Entropy for Unitary Operators

大阪教育大学 長田 まり桑 (Marie Choda)

Abstract

We define the entropy $S(u)$ for an $n \times n$ unitary matrix $u$, and by using the values of $S(u)$ we characterize the notion of mutual orthogonality between two maximal abelian subalgebras of $M_n(\mathbb{C})$. We apply these method to unitaries in type $\text{II}_1$-factors $M$ and characterize the notion of commuting square condition between two subfactors of $M$ with the Jones index 2.

1 Introduction

This is a continuation of my following two reports:

(講究 1) 数理解析研究所講究録 1819, Entropy via partitions of unity, pp 9–21;

(講究 2) 数理解析研究所講究録 1820, A representation of unital completely positive maps, pp 11–24.

There are several notions which describe some relative position between two subalgebras of operator algebras. As one of such notions for relations between two subalgebras of finite von Neumann algebras, Popa introduced the notion of mutually orthogonal subalgebras (definition below) in [15]. By the terminology complementarity, the same notion is investigated in the theory of quantum systems (see [12] for example).

We are interested in to give numerical characterizations for the notion of mutually orthogonality between two isomorphic subalgebras. The most primary interest would be the case where two subalgebras of some full matrix algebra, both of which are either maximal abelian subalgebras or isomorphic to also some full matrix algebra. In such the cases, two subalgebras are
connected by some unitary, and we would like to know how such a unitary plays a key role.

Our motivation for this work arises from the following fact: To give a numerical characterizations for the notion of mutually orthogonality, in the previous paper [1], as one of way, we defined a constant \( h(A|B) \) for two subalgebras \( A \) and \( B \) of a finite von Neumann algebra, and explained the relative position between maximal abelian subalgebras \( A \) and \( B \) of the algebra \( M_n(C) \) of \( n \times n \) complex matrices by using the values of \( h(A|B) \). This \( h(A|B) \) is a slight modification of Connes-Størmer relative entropy \( H(A|B) \) in [4] (cf. [10]).

If \( A_1 \) and \( A_2 \) are maximal abelian subalgebras of \( M_n(C) \), then there exists a unitary \( u \) in \( M_n(C) \) such that \( A_2 = uA_1u^* \), (which we denote by \( u(A_1, A_2) \)), and we showed that \( A_1 \) and \( A_2 \) are mutually orthogonal if and if \( h(A_1|A_2) = H(b(u(A_1, A_2))) = \log n \) in [1, Corollary 3.2] (cf. (講究 1)), where \( H(b(u(A_1, A_2))) \) is the entropy defined in [16] for the unistochastic matrix \( b(u(A_1, A_2)) \) induced by the unitary \( u = u(A_1, A_2) \). This means that \( A_1 \) and \( A_2 \) are mutually orthogonal if and if the value \( h(A_1|A_2) \) is maximal and equals to the logarithm of the dimension of the subalgebras. Also we had in [2] related results in the case of subfactors of the type \( II_1 \) factors. We remark that it does not hold in general that \( H(A_1|A_2) = h(A_1|A_2) \) (see, for example [13, Appendix]).

Next when \( A_1 \) and \( A_2 \) are subalgebras of \( M_n(C) \), both of which are isomorphic to also some full matrix algebra \( M_k(C) \), our discussion in (講究 1) (see, also [3, Section 3]) was as the followings: The algebra \( M_n(C) \) is decomposed into the tensor product: \( M_n(C) = M_m(C) \otimes M_k(C) \) for some integers \( m \) with \( n = mk \), and also \( A_1 \) and \( A_2 \) are connected by some unitary \( u \in M_n(C) \). By decomposing such a unitary \( u \) into the tensor product form, we gave a finite set \( U \) satisfying the property called finite operational partition of unity so that a density matrix \( \rho(U) \) thanks by the method of Lindblad [9]. By using the von Neumann entropy for \( \rho(U) \) in place of relative entropy \( h(A_1|A_2) \), we showed that \( A_1 \) and \( A_2 \) are mutually orthogonal if and only if the von Neumann entropy of the density matrix \( \rho(U) \) takes the maximum value \( 2 \log n \), which is the logarithm of the dimension of the subfactors.

