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Structure and stability of stationary solutions to a cross-diffusion equation

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1 Introduction

This is a joint project with Yuan Lou (Ohio State University), Wei.-Ming Ni (University of Minnesota and East China Normal University) concerning mathematical analysis, and Masaharu Nagayama (Hokkaido University), Tatsuki Mori (Ryukoku University) concerning numerical computation.

In an attempt to model segregation phenomena in population dynamics, Shigesada, Kawasaki and Teramoto [7] in 1979 incorporated the inter-competition system. In particular, the following system was proposed

$$\begin{cases} u_t = \Delta[(d_1+\rho_{11}u+\rho_{12}v)u]+u(a_1-b_1u-c_1v), & \text{in } \Omega\times(0,\infty), \\ v_t = \Delta[(d_2+\rho_{21}u+\rho_{22}v)v]+v(a_2-b_2u-c_2v), & \text{in } \Omega\times(0,\infty), \\ \dfrac{\partial u}{\partial n}=\dfrac{\partial v}{\partial n}=0, & \text{on } \partial\Omega\times(0,\infty), \\ u(x,0)=u_0(x), v(x,0)=v_0(x), & \text{in } \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain $R^N (N\geq 1)$ with smooth boundary $\partial\Omega$. Here u and v represent the densities of two competing species. The constants a_j, b_j, c_j and d_j $(j=1,2)$ are all positive, where a_1, a_2 denote the intrinsic growth rates of these two species, b_1 and c_2 account for intra-specific competitions while b_2, c_1 account for inter-specific competitions, and d_1, d_2 are their diffusion rates. The constants ρ_{11}, ρ_{22} represent intra-specific population pressures, also known as self-diffusion rates, and ρ_{12}, ρ_{21} are the coefficients of inter-specific population pressures, also known as cross-diffusion rates.

For convenience, we set $A:=a_1/a_2$, $B:=b_1/b_2$, $C:=c_1/c_2$. If $B<C$, we call it the strong competition case and $B>C$ the weak competition case.

If $\rho_{11}=\rho_{12}=\rho_{21}=\rho_{22}=0$, then (1.1) is the classical Lotka-Volterra competition diffusion system with Neumann boundary condition

$$\begin{cases} u_t = d_1\Delta u+u(a_1-b_1u-c_1v), & \text{in } \Omega\times(0,\infty), \\ v_t = d_2\Delta v+v(a_2-b_2u-c_2v), & \text{in } \Omega\times(0,\infty), \\ \dfrac{\partial u}{\partial n}=\dfrac{\partial v}{\partial n}=0, & \text{on } \partial\Omega\times(0,\infty), \\ u(x,0)=u_0(x), v(x,0)=v_0(x), & \text{in } \Omega. \end{cases} \tag{1.2}$$

It is well known that in the "weak competition" case, i.e.

$$B > A > C,$$

the constant steady state $(u_*, v_*) = \left(\frac{a_1c_2-a_2c_1}{b_1c_2-b_2c_1}, \frac{b_1a_2-b_2a_1}{b_1c_2-b_2c_1}\right)$ is globally asymptotically stable regardless of the diffusion rates d_1 and d_2. This implies, in particular, that no nonconstant steady state can exist for any diffusion rates d_1, d_2.

On the other hand, it seems not entirely reasonable to add just diffusions to models in population dynamics, since individuals do not move around completely randomly. In particular, while modeling segregation phenomena for two competing species one must take into account the cross-diffusion pressures

$$\begin{cases} u_t = \Delta[(d_1 + \rho_{12}v)u] + u(a_1 - b_1u - c_1v), & \text{in } \Omega \times (0, \infty), \\ v_t = \Delta[(d_2 + \rho_{21}u)v] + v(a_2 - b_2u - c_2v), & \text{in } \Omega \times (0, \infty), \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0, & \text{on } \partial\Omega \times (0, \infty), \\ u(x, 0) = u_0(x), v(x, 0) = v_0(x), & \text{in } \Omega. \end{cases} \tag{1.3}$$

Mimura and his collaborators started mathematical analysis around 1980 (see, e.g. Mimura [4]). Considerable work has been done concerning the global existence of solutions to systems (1.3) under various hypotheses. A priori estimates are crucial to obtain the global existence. As for recent progress including stationary problems, see Ni [5], Ni [6], Yagi[9] and Yamada [10].

2 Limiting equation

We first focus on the effect of cross-diffusion on steady states. To illustrate the significance of cross-diffusions, we again go to the weak competition case (i.e. $B > A > C$) since in this case (1.3) has no nonconstant steady states if both $\rho_{12} = \rho_{21} = 0$. Lou–Ni [1],[2] show that, indeed, if one of the two cross-diffusion rates, say ρ_{12}, is large, then (1.3) will have nonconstant steady states provided that d_2 belongs to a proper range. On the other hand, if both ρ_{12} and ρ_{21} are small, then (1.3) will have no nonconstant steady states under the condition $B > A > C$. This shows the cross-diffusion does seem to help create patterns.

In the strong competition case, i.e. $B < A < C$, even the situation of steady states solutions of (1.2) becomes more interesting. Cross-diffusion still have similar effects in help creating nontrivial patterns of (1.3).

