

## The Fibonacci sequence in nature implies thermodynamic maximum entropy

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### Abstract

The physical and mathematical reasons that could explain the appearance of Fibonacci numbers and the golden ratio in nature had been veiled until 1981, when Yasuichi Horibe submitted “An entropy view of Fibonacci trees” to *Fibonacci Quarterly*. As he used Shannon’s information entropy formula, Horibe’s reasoning, unfortunately, was not considered thermodynamic. In 2011, I tried a new interpretation of Clausius entropy for a non-equilibrium state and conceived of using thermodynamic  $F$  instead of  $W$  in the Boltzmann’s formula for the equilibrium state:  $S = k \log W$ .  $F$  is the conditional probability that implies the total number of possible state shifts at non-equilibrium. Then, I found that  $F$  in a one-dimensional information system, such as “010101...,” produces the Fibonacci sequence as the maximum entropy values. When I presented the mathematical findings on  $F$  at the Fibonacci Meeting of Japan in 2012, Haruo Hosoya found that the calculation of  $F$  was the same as that of the Topological Index (now also called the Hosoya Index), which had been proposed in 1971. This coincidence indicates that the Topological Index might provide the maximum entropy values of molecular surface electrons. The maximum entropy, for life in nature, has these advantages: the full use of resources (the most cost-efficient performance), non-reactiveness (safety), and the largest capacity for data collection (adaptive intelligence). This is the reason that Fibonacci numbers appear often in nature and why designs in nature that follow the golden ratio appeal to us.

### Introduction

A human’s ability to understand nature is strictly limited by its sensory organs. Nature’s image in a human’s mind is a virtual world, reconstructed from sensory data, and is not actual nature itself. The region of the brain responsible for feelings produces an emotional response to sensory data soon after they are acquired. In contrast, the region of the brain responsible for rational thinking prefers to segregate the sensory data into elementary parts and basic relations that can be used to reconstruct the virtual image. As the brain’s operation is restricted by the poor function of the neural network in a volume space of 1.5 liters, a human’s rational brain is good at simplifying complex relations, but poor at understanding complex relations as they are. Although I do not know of any scholarly argument about the interpretation difficulty, the bias for mixed gas might be considered as one example.

Let there be 10 liters of air, composed of nitrogen (78%mol), oxygen (21%mol), and argon (1%mol) at 1 atm and 298 K (room temperature). As we cannot know the true pressures and volumes of these gases, we must fix pressure or volume in order to simplify the situation. We tend to think that each gas has the same fixed volume of 10 liters and different partial pressures. However, it is possible to reason that each gas has a different partial volume and the same fixed pressure of 1 atm. The latter simplification is consistent with Blaise Pascal’s law of equal pressure. The values for both the simplified combinations of pressure and volume would then be significantly different from actual values. In fact, I am not sure Argon (only 1%mol) has a volume of 10 liters or a pressure of 1 atm. Therefore, such a simplification bias is carried out to make scientific calculations easy, simple, and practical. A human’s analytical

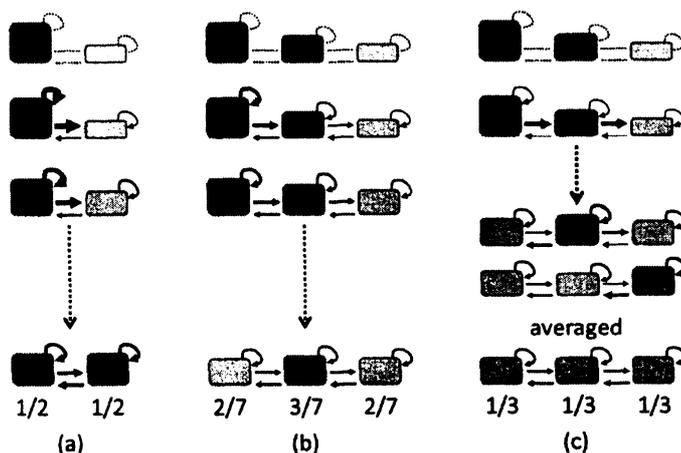
understanding of nature cannot exclude this kind of simplification bias.

We can say that the same kind of bias exists for the statistic interpretation of Clausius thermodynamic entropy. The “principle of equal a priori probability,” postulated by Gibbs, is one of such simplifications. This postulation correctly implies (almost) equal state-existence probability (EP) for all accessible microstates of a system at equilibrium. The word “equilibrium” implies that although the system tends to keep the same state in macro-scales, it has continuous dynamic shifts from one microstate to another. As volume is a counter-factor of pressure in the mixed-gas problem, similarly equal state-shift probability (SP) for all successive microstates is a counter of equal EP. However, Boltzmann and Gibbs used the simplification of equal EP at equilibrium without making a clear reference to equal SP.

