^* := \{z^* = z/|z|^2; z \in D\}$. Since w_0^* is nonnegative, we see that $w_0^*(z^*) = d_0 G_0^{D^*}(z^*)$ for some constant $d_0 \geq 0$. Thus

$$w_0(z) = d_0 G_0^{D^*}(z^*) = d_0 G_0^{D^*}(z/|z|^2).$$

If $d_0 > 0$, this function has logarithmic growth at $z = \infty$, while v_0 has no such a singularity at $y = 0$. This implies that $d_0 = 0$, that is, $w_0 \equiv 0$. Since w_1 satisfies the same equation as w_0 and must be nonnegative, we have

$$w_1(z) = d_1 G_0^{D^*}(z^*) = d_1 G_0^{D^*}(z/|z|^2), \tag{2.9}$$

where $d_1 \geq 0$ is some undetermined constant.

We compare the expansions in the region $|x| \sim \varepsilon^{\frac{1}{2}} \delta_\varepsilon^{\frac{1}{2}}$. By putting $z = \varepsilon^{-\frac{1}{2}} \delta_\varepsilon^{\frac{1}{2}} \hat{x}$ in (2.9) and $y = \varepsilon^{\frac{1}{2}} \delta_\varepsilon^{-\frac{1}{2}} \hat{x}$ in (2.2), we have

$$\begin{aligned} w_1(\varepsilon^{-\frac{1}{2}} \delta_\varepsilon^{\frac{1}{2}} \hat{x}) &= d_1 \left(\frac{1}{2\pi} \log |\varepsilon^{-\frac{1}{2}} \delta_\varepsilon^{\frac{1}{2}} \hat{x}| - H_0^{D^*} \left(\varepsilon^{\frac{1}{2}} \delta_\varepsilon^{-\frac{1}{2}} \frac{\hat{x}}{|\hat{x}|^2} \right) \right) \\ &\sim d_1 \left(\frac{1}{2\pi} \log(\varepsilon^{-1} \delta_\varepsilon) + \frac{1}{2\pi} \log |\varepsilon^{\frac{1}{2}} \delta_\varepsilon^{-\frac{1}{2}} \hat{x}| - H_0^{D^*}(0) \right), \\ v_0(\varepsilon^{\frac{1}{2}} \delta_\varepsilon^{-\frac{1}{2}} \hat{x}) &= \log \frac{8(1-\rho^2)}{(\varepsilon^{\frac{1}{2}} \delta_\varepsilon^{-\frac{1}{2}} \hat{r})^2 \{ (\varepsilon^{\frac{1}{2}} \delta_\varepsilon^{-\frac{1}{2}} \hat{r}) + (\varepsilon^{\frac{1}{2}} \delta_\varepsilon^{-\frac{1}{2}} \hat{r})^{-1} - 2\rho \cos(\hat{\theta} - \gamma) \}^2} \\ &\sim \log\{8(1-\rho^2)\}. \end{aligned}$$

Thus, assuming that two expansions $\log 1/(\delta_\varepsilon^2 \lambda_\varepsilon) + v_0(y) + \delta_\varepsilon v_1(y) + \cdots$ and $w_0(z) + \delta_\varepsilon w_1(z) + \cdots$ give the same expansion in $|x| \sim \varepsilon^{\frac{1}{2}} \delta_\varepsilon^{\frac{1}{2}}$, we deduce

$$v_1(y) = \frac{d_1}{2\pi} \log r + a_2 + o(1) \quad \text{as} \quad r \to 0 \tag{2.10}$$

and

$$\log \frac{8(1-\rho^2)}{\delta_\varepsilon^2 \lambda_\varepsilon} - \delta_\varepsilon a_2 = d_1 \delta_\varepsilon \left(\frac{1}{2\pi} \log(\varepsilon^{-1} \delta_\varepsilon) - H_0^{D^*}(0) \right). \tag{2.11}$$

Here $a_2 \in \mathbb{R}$ is a constant determined later.