Here, we pick up another kind of decomposition for the algebra \( M_n(C) \). That is the crossed product decomposition \( M_n(C) = D_n(C) \times_\alpha \mathbb{Z}_n \) of the diagonal matrices \( D_n(C) \) by the integer group \( \mathbb{Z}_n \) with respect to the action \( \alpha \) with \( \alpha(e_i) = e_{i+1} \) (mod \( n \)), where \( \{e_1, e_2, \ldots , e_n\} \) are mutually orthogonal
minimal projections of the maximal abelian subalgebra $D_n(\mathbb{C})$ of $M_n(\mathbb{C})$.

More generally, we consider the crossed product $M$ of a finite von Neumann algebra $N$ by a finite group $G$. We decompose a given unitary $u$ in $M$ into the crossed product form. Then a family of positive operators in $N$ appears as the coefficients of $u$ in the crossed product decomposition, and the family is a finite partition of unity in the sense of Connes-Størmer [4] (see [10, 1.3]). By considering the von Neumann entropy for these positive operators, we introduce the entropy $S(u)$ for the unitary $u$. We characterize the mutual orthogonality for a pair $\{A, B\}$ of maximal abelian subalgebras of $M_n(\mathbb{C})$ by the value $S(u)$ for a unitary $u$ with $B = uAu^*$. We also apply these method to unitaries in type $II_1$-factors $M$ and characterize the notion of commuting square condition between two subfactors of $M$ with the Jones index 2.

2 Preliminaries

2.1 Entropy function $\eta$.

The entropy function $\eta$ is defined on the interval $[0, 1]$ by

$$\eta(t) = -t \log t \quad (0 < t \leq 1) \quad \text{and} \quad \eta(0) = 0. \quad (2.1)$$

Let $\lambda = \{\lambda_1, \cdots, \lambda_n\}$ be a finite family of real numbers. We call the $\lambda$ a finite partition of 1 if $\lambda_i \geq 0$ and $\sum_i \lambda_i = 1$. The entropy $H(\lambda)$ for $\lambda$ is given by

$$H(\lambda) = \eta(\lambda_1) + \cdots + \eta(\lambda_n).$$

The function $\eta$ is strictly concave, that is,

$$\sum_{i=1}^{n} t_i \eta(s_i) \leq \eta(\sum_{i=1}^{n} t_i s_i) \quad (2.2)$$

holds whenever $s_i \in [0, 1]$ and for real numbers $t_i \geq 0$ with $\sum_{i=1}^{n} t_i = 1$, and equality holds if and only if $s_i = s_j$ for all $i = 1, \cdots, n$. This implies that

$$H(\lambda) \leq \log n \quad (2.3)$$

and $H(\lambda) = \log n$ if and only if $\lambda_i = 1/n$ for all $i = 1, \cdots, n$. 
2.2 Finite partition of unity

Let $A$ be a unital $C^*$-algebra. There are two kind of notions for a finite system, which are named by a finite partition of unity, as the followings: The first one was given by Connes-Størmer and the second one was given by Lindblad (See [10] or [11]).

A finite subset $\{x_1, \ldots, x_k\}$ of $A$ is called a finite partition of unity if they are nonnegative operators in $A$ which satisfy that

$$\sum_{i=1}^{n} x_i = 1_A.$$ 

A finite subset $X = \{x_1, \ldots, x_k\}$ of $A$ is called a finite operational partition in $A$ of unity of size $k$ if

$$\sum_{i}^{k} x_i^* x_i = 1_A.$$ 

If $\phi$ is a state of $A$, then the density matrix $\rho_\phi[X]$ for a finite operational partition $X$ associated with $\phi$ is the matrix such that the each $(i, j)$-coefficient $\rho_\phi[X](i, j)$ is given by $\rho_\phi[X](i, j) = \phi(x_j^* x_i)$. When $L$ is a finite von Neumann algebra and that $\tau_L$ is a fixed faithful normal tracial state of $L$, to a finite operational partition $X$ in $L$ of unity of size $k$, we associate a $k \times k$ density matrix $\rho_{\tau_L}[X]$. We denote this matrix simply by $\rho[X]$, that is, the $(i, j)$-coefficient $\rho[X](i, j)$ of $\rho[X]$ is given by

$$\rho[X](i, j) = \tau_L(x_j^* x_i), \quad i, j = 1, \ldots, k.$$ 

We gave several examples about this entropy in (講究 1) and (講究 2).