However, I found that under a very specific condition at non-equilibrium, we can use both types of simplification: equal SP and equal EP. By using these simplifications, we can count the total number of accessible microstates at non-equilibrium and the total number of possible state shifts at non-equilibrium. Both numbers have the same value, although I refer to the value as “ $F$ ” instead of “ $W$ ,” which Boltzmann used for equilibrium. The concept of  $F$  is far more easy and simple to understand than the concept of  $W$ . The  $F$  interpretation of the thermodynamic entropy will make it easier for even junior-high school students to learn the concept of dissipative structures that are essential to understand nature.

#### Thermodynamic $F$ instead of $W$

Let us consider an example of state shifts from non-equilibrium to equilibrium (Fig. 1). Fig. 1(a) shows a system that has two states. Each state has two optional paths: the shift to itself and the shift to the other. All values of SP from the same state must be equal; this is a postulation. First, for example, the EP of the left state is 0.9 and the EP of the right state is 0.1. Then, the flow (state shift) from the left to the right must be larger than the counter-flow. This non-equal probability shift will continue until the system has equal SP for all states. This simple system has equal SP and equal EP at equilibrium.

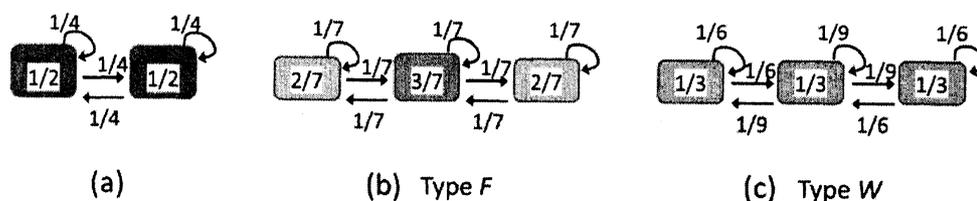


**Fig. 1. Two types of equilibrium for easy counting.**

The darkness and height of each state indicates a degree of EP. The largeness of each arrow from a state indicates the degree of SP.

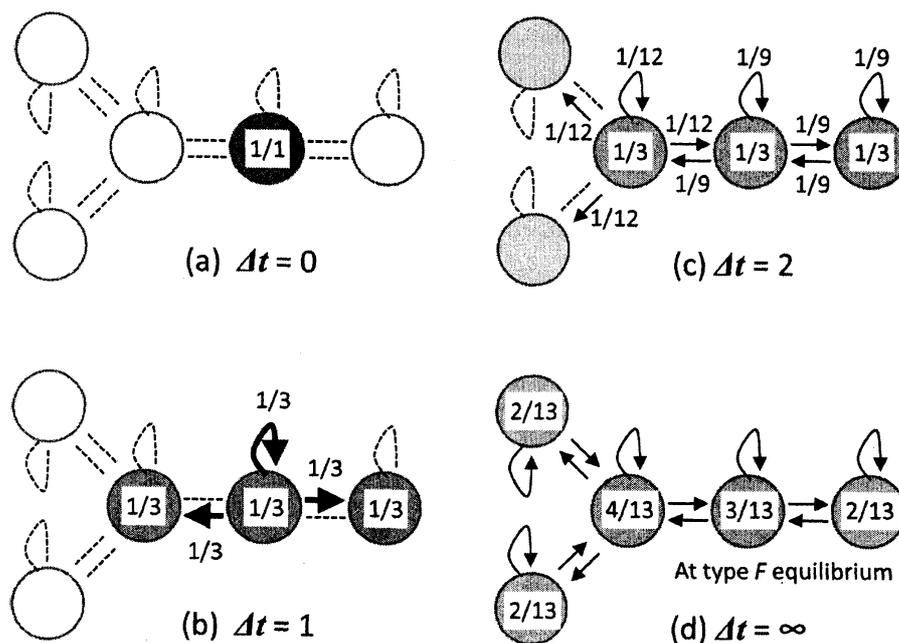
Figs. 1(b) and 1(c) show a rather complex system composed of three states. This system

has two types of equilibrium. Fig. 1(b) shows the equilibrium with equal flow and non-equal existence (type *F* equilibrium). Fig. 1(c) shows the equilibrium with non-equal SP and equal EP (type *W* equilibrium). Fig. 2 shows SP and EP values of both equilibrium types. As is easy to see, the SP values of type *W* equilibrium are not well balanced. This implies that we can get equal EP values just as an averaged probability, as shown in Fig. 1(c). To keep equal EP values intentionally, the system needs a statistical enhancement of flow. Use of the equilibrium model of type *W* is very limited, but it was enough for Boltzmann to easily count the total number of accessible states of a system at equilibrium.



**Fig. 2. Calculation of SP.**

When I was examining some asymmetric systems composed of several states, I found a very specific stage where both equal shift and equal existence are relevant. I let any shift take 1 time unit ( $1\Delta t$ ) and calculated conditional probabilities of flows and states step by step. Fig. 3 displays one such experiment.



**Fig. 3. Conditional probability at non-equilibrium.**

As Fig. 3(b) shows, just at the moment when any change starts to occur in a system, we can think both of equal SP and equal EP, and both probabilities have the same value. The total number of flow branches of Fig. 3(b) is three, and the total number of accessible states is also three. Probability of an arrow is  $1/3$  and probability of a state is also  $1/3$ . This is a very simple relation that everyone can recognize. I call this number three *F* instead of *W*.