Now we solve (2.3) under the conditions (2.7) and (2.10). First we observe that the functions

$$\Phi_{\rho,\gamma,1}(z) = \frac{r - r^{-1}}{r + r^{-1} - 2\rho\cos(\theta - \gamma)},$$
$$\Phi_{\rho,\gamma,2}(z) = \frac{2\cos(\theta - \gamma) - \rho(r + r^{-1})}{r + r^{-1} - 2\rho\cos(\theta - \gamma)},$$
$$\Phi_{\rho,\gamma,3}(z) = \frac{\sin(\theta - \gamma)}{r + r^{-1} - 2\rho\cos(\theta - \gamma)}$$

are bounded solutions of (2.3). Furthermore, every bounded solution of (2.3) is linear combination of these solutions (see [4], [5]). We also observe what is necessary to solve the equation. Suppose that (2.3), (2.10) and (2.7) has a solution. By a simple computation, we have

$$\Phi_{\rho,\gamma,j}(z) = \begin{cases} 1 + 2\rho r^{-1}\cos(\theta - \gamma) + O(r^{-2}) & (j = 1) \\ -\rho\{1 - 2(\rho^{-1} - \rho)r^{-1}\cos(\theta - \gamma)\} + O(r^{-2}) & (j = 2) \\ r^{-1}\cos(\theta - \gamma) + O(r^{-2}) & (j = 3) \end{cases} \quad \text{as } r \to \infty,$$

$$\Phi_{\rho,\gamma,j}(z) = \begin{cases} -1 + 2\rho r\cos(\theta - \gamma) + O(r^{2}) & (j = 1) \\ -\rho\{1 - 2(\rho^{-1} - \rho)r\cos(\theta - \gamma)\} + O(r^{2}) & (j = 2) \\ r\cos(\theta - \gamma) + O(r^{2}) & (j = 3) \end{cases} \quad \text{as } r \to 0.$$

Hence, as $r \to \infty$,

$$r\left(\frac{\partial w_1}{\partial r}\Phi_{\rho,\gamma,j} - w_1\frac{\partial \Phi_{\rho,\gamma,j}}{\partial r}\right)$$
$$= \begin{cases} -c_0\mu r\cos(\theta - \tau) - \dfrac{c_1}{2\pi} - 4c_0\rho\mu\cos(\theta - \tau)\cos(\theta - \gamma) + o(1) & (j = 1) \\ -\rho\{-c_0\mu r\cos(\theta - \tau) - \dfrac{c_1}{2\pi} & \\ \qquad + 4c_0(\rho^{-1} - \rho)\mu\cos(\theta - \tau)\cos(\theta - \gamma)\} + o(1) & (j = 2) \\ -2c_0\mu\cos(\theta - \tau)\sin(\theta - \gamma) + o(1) & (j = 3) \end{cases},$$

and as $r \to 0$,

$$r\left(\frac{\partial w_1}{\partial r}\Phi_{\rho,\gamma,j} - w_1\frac{\partial \Phi_{\rho,\gamma,j}}{\partial r}\right) = \begin{cases} -d_1/(2\pi) + o(1) & (j = 1) \\ -(\rho d_1)/(2\pi) + o(1) & (j = 2) \\ o(1) & (j = 3) \end{cases}.$$

Thus multiplying both sides of (2.3) by $\Phi_{\rho,\gamma,j}$ and integrating give

$$0 = \left[\int_0^{2\pi} r\left(\frac{\partial w_1}{\partial r}\Phi_{\rho,\gamma,j} - w_1\frac{\partial \Phi_{\rho,\gamma,j}}{\partial r}\right)d\theta\right]_{r=0}^{\infty}$$

$$= \begin{cases} -c_1 - 4\pi c_0 \rho\mu \cos(\gamma - \tau) + d_1 & (j = 1) \\ -\rho\{-c_1 + 4\pi c_0(\rho^{-1} - \rho)\mu\cos(\gamma - \tau) - d_1\} & (j = 2) \\ 2\pi c_0 \mu \sin(\gamma - \tau) & (j = 3) \end{cases}.$$

Note that $\mu > 0$ from (2.1) and $d_1 \geq 0$. Therefore the above relations yield

$$\gamma = \tau,$$
$$c_1 = 2\pi c_0 \mu \left(\frac{1}{\rho} - 2\rho\right) = 16\pi^2\mu\left(\frac{1}{\rho} - 2\rho\right),$$
$$d_1 = \frac{2\pi c_0 \mu}{\rho} = \frac{16\pi^2\mu}{\rho}.$$