2.3 The von Neumann entropy.

Let $\rho \in M_n(\mathbb{C})$ be a positive semidefinite matrix with $\rho \leq 1$, where $1$ is the identity matrix. The von Neumann entropy $S(\rho)$ is given by

$$S(\rho) = \text{Tr}(\eta(\rho)),$$ 

that is $S(\rho) = \sum_{i=1}^{n} \eta(\lambda_i)$, where $\lambda = \{\lambda_i\}_{i=1}^{n}$ is the eigenvalues of $\rho$. By a density matrix, we mean a positive semidefinite matrix $\rho$ such that $\text{Tr}(\rho) = 1$. If $\rho$ is a density matrix, then the eigenvalues of $\rho$ is a finite partition of $1$. 

2.4 Representation that $M_n(\mathbb{C}) = D_n(\mathbb{C}) \times_{\alpha} \mathbb{Z}_n$

Let $x$ be a $n \times n$ complex matrix. We denote by $x_{ij}$ the $\{i,j\}$-component of $x$. Let $v \in M_n(\mathbb{C})$ be the unitary matrix such that $v_{ij} = \delta_{i,j-1}1 \text{ (mod n)}$, for all $i, j = 1, \cdots, n$. The conditional expectation $E_D$ of $M_n(\mathbb{C})$ on the algebra of the diagonal matrices $D_n(\mathbb{C})$ is given by the following form:

$$E(x) = \sum_{i=1}^{n} \epsilon_i x \epsilon_i, \quad (x \in M_n(\mathbb{C})),$$

where $\epsilon_i$ is the diagonal matrix whose $\{j,j\}$-component is $\delta_{i,j}1$.

For a given $u \in M_n(\mathbb{C})$, we denote by $u_j$ the diagonal matrix such that each $\{i,i\}$-component is $u_{i,j+i} \text{ (mod n)}$ for all $j = 0, 1, \cdots, n-1$ and $i = 1, \cdots, n$. Then $u$ is represented as

$$u = u_0 + u_1 v + u_2 v^2 + u_3 v^3 + \cdots + u_{n-1} v^{n-1}.$$

This means that the algebra $M_n(\mathbb{C})$ is decomposed into the crossed product $D_n(\mathbb{C}) \times_{\alpha} \mathbb{Z}_n$ of $D_n(\mathbb{C})$ by the cyclic group $\mathbb{Z}_n$ with respect to the action $\alpha$ with $\alpha(\epsilon_i) = \epsilon_{i+1}$. Each $u_j$ is also determined by the form that $u_j = E_D(\epsilon_i v^{-j})$ for all $j = 0, 1, \cdots, n-1$.

2.4.1 Example.

As an example, we can see the above fact as the followings:

$$v = \begin{bmatrix} 0 & 0 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ 0 & \cdots & 1 & 0 \end{bmatrix}$$
and

\[
\begin{bmatrix}
  u_{11} & u_{12} & \cdots & u_{1n} \\
  u_{21} & u_{22} & \cdots & u_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  u_{n1} & u_{n2} & \cdots & u_{nn}
\end{bmatrix}
\]

\[
\begin{bmatrix}
  u_{11} & 0 & \cdots & 0 \\
  0 & u_{22} & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & u_{nn}
\end{bmatrix}
\]

\[
\begin{bmatrix}
  u_{1n} & 0 & \cdots & 0 \\
  0 & u_{21} & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & u_{n \cdot n-1}
\end{bmatrix}
\]

\[
= u_0 + u_1 v + u_2 v^2 + \cdots + u_{n-1} v^{n-1}.
\]

Here

\[
\begin{bmatrix}
  u_{1 \cdot j+1} & 0 & \cdots & 0 \\
  0 & u_{2 \cdot j+2} & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & u_{nj}
\end{bmatrix}, \quad (j = 0, 1, \cdots, n - 1).
\]

3 Entropy for unitary operators via crossed product decomposition

In this section, we assume that $M$ is given as the crossed product $M = N \times_{\alpha} G$ of a finite von Neumann algebra $N$ by a finite group $G$ with respect to a freely acting $\tau_N$-preserving (i.e., $\tau_N \circ \alpha_g = \tau_N$, for all $g \in G$) action $\alpha$ of $G$ on $N$. Here $\tau_N$ is a fixed normal faithful tracial state of $N$. We regard $N$ as a von Neumann subalgebra of $M$, then we have a unitary representation
$v$ of $G$ to $M$ such that $\alpha_g(y) = v_g y v_g^*$ for all $g \in G$, $y \in N$. Each $x \in M$ is uniquely written as

$$x = \sum_{g \in G} x_g v_g, \quad x_g \in N$$

(3.1)