Let me present a clear definition of  $W$  and  $F$ . Boltzmann's  $W$  implies the "total number of accessible microstates for a system at equilibrium" under the postulation of the "principle of equal probability of existence."  $F$  implies "total number of shifts (flows, paths, branches, changes,...) from a microstate of a system at non-equilibrium to its accessible (successive) microstates within a moment of the system" under postulation of the "principle of equal probability of shift and existence." This number is the same as the "total number of 'branchable' (accessible, 'shiftable',...) microstates for a system within a moment of the system at non-equilibrium" under the same postulation. Further, "a moment ( $1\Delta t$ )" is a minimal time unit of the system. Later, it will be shown that  $\Delta t$  implies  $\Delta F$ .

Next, I started to examine if  $F$  can work well as the entropy of a system at non-equilibrium and at equilibrium.

### $F$ entropy of a one-dimensional information system

For the first experiment, I used a one-dimensional information system such as "ABABBBAABCC," where several kinds of elements arranged themselves in a line. I assumed that all elements are conserved, exchange of neighboring elements could occur within one unit time of the system (1 s here), and possibilities of all exchanges are equal. Then, I counted how many different arrangements the system can have in 1 s. Obviously, exchange of the same kind of element is as good as the nonoccurrence of the exchange. It is essential to find the axes for exchange and the count combination of axes.

For example, "ABABBBAABCC" is "A|B|A|BBB|AA|B|CC," and it can be thought of as "|||×|×||" ("|" implies an axis for exchange, and "×" implies non-interference between neighboring axes). "|||" can be of five kinds of combinations, such as "nnn," "|nn," "|n|," "n|n," and "nn|" ("n" implies no exchange). "|" can be of two kinds: "n" and "|." "||" can be of three kinds, which are "nn," "|n," and "n|." Hence, the "|||×|×||" can have  $5 \times 2 \times 3 = 30$  kinds of combinations. The  $F$  entropy of the "ABABBBAABCC" is 30.

I noticed that a configuration pattern such as "010101..." has the maximum  $F$  entropy. I counted from "01" to "0101010" and obtained the sequence "2, 3, 5, 8, 13, and 21." After some investigation, I could determine this to be the same as the Fibonacci sequence.

Although my knowledge in mathematics and physics is limited, my knowledge in medicine and biology is sufficient to understand the meaning of the result of the first experiment on  $F$  entropy. The reason for the many appearances of the Fibonacci sequence in nature is quite clear: The Fibonacci sequence implies maximum thermodynamic entropy.

Life in nature has many kinds of one-dimensional information systems. DNA is a one-dimensional sequence of nucleotides, and a protein is a one-dimensional sequence of amino acids. Oral language is a one-dimensional sequence of voices.

Fibonacci numbers are often shown in reproduction, an essential function for life. This implies that life uses a one-dimensional information system for reproduction. A one-dimensional information system is primitive, basic, tough, easy to maintain, inexpensive to replicate, and reliable, as compared to two- or three-dimensional information systems.

Later, I learned that the calculation method for  $F$  entropy is the same as the one for the Hosoya Index (combinations of non-neighboring bonds that connect vertices of molecular graphs), and Horibe had proved that the Fibonacci sequence implies maximum informational entropy. This knowledge leads us to the concept that probability can be used to integrate the

Hosoya Index of chemistry, the entropy of thermodynamics, and the entropy of information theory.

Thus, the successful explanation of the Fibonacci numbers in nature by  $F$  entropy demands reconsideration of (statistical) thermodynamics today.

When I found  $F$  entropy, I was not confident of its scientific orthodoxy, so to speak. As I was a layman, the restriction due to the second law of thermodynamics was a severe encumbrance to my investigation. While  $W$  entropy by Boltzmann needs the second law to explain irreversibility in spontaneous processes,  $F$  entropy does not need the law, and furthermore, it denies the classical concept of the law.

However, the pure probable concept of entropy illustrated in several books by Arieh Ben-Naim freed me completely from the irrational approach to entropy. Everything is constantly changing. Entropy is just a possible degree of randomness in such a change. Entropy is first a problem of probability than anything else. The  $F$  entropy hypothesis was encouraged by Ben-Naim, but it has not been validated by him as yet. The following is my idea of  $F$ , based on the  $F$  entropy hypothesis described above.