Conversely, it can be checked that the function

$$V(y) = -c_0\mu\left\{\left(\frac{1}{\rho} - \rho\right)\Phi_{\rho,\tau,1}(y)\log r + \Phi_{\rho,\tau,1}(y)\log r - \frac{1}{\rho} + r\cos(\theta - \tau)\right\}$$

is a solution of (2.3), (2.10), (2.7) provided that γ, c_1 and d_1 satisfy the above relations. Thus, by setting $a_2 = a_3 = (c_0\mu)/\rho$, we see that v_1 is given by

$$v_1(y) = V(y) + \alpha\Phi_{\rho,\tau,3}(y),$$

where $\alpha \in \mathbb{R}$ is an arbitrary constant.

From (2.8) and (2.11), it can be shown that

$$\delta_\varepsilon = \frac{\rho}{2\pi\mu}\frac{\log\log\frac{1}{\varepsilon}}{\log\frac{1}{\varepsilon}}(1 + o(1))$$

as $\varepsilon \to 0$. Hence setting $\eta_\varepsilon = 2\pi\mu\delta_\varepsilon/\rho$, we have

$$\eta_\varepsilon = \frac{\log\log\frac{1}{\varepsilon}}{\log\frac{1}{\varepsilon}}(1 + o(1)),$$
$$\lambda_\varepsilon = \frac{4\rho^2(1-\rho^2)e^{8\pi H_0^\Omega(0)}}{\mu\pi}\eta_\varepsilon^2(1 + o(1)).$$

This indicates that u_ε appears through a saddle-node bifurcation when $\rho \sim 1/\sqrt{2}$.

Finally we discuss how the constant α is determined. From the formal expansion obtained above, the solution u_ε is expected to expand as

$$u_\varepsilon(x) = \log\frac{1}{\delta_\varepsilon^2\lambda_\varepsilon} + v_0(y) + \delta_\varepsilon V(y) + \alpha\delta_\varepsilon\Phi_{\rho,\tau,3}(y) + (h.o.t.)$$

provided that $|y| \sim 1$. This expansion is valid only in the region $|y| \sim 1$, and therefore we add a correction term to obtain an approximation in the whole region of Ω_ε. We define a correction function v_c as a solution of

$$\begin{cases} \Delta v_c = 0 & \text{in} \quad \delta_\varepsilon^{-1}\Omega_\varepsilon, \\ v_c = -\log \dfrac{1}{\delta_\varepsilon^2 \lambda_\varepsilon} - v_0 - \delta_\varepsilon V & \text{on} \quad \partial(\delta_\varepsilon^{-1}\Omega_\varepsilon). \end{cases}$$

Then one can show that

$$|v_c(y)| \le C\left(\varepsilon r^{-1} + \delta_\varepsilon^2 r^2\right)$$

for all $y \in \delta_\varepsilon^{-1}\Omega_\varepsilon$, and

$$v_c(y) = \delta_\varepsilon^2 \xi(y) + o(\delta_\varepsilon^2)$$

locally uniformly for $y \in \mathbb{R} \setminus \{0\}$ as $\varepsilon \to 0$. Here $C > 0$ is a constant independent of ε and ξ is a function determined by the regular part of a Green's function in Ω (we omit the detail of ξ). Consequently, we obtain the expansion

$$u_\varepsilon(x) = \log \frac{1}{\delta_\varepsilon^2 \lambda_\varepsilon} + U_\varepsilon(y) + \alpha\delta_\varepsilon \Phi_{\rho,\tau,3}(y) + r_\varepsilon(y),$$

where $U_\varepsilon = v_0 + \delta_\varepsilon V + v_c$ and r_ε is a remainder term. r_ε is expected to be small on whole domain Ω_ε in some appropriate topology.