The conditional expectation $E_N$ of $M$ onto $N$ is given by $E_N(x) = x_1$, where 1 is the unit of $G$, and it holds, for each $x \in M$, that

$$x_g = E_N(x v_g^*) \quad \text{for all} \quad g \in G.$$  

(3.2)

The trace $\tau_N$ is extended to the trace $\tau_M$ of $M$ by $\tau_M = \tau_N \circ E_N$.

If $u \in M$ is a unitary, then the family $\{u_g; g \in G\} \subset N$ satisfies that

$$\sum_{g \in G} u_g u_g^* = 1_N \quad \text{and} \quad \sum_{g \in G} u_g \alpha_k(u_{hk^{-1}}) = 0, \quad (k \neq 1).$$

(3.3)

By means of the family $\{u_g u_g^*; g \in G\}$, which is a finite partition of unity in $N$, we define the entropy $S(u)$ as the followings:

3.1 Definition.

For a unitary $u \in M = N \times_\alpha G$, let

$$S(u) = \sum_{g \in G} \tau_N \eta(u_g u_g^*).$$

(3.4)

3.1.1 Case of type $I_n$ factors.

First, we take up the case of the type $I_n$ factor $M$. Let $A$ be a maximal abelian subalgebra of $M$. Then $M$ is isomorphic to the matrix algebra $M_n(\mathbb{C})$ and $A$ is isomorphic to the algebra of diagonal matrices $D_n(\mathbb{C})$. The $M$ is represented as the the crossed product of $A$ by the group $\mathbb{Z}_n$ with respect to $\alpha$:

$$M = A \times_\alpha \mathbb{Z}_n.$$  

Here, the automorphism $\alpha$ of $A$ is given by

$$\alpha(e_i) = e_{i+1}, \quad (\text{mod } n)$$

for a mutually orthogonal minimal projections $\{e_1, e_2, \cdots, e_n\}$ of $A$. Let $\{e_{ij}; i, j = 1, 2, \cdots, n\}$ be a system of a matrix units of $M$ with $e_{ii} = e_i, (i =
1, 2, \cdots , n). Then the unitary \( v = \sum_{i=1}^{n} e_{i-1} \) in \( M \) satisfies that \( \alpha(a) = vav^{*} \) for all \( a \in A \). For the decomposition of a unitary \( u \in M \):

\[
\begin{align*}
u = \sum_{j=0}^{n-1} u_j v^j.
\end{align*}
\]

the entropy \( S(u) \) is nothing else but the average of the von Neumann entropy for \( \{S(u_j u_j^*) : j = 0, 1, \cdots, n - 1\} \), that is

\[
S(u) = \sum_{j=0}^{n-1} \tau_A \eta(u_j u_j^*) = \frac{1}{n} \sum_{j=0}^{n-1} \text{Tr} \eta(u_j u_j^*) = \frac{1}{n} \sum_{j=0}^{n-1} S(u_j u_j^*).
\]

### 3.2 Mutually orthogonal maximal abelian subalgebras

Let \( A \) and \( B \) be two maximal abelian subalgebras of \( M = M_n(\mathbb{C}) \). Then there exists a unitary \( u \) with \( B = uAu^{*} \). By using the crossed product decomposition \( M = A \times_{\alpha} \mathbb{Z}_{n} \), we gave a characterization in [3] for the mutually orthogonality of \( A \) and \( B \) via the von Neumann entropy \( S(u) \).