### Entropy [rule] and entropy [value]

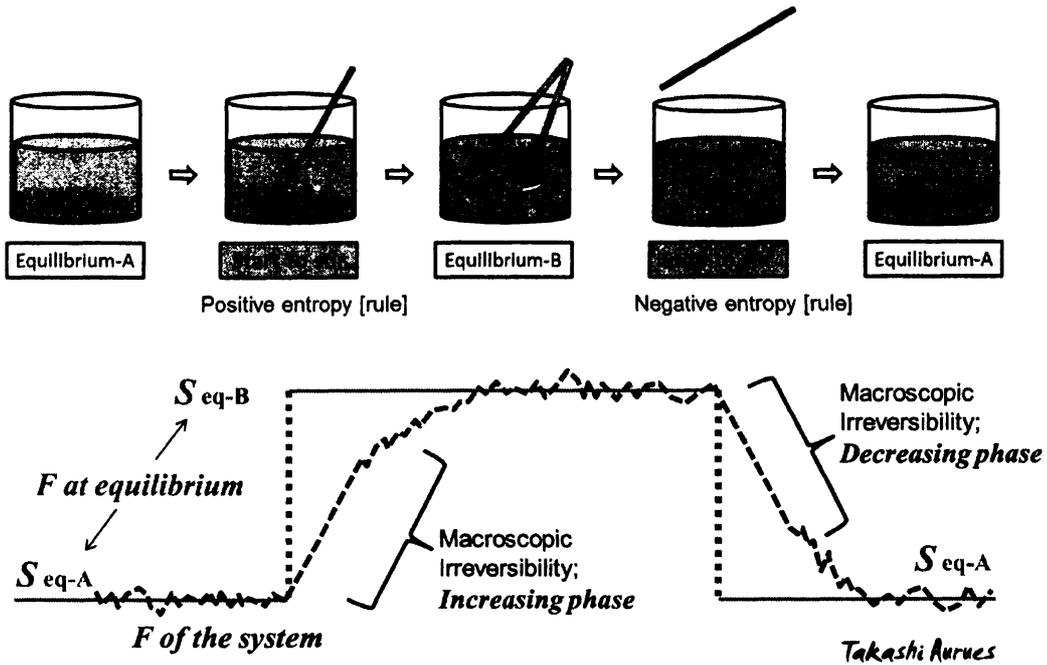
Entropy [rule] of a system is defined as restrictive rules for randomness of the system change. Entropy [value] of the system is defined as a degree of randomness determined by the entropy [rule] of the system. The positive entropy [rule] of a system is defined as a change on the entropy [rule] of the system that increases the entropy [value] of the system. The negative entropy [rule] of the system is defined as a change in the entropy [rule] of the system that decreases the entropy [value] of the system. If any change in the entropy [rule] occurs, a change in entropy [value] follows. Human's rational brain prefers to think analytically in this manner. You will soon realize that these terms are very useful.

### Entropy in macro-scales

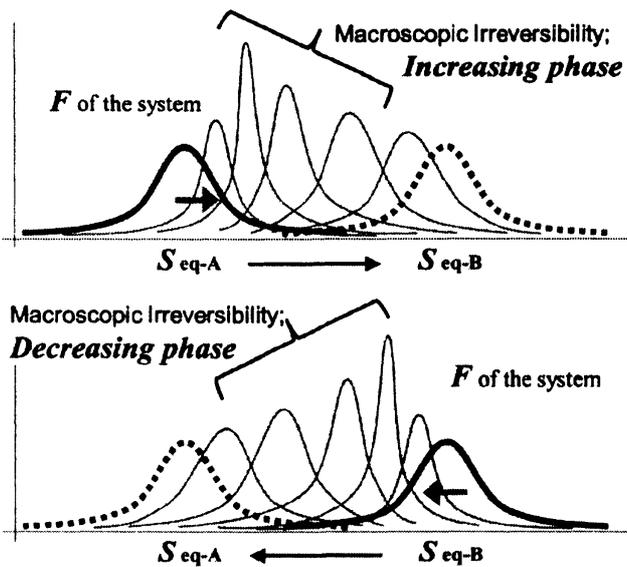
In any spontaneous change, entropy in macro-scales not only increases, but also decreases irreversibly. Let me directly discuss the second law of thermodynamics. An illustration of a potential heavy-particle-stir experiment is given in Fig. 4. Imagine that you stir up heavy particles at the bottom of a beaker that is two-thirds full of water.

This system has two levels of equilibrium (eq-A and eq-B). Stirring is the positive entropy [rule] of the system. While water is being stirred, the entropy [value] at the equilibrium of the system is  $S_{\text{eq-B}}$ . To stop stirring is the negative entropy [rule] of the system. When you stop stirring, the entropy [value] at equilibrium is  $S_{\text{eq-A}}$ . These are matters of equilibrium with the classical symbol  $S$  used for the entropy [value] at equilibrium, as shown in Fig. 4.

However, the actual change in the particles progresses step by step. You will recognize two kinds of irreversibility shown in Figs. 4 and 5. The direction of entropy change must not be fixed in the second law. If the system is at a lower entropy [value] than the entropy [value] that the entropy [rule] of the system determines at equilibrium, the actual entropy [value] of the system increases macro-irreversibly. Further, if the system is at a higher entropy [value] than the entropy [value] at the equilibrium of the system, its actual entropy [value] decreases macro-irreversibly.