We set $\eta_\varepsilon(y) = \alpha\delta_\varepsilon\Phi_{\rho,\tau,3}(y) + r_\varepsilon(y)$ and substitute the above expansion into (LG). Then the equation is rewritten as

$$\mathcal{L}(\eta_\varepsilon) + F(\eta_\varepsilon) + R_\varepsilon = 0,$$

where

$$\begin{aligned} \mathcal{L}(\eta_\varepsilon) &= \Delta\eta_\varepsilon + e^{U_\varepsilon}\eta_\varepsilon, \\ F(\eta_\varepsilon) &= e^{U_\varepsilon}(e^{\eta_\varepsilon} - 1 - \eta_\varepsilon), \\ R_\varepsilon &= \Delta U_\varepsilon + e^{U_\varepsilon}. \end{aligned}$$

To determine the constant α, we multiply the above equation by $\Phi_{\rho,\tau,3}$ and integrate over $\delta_\varepsilon^{-1}\Omega_\varepsilon$. Then we have

$$\begin{aligned} \int_{\delta_\varepsilon^{-1}\Omega_\varepsilon} \mathcal{L}(\eta_\varepsilon)\Phi_{\rho,\tau,3}dx &\sim \int_{\delta_\varepsilon^{-1}\Omega_\varepsilon} \eta_\varepsilon \mathcal{L}(\Phi_{\rho,\tau,3})dx \\ &\sim \alpha\delta_\varepsilon \int_{\delta_\varepsilon^{-1}\Omega_\varepsilon} (e^{U_\varepsilon} - e^{v_0})\Phi_{\rho,\tau,3}^2 dx \\ &\sim \alpha\delta_\varepsilon^2 \int_{\mathbb{R}^2} e^{v_0} V \Phi_{\rho,\tau,3}^2 dx \\ &= \frac{4\pi^2\mu}{\rho}\alpha\delta_\varepsilon^2, \\ \int_{\delta_\varepsilon^{-1}\Omega_\varepsilon} F(\eta_\varepsilon)\Phi_{\rho,\tau,3}dx &\sim \int_{\delta_\varepsilon^{-1}\Omega_\varepsilon} e^{U_\varepsilon}\eta_\varepsilon^2 \Phi_{\rho,\tau,3}dx \end{aligned}$$

$$
\begin{aligned}
&\sim \alpha^2\delta_\varepsilon^2 \int_{\mathbb{R}^2} e^{v_0}\Phi_{\rho,\tau,3}^3 dx \\
&= 0.
\end{aligned}
$$

From the definition of U_ε, we see that

$$
R_\varepsilon = e^{v_0}(e^{\delta_\varepsilon V + v_c} - 1 - \delta_\varepsilon V).
$$

Hence

$$
\begin{aligned}
\int_{\delta_\varepsilon^{-1}\Omega_\varepsilon} R_\varepsilon \Phi_{\rho,\tau,3} dx &\sim \int_{\delta_\varepsilon^{-1}\Omega_\varepsilon} e^{v_0}\{v_c + (\delta_\varepsilon V + v_c)^2\}\Phi_{\rho,\tau,3} dx \\
&\sim \delta_\varepsilon^2 \int_{\mathbb{R}^2} e^{v_0}(\xi + V^2)\Phi_{\rho,\tau,3} dx \\
&= \delta_\varepsilon^2 \int_{\mathbb{R}^2} e^{v_0}\xi\Phi_{\rho,\tau,3} dx.
\end{aligned}
$$

Thus α is given by

$$
\alpha = \frac{\rho}{4\pi^2\mu}\int_{\mathbb{R}^2} e^{v_0}\xi\Phi_{\rho,\tau,3} dx.
$$

At the end, we summarize what we obtained.

Main Result 1. *Assume* (2.1). *Then, for small ε and $\rho \in (0,1)$, we can construct a "formal" solution $(\lambda_\varepsilon, u_\varepsilon)$ of* (LG) *with the following expansion :*

$$
\left.
\begin{aligned}
\lambda_\varepsilon &\sim \frac{4\rho^2(1-\rho^2)e^{8\pi H_0^\Omega(0)}}{\mu\pi}\left(\frac{\log\log\frac{1}{\varepsilon}}{\log\frac{1}{\varepsilon}}\right)^2, \\
u_\varepsilon(x) &\sim \log\frac{1}{\delta_\varepsilon^2\lambda_\varepsilon} + v_0(\delta_\varepsilon^{-1}x) + \delta_\varepsilon v_1(\delta_\varepsilon^{-1}x) + v_c(\delta_\varepsilon^{-1}x)
\end{aligned}
\right\}
\quad \textit{as} \quad \varepsilon \to 0.
$$

Here constants and functions are chosen suitably as discussed above.

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