#### 3.2.1 Theorem

Let \( A \) and \( B \) be maximal abelian subalgebras of \( M \). Let \( u \) be a unitary in \( M \) with \( B = uAu^{*} \). Then the following are equivalent:

1. \( A \) and \( B \) are mutually orthogonal;
2. \( u_j u_j^* = \frac{1}{n} 1_A \), \( j = 0, 1, 2, \cdots, n - 1 \);
3. the entropy \( S(u) \) takes the maximal value:

\[
S(u) = \log n = \max \{ S(w) \mid w \in M, \text{unitary} \}.
\]

#### 3.2.2 Complex Hadamard matrix

A unitary matrix \( u \in M_n(\mathbb{C}) \) is called a complex Hadamard matrix in [7] if all entries \( u(i, j) \) of \( u \) have the same modulus, that is \( |u(i, j)| = 1/\sqrt{n} \) for all \( i, j = 1, \cdots, n \). For a \( u \in M_n(\mathbb{C}) \), Sunder and Jones gave the characterization in [7, 5.2.2] that \( D_n(\mathbb{C}) \) and \( uD_n(\mathbb{C})u^{*} \) are mutually orthogonal if and only if \( u \) is a complex Hadamard matrix.
It is clear that if \( u \) is a complex Hadamard matrix then \( S(u) = \log n \) and the above proof for \( (3) \Rightarrow (2) \) of Theorem 4.2.1 shows that the converse is true. Hence we have that \( u \) is a complex Hadamard matrix if and only if \( S(u) = \log n \). See [7, 5.2.2] for examples of complex Hadamard matrices.

### 3.3 Commuting squares and the entropy for unitaries

In this section, let us see how the discussion in the section 3 develops in the framework of \( \Pi_1 \) factors.

For a subfactor \( N \) of a finite factor \( M \), Jones ([6]) introduced the index \([M : N]\), the set of all values of which is \( \{4 \cos^2 \frac{\pi}{n}; n = 3, 4, 5, \cdots \} \cup [4, \infty) \).

The most typical example of orthogonal pairs of subfactors in \( \Pi_1 \) factors will be the followings:

#### 3.3.1 Example of orthogonal pair of subfactors in \( \Pi_1 \) factors.

Let \( \lambda \) be a real number such that

\[
\lambda^{-1} \in \{4 \cos^2 \frac{\pi}{n}; n = 3, 4, 5, \cdots \} \cup [4, \infty),
\]

and let \( \cdots, e_{-1}, e_0, e_1, e_2, \cdots \) be a sequence of projections with the properties:

\[
e_i e_{i=\pm 1} e_i = \lambda e_i, \quad \text{and} \quad e_i e_j = e_j e_i \quad \text{if} |i - j| \geq 2.
\]

Such the sequences of projections appeared in the step of Jones construction of subfactors ([6]).

Let \( M_\infty \) be the von Neumann algebra generated by \( \{e_i | i \in \mathbb{Z}\} \), then \( M_\infty \) is the hyperfinite \( \Pi_1 \) factor, and the unique trace \( \tau \) of \( M_\infty \) is given by the property:

\[
\tau(we_k) = \lambda \tau(w), \quad (w \in \text{alg}\{e_k | j < k\}).
\]

Let \( N \) be the von Neumann subalgebra of \( M_\infty \) generated by \( \{e_i | i < 0\} \), and let \( L \) be the von Neumann subalgebra of \( M_\infty \) generated by \( \{e_i | i \geq 0\} \). Then \( \{N, L\} \) is a mutually orthogonal pair of subfactors in \( M_\infty \).

We remark that \([M_\infty : N] = [M_\infty : L] = \infty\), and we would like to discuss in the framework of subfactors with finite index.
3.3.2 Commuting square condition

From now, assume that $M$ is a type II$_1$-factor. If $N$ is a von Neumann subalgebra of $M$, then we have always the unique faithful normal conditional expectation $E_N : M \to N$. Let $N_1$ and $N_2$ be von Neumann subalgebras of $M$. In a connection with Jones index theory ([6]), the notion of a *mutual orthogonal pair* was generalized by Goodman-Harpe-Jones ([5]) to the notion of a pair satisfying the *commuting square condition*. The diagram

\[
\begin{array}{ccc}
N_1 & \subset & M \\
\cup & & \cup \\
N_3 & \subset & N_2
\end{array}
\]

is said to be a commuting square if

\[E_{N_1}E_{N_2} = E_{N_2}E_{N_1} \quad \text{and} \quad N_3 = N_1 \cap N_2.\]

We say that a pair \( \{N_1, N_2\} \) satisfies the commuting square condition if \( E_{N_1}E_{N_2} = E_{N_2}E_{N_1} \). Of course, the pair \( \{N_1, N_2\} \) satisfies the commuting square condition if \( N_1 = N_2 \). We say a pair \( \{N_1, N_2\} \) is nontrivial if \( N_1 \neq N_2 \), and we are interested in non-trivial pairs of subfactors which satisfy the commuting square condition.