**Fig. 4. A stirring experiment showing macroscopic irreversibility.**



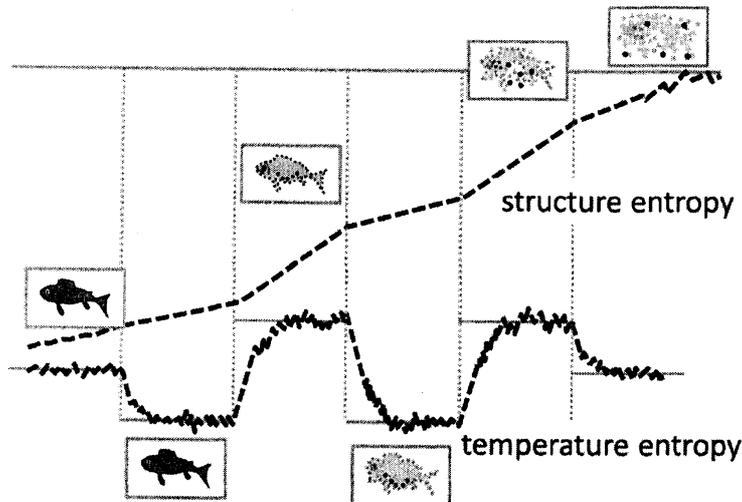
**Fig. 5. Increasing and decreasing macroscopic irreversibility.**

In nature, there are many different kinds of entropy. This easy example will help you understand entropy very well.

A fish in soup has temperature and structure entropy

Put some fresh fish into soup at room temperature. Then, cool it down, boil it up, cool it down, and boil it up again. Finally, let it cool down to room temperature. Fig. 6 shows the change in temperature entropy and structure entropy of the fish. Instead of abolishing the second law, I

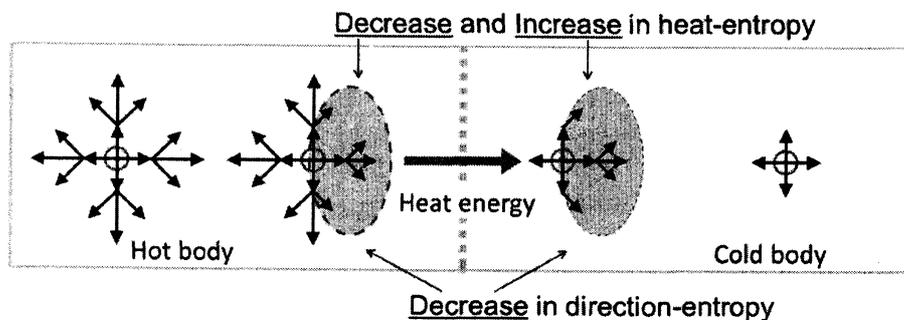
would like to attempt to reform it by re-examining the relationship between energy and entropy.



**Fig. 6. Temperature and structure entropy of fish in soup.**

#### Thermodynamic and elementary entropy

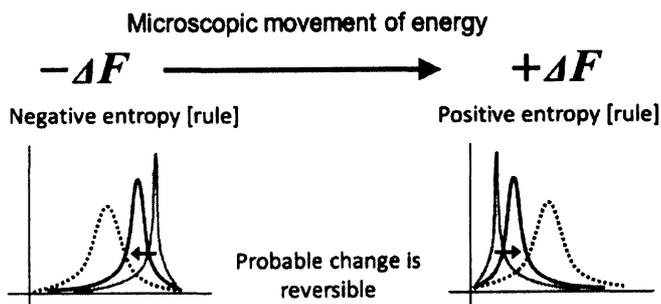
Thermodynamic entropy is a complex of some kinds of elementary entropy. Fig. 7 shows thermal movements and directions of mass points schematically. At the contact surface between a hot body and a cold body, heat energy starts to move from the hot body to the cold one irreversibly. Fig. 7 displays the heat-entropy change and direction-entropy change of mass points. Classical Clausius entropy is a complex of several kinds of entropy. Hosoya entropy is one of such entropies.



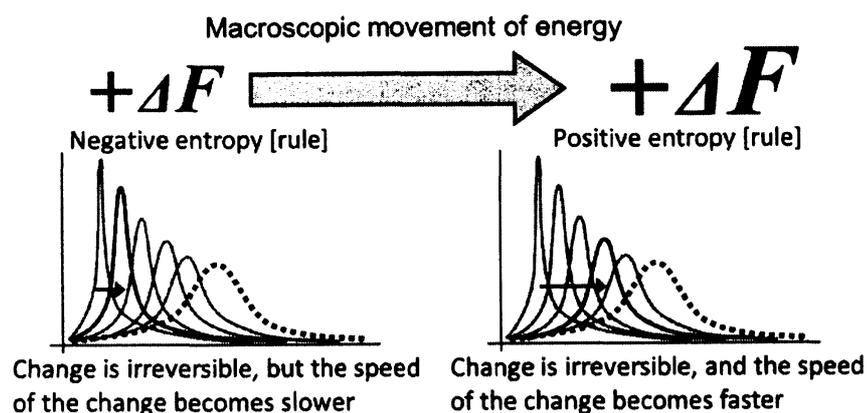
**Fig. 7. Movement of heat energy and change in entropy.**