3.3.3 Index 2 subfactors

Here, we replace the notion of *mutual orthogonality* to that of *commuting square condition*, and show that, for a pair of finite index subfactors, the entropy \( S(u) \) plays a key role in our characterization similarly in the section 3.

As the first non-trivial subfactor \( N \) of \( M \), index 2 subfactors appear. The index 2 subfactor is unique up to the conjugacy and it is the biggest subfactor from the point of view of the index theory. By replacing the mutual orthogonality to the commuting square condition, we study the pairs of the biggest subfactors, that is the index 2 subfactors.

Jones picked up the index 2 subfactors \( N \) of \( M \) in [8, Chapter 3] and investigated properties of \( N \cap uNu^* \), where \( u \) is a unitary in \( M \) satisfying some condition. For such a pair \( \{N, u\} \) of the index 2 subfactor \( N \) and \( u \), we characterize the commuting square condition for the pair \( \{N, uNu^*\} \).
Let \( N \) be a subfactor of \( M \) such that \([M : N] = 2\). As Jones showed in [6], \( M \) is decomposed into the crossed product of \( N \) by the group of an outer automorphism \( \alpha \) of \( N \) with the period 2:

\[
M = N \times_\alpha \mathbb{Z}_2.
\]

Then there exists a self-adjoint unitary \( v \) in \( M \) such that

\[
\alpha(x) = vxv^* \quad \text{for all} \quad x \in N
\]

and \( M \) is represented as \( M = N \oplus Nv \). Also we have a projection \( e \) in \( N \) such that

\[
\alpha(e) = e^\perp = 1 - e.
\]

The uniqueness of the trace \( \tau \) of a \( \text{II}_1 \) factor implies that \( \tau(e) = 1/2 \).

Let \( A \) be the von Neumann subalgebra of \( N \) generated by \( e \):

\[
A = Ce \oplus Ce^\perp = Ce \oplus C(1 - e).
\]

Each unitary \( u \) in \( M \) is decomposed into the form

\[
u = u_0 + u_1 v, \quad u_0, u_1 \in N
\]

and

\[
u_0 = E_N(u), \quad u_1 = E_N(uv).
\]

We call the \( \{u_0, u_1\} \) the coefficients of \( u \).

We restrict our attention to the unitaries \( u \in M \) such that the coefficients of \( u \) are contained in the abelian subalgebra \( A \) of \( M \), and we characterized in [3] the commuting square condition for \( \{N, uNu^*\} \) via the entropy \( S(u) \) as follows:

3.3.4 Theorem.

Let \( N \) be a \( \text{II}_1 \) factor and \( \alpha \) be an outer automorphism of \( N \) with the period 2. Let \( M \) be the crossed product \( M = N \times_\alpha \mathbb{Z}_2 \). Assume that \( A \) is the above 2-dimensional subalgebra of \( N \). Then for a unitary \( u \in M \) whose coefficients \( u_0 \) and \( u_1 \) are contained in \( A \), the following conditions are equivalent:

1. \( N \) and \( uNu^* \) satisfy the non-trivial commuting square condition;
2. \( u_j u_j^* = \frac{1}{2} 1_N \), \( j = 0, 1 \);
3. \( S(u) = \log 2 \).
3.3.5 Remark.

1. The notion of the Jones index for subfactors are generalized to some constant for subalgebras by of Pimsner-Popa([14]). Since the above $A$ is a 2-dimensional *-subalgebra of $N$ and $\alpha(A) = A$, it implies by [8, Lemma 3.10] that under the above assumption for the coefficients $\{u_0, u_1\}$ of $u$, that is $\{u_0, u_1\} \subset A$, the von Neumann subalgebra $N \cap uNu^*$ of $M$ is of finite index in the sense of Pimsner-Popa([14]).

2. If $N_1$ and $N_2$ are subfactors with finite index of a type $\text{II}_1$ factor $M$ and if $N_1$ and $N_2$ are mutually orthogonal, then $N_1 \cap N_2 = \mathbb{C}1$ so that the index is infinite: $[M : N_1 \cap N_2] = \infty$.

References


marie@cc.osaka-kyoiku.ac.jp