#### An energy movement carries probability

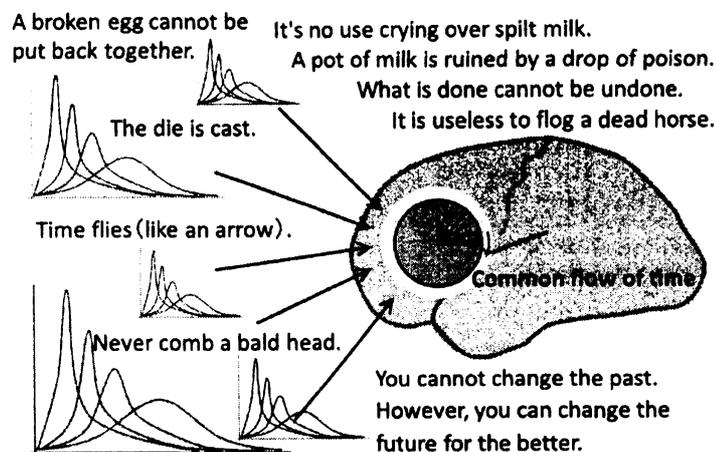
Fig. 7 suggests any energy movement from one place to another changes both the entropy [rules] at the old place and the new. This seems to suggest that energy carries entropy. Fig. 8 shows the change in entropy from a microscopic energy movement, and Fig. 9 shows an entropy change from a macroscopic energy movement.



**Fig. 8. Entropy [rule] change from a microscopic movement of energy.**



**Fig. 9. Entropy [rule] change from a macroscopic movement of energy.**

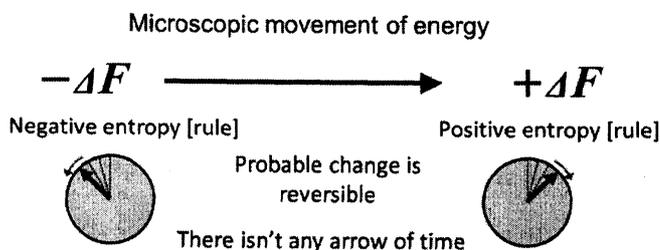


**Fig. 10. The illusory arrow of time.**

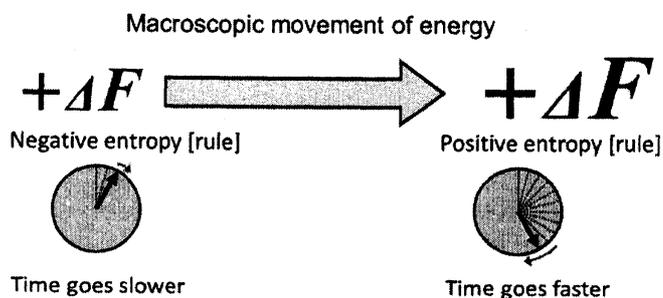
All elementary changes in entropy are reversible. In macroscopic energy movement, several microscopic reverse changes in entropy are covered by larger-sized microscopic counter-changes in entropy. All changes are averaged to make a macroscopic irreversible change. As Arieh Ben-Naim says, Boltzmann first understood this property to establish statistical thermodynamics.

Time-dependent irreversibility makes a virtual clock in a human's brain. The common flow of time in space ("arrow of time") is just an illusion (Fig. 10). Hence,  $\Delta F$  of a system is  $\Delta t$  of the system.

Fig. 11 is a revised Fig. 8, and Fig. 12 is a revised Fig. 9 using the clock model as the entropy change.



**Fig. 11. Time comes and goes by microscopic movement of energy.**



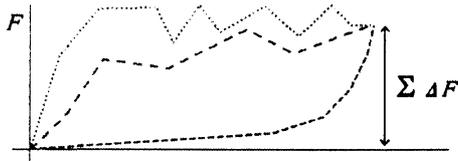
**Fig. 12. Time goes slower and faster by the macroscopic movement of energy.**

Entropy is as conservative as energy, but in a different manner

As has been shown above, the change in the amount of entropy ( $\Delta F$ ) has a close relationship with the amount of energy movement that has caused the entropy change. The laws of thermodynamics could be rewritten as the following:

First law:

1. In any system, there is a conservative quantity that never increases or decreases before or after any qualitative change of the system. We call it "energy." (Energy is space restricted, probably by entropy [rule].)
2. In any system, there is a quality that has a conservative quantity that *the total amount of quantitative changes* ( $\Sigma\Delta F$ ) of the system is kept constant, independent of any process of the system. We call it "entropy" (Fig. 13).
3. Energy and entropy are classified. Exchange between different types of energy or between different types of entropy is restricted. The amount of entropy that energy carries has a quantitative relationship with the type of energy and the type of entropy.



**Fig. 13. Sum of  $\Delta F$  is conserved independent of process.**

Second law:

1. In a system composed of micro-reversible changes, an improbable condition transfers spontaneously and irreversibly in macro-scales to the probable condition of the system. (What is probable is apt to occur. This macro-irreversibility is a statistical property of the system.)

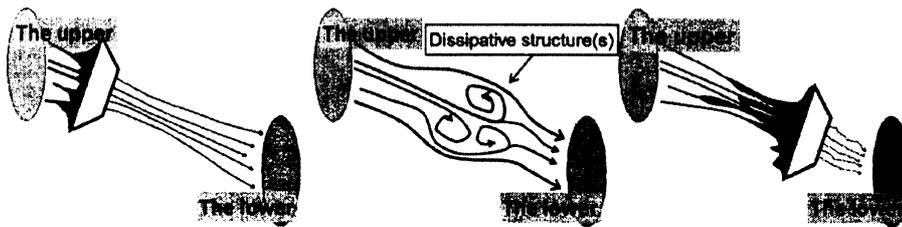
2. No movement of energy without friction against the movement. (Energy carries entropy. Any movement of energy accompanies friction through which entropy of the energy transfers to other energy. Friction here has wide meaning beyond mechanical friction. Friction is the relationship with surroundings. Without any relationship with the world, an object cannot exist in the world. Any existence in a world has friction with the world. Some structures may collapse due to friction, but other structures may grow due to friction, such as a typhoon.)

3. Any movement of energy is absolutely irreversible. (It is impossible to measure this absolute irreversibility by friction as we are a part of the universe.)

Movement of energy, transfer of entropy, and friction are different aspects of the same change. Now, we are ready to understand dissipative structure(s). Life in nature is representative of a dissipative structure. If the Fibonacci sequence implies maximum entropy, it is very natural for life to express Fibonacci numbers in its structures.

### Thermodynamic mechanism of dissipative structures

The easiest dissipative structure to understand is a whirlpool in a river (Fig. 14). If water flow from the upper reaches of the river stops, or if water flow to the lower reaches of the river is stopped, the whirlpool disappears. This is the fate of any dissipative structure.



**Fig. 14. A whirlpool in a river disappears without water flow.**

It is interesting to ask why Clausius gave attention only to the irreversible increase in entropy. The reason is simple. We live in the universe at non-equilibrium, which is continuously changing toward a higher entropy state of equilibrium. In such an entropy-increasing-type space, micro-energy movement that increases entropy of the space occurs much more regularly than micro-energy movement that decreases entropy of the space. Almost all inorganic changes appear to increase the entropy of the world except for a few reactions that decrease entropy. Without such exceptions, cannot life exist in an entropy-increasing-type space?

Yes! A dissipative structure is a formation that keeps its general appearance by discarding its rising entropy. Fig. 15 shows the mechanism by which a dissipative structure takes away excess entropy auto-produced in its body. Fig. 16 is an alternative representation of Fig. 14.

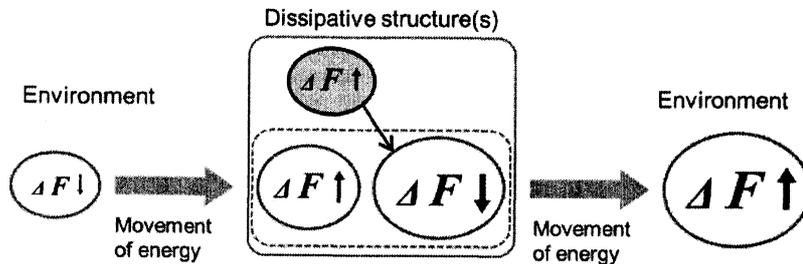


Fig.15. Mechanism of discarding entropy from a body.

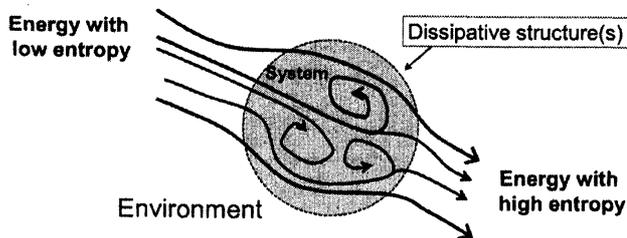


Fig. 16. A dissipative structure exists only in energy flow.

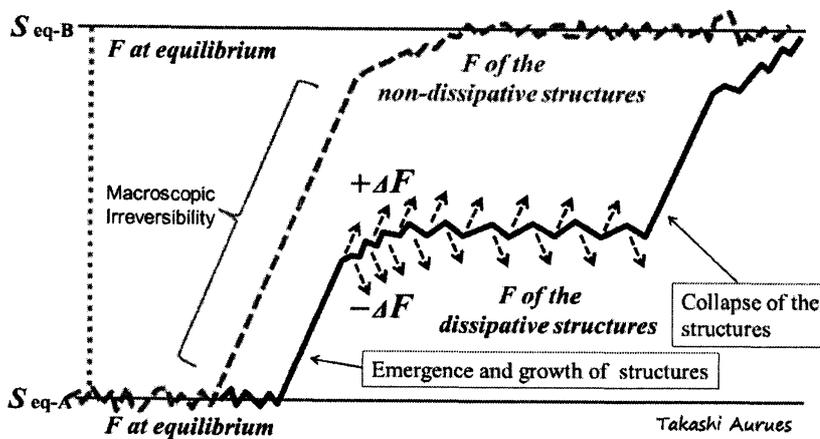
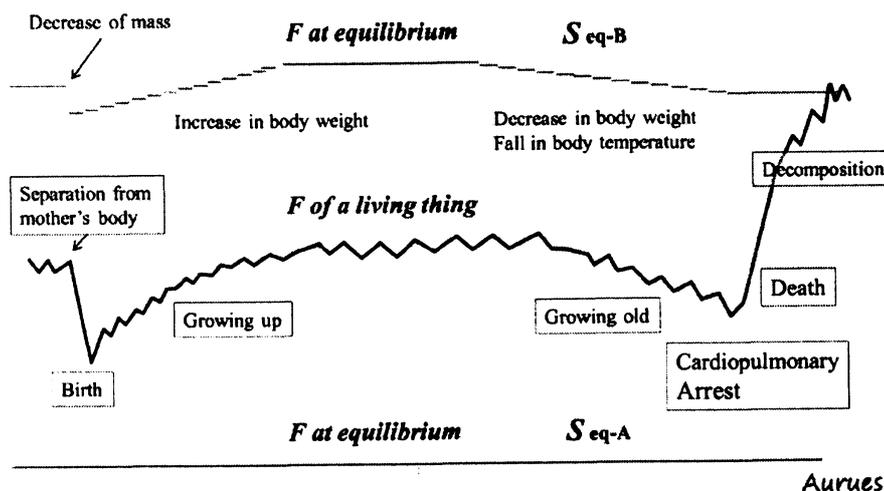


Fig. 17. A dissipative structure exists only in irreversibility.

Now you can understand your whole life (Figs. 17 and 18). Not only life but also almost all of existence and all phenomena in our universe are dissipative structures. Everything that has an end is a dissipative structure. Hence, even a positron is a dissipative structure. Nations are also dissipative structures. Further, what is important is that Charles Darwin's evolutionary theory can be applied to all dissipative structures. Without evolutionary change, no dissipative structure can continue for long. Buddha might have been the first person who realized the fate of a dissipative structure.



**Fig. 18. Sample life cycle of a human.**

The final postulation of my idea of  $F$  might be paradoxical. All changes in nature occur “spontaneously.” What occurs next is definitely determined, but in a probable manner. Hence, it would seem that Pierre-Simon Laplace’s demon of scientific determinism can live with dice.

### Acknowledgments

Special thanks to: Shigeru Nakamura, who invited me to the Fibonacci Meeting of Japan on 23th March 2012; Arieh Ben-Naim, who liberated me from the second law; Haruo Hosoya, who found the common property between the Topological Index and  $F$  entropy; and Masumi Kawasaki, who not only understood the important relationship between the Fibonacci sequence and entropy, but also introduced me to Shigeru Furuichi, who made a rediscovery of Horibe’s achievements and invited me to this meeting at RIMS in Kyoto University.

### References

- [1] H. Hosoya “Topological Index. A Newly Proposed Quantity Characterizing the Topological Nature of Structural Isomers of Saturated Hydrocarbons” *Bulletin of the Chemical Society of Japan*, 44 (1971) 2332-2339.
- [2] Y. Horibe “An Entropy View of Fibonacci Trees” *The Fibonacci Quarterly* 20 (1982) 168-178
- [3] Y. Horibe “Notes on Fibonacci Trees and Their Optimality” *The Fibonacci Quarterly* 21 (1983) 118-128
- [4] Y. Horibe “Entropy of Terminal Distributions and the Fibonacci Trees” *The Fibonacci Quarterly* 26 (1988) 135-140
- [5] Arieh Ben-Naim “Entropy Demystified” World Scientific Pub Co Inc.; Expanded edition (2008)
- [6] Arieh Ben-Naim “A Farewell to Entropy” World Scientific Pub Co Inc. (2008)
- [7] Arieh Ben-Naim “Discover Entropy and the Second Law of Thermodynamics” World Scientific Pub Co Inc. (2009)
- [8] Arieh Ben-Naim “Entropy and the Second Law: Interpretation and Misinterpretations” World Scientific Pub Co Inc. (2012)
- [9] Kyoto20121113Aurues.v04.pdf and .ppt: used at the workshop. (*Japanese*)
- [10] 中村滋著「フィボナッチ数の小宇宙 (改訂版)」日本評論社, 2008年第2版
- [11] 細矢治夫著「トポロジカル・インデックス. フィボナッチ数からピタゴラスの三角形までをつなぐ新しい数学」日本評論社, 2012年
- [12] 堀部安一著「情報エントロピー論 (第2版)」森北出版, 1997年第2版, 「付録: 黄金比とエントロピー」
- [13] 堀部安一著「黄金比とエントロピー」数理科学 No.294, サイエンス社, 1987年
- [14] アリー ベン・ナウム著, 中嶋一雄訳「エントロピーがわかる」講談社ブルーバックス B-1690, 2010年

### Appendix

$F = 2^{F_s}$ . So  $F_s = \log_2 F$ , where  $F_s$  is entropy computed using Shannon’s formula on shift probability. This equation implies thermodynamic entropy is information entropy.