The following two theorem are due to Lou-Ni [1], [2].

Theorem 2.1 *([2]) Suppose for simplicity that $\rho_{21} = 0$. Suppose further that $B \neq A \neq C$, $n \leq 3$ and $\frac{a_2}{d_2} \neq \lambda_k$ for any $k \geq 1$, where λ_k is the kth eigenvalue of $-\Delta$ on Ω with zero Neumann boundary data. Let (u_j, v_j) be a nonconstant steady state solution of (1.3) with $\rho_{12} = \rho_{12,j}$. Then by passing to a subsequence if necessary, either (i) of (ii) holds as $\rho_{12,j} \to \infty$:*

(i) $(u_j, \frac{\rho_{12,j}}{d_1} v_j) \to (u, v)$ uniformly, $u > 0$, $v > 0$, and

$$\begin{cases} d_1 \Delta[(1+v)u] + u(a_1 - b_1 u) = 0 & \text{in } \Omega, \\ d_2 \Delta v + v(a_2 - b_2 u) = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases}$$

(ii) $(u_j, v_j) \to (\frac{\tau}{v}, v)$ uniformly, τ is a positive constant, $v > 0$, and

$$\begin{cases} \displaystyle\int_\Omega \frac{\tau}{v}\left(a_1 - b_1 \frac{\tau}{v} - c_1 v\right) \mathrm{d}x = 0, & \\ d_2 \Delta v + v(a_2 - c_2 v) - b_2 \tau = 0 & \text{in } \Omega, \\ \dfrac{\partial v}{\partial n} = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

Their proofs of obtaining the above limiting equations are quite hard and lengthy. The most important step in the proof is to obtain a priori bounds on steady states of (1.3) that are independent of ρ_{12}.

It seems from numerical computations that solutions of the case (i) is not directly related with stable solutions of the original equation with sufficiently large ρ_{12}. However, we observe numerically that solutions of the case (ii) is closely related with the original equation with sufficiently large ρ_{12}.

Thus, we will concentrate on the case (ii). Now, we consider the 1-dimensional case with $\Omega = (0, 1)$. The limiting equation becomes as follows:

$$\begin{cases} \displaystyle\int_0^1 \frac{1}{v}\left(a_1 - b_1 \frac{\tau}{v}\right) \mathrm{d}x - c_1 = 0, \\ d_2 v_{xx} + v\left(a_2 - b_2 \frac{\tau}{v} - c_2 v\right) = 0, \quad \text{in } (0,1), \\ v_x(0) = v_x(1) = 0, \\ v > 0, \quad \text{in } (0,1). \end{cases} \tag{2.2}$$

3 Structure and stability in 1-dimensional case

Due to the scaling and reflection properties of solutions to autonomous ordinary differential equations, all solutions to the (2.2) are obtained by several reflections and a suitable re-scaling from solutions of the following system:

$$\begin{cases} \displaystyle\int_0^1 \frac{1}{v}\left(a_1 - b_1\frac{\tau}{v}\right) dx - c_1 = 0, \\ d_2 v_{xx} + v\left(a_2 - b_2\frac{\tau}{v} - c_2 v\right) = 0 \ \text{ in } (0,1), \\ v_x(0) = v_x(1) = 0, \\ v > 0, \ \text{ and } \ v_x > 0, \ \text{ in } (0,1). \end{cases} \tag{3.1}$$

Now, we will discuss about the structure of stationary solutions and their stability.

This system (3.1) consists of a nonlinear elliptic equation and an integral constraint. As far as existence and non-existence in one dimensional domain are concerned, Lou-Ni-Yotsutani [3] obtained nearly complete knowledge. They also obtained the precise qualitative behavior of solutions to this limiting system as the diffusion rate varies.

Their basic approach is to convert the problem of solving the system to a problem of solving its "representation" in a different parameter space. This is first done *without* the integral constraint, and then they use the integral constraint to find the "solution curve" in the new parameter space. This turns out to be a powerful method as it gives fairly precise information about the solutions.

We have recently made clear the remained delicate parts due to the explicit representation by elliptic functions.

We summarized the structure of solutions of (3.1). We concentrate on the case

$$B < C \quad \text{(strong competition case)}.$$

The following two theorem are due to [3].

Theorem 3.1 (Existence) *Suppose that $B < C$. If*

$$\max\left\{0, \frac{B+C-2A}{C-B}\right\} \frac{a_2}{\pi^2} < d_2 < \frac{a_2}{\pi^2},$$

then there exists a solution $(v(x), \tau)$ of (3.1).

Theorem 3.2 (Nonexistence) *Suppose that $B < C$.*

(i) If $d_2 \geq \frac{a_2}{\pi^2}$, then there exists no solution of (3.1).

(iii) If $A < B$, there exists no solution of (3.1).

(iii) If $B \leq A < \frac{B+C}{2}$, then there exists a $d_2^* = d_2^*(A, B, C, a_2) > 0$ such that there exists no solution of (3.1) for $d_2 \in (0, d_2^*]$.

We see that the above theorem is sharp by the following theorems. The existence region depending on the the ratio C/B. The situation drastically changes at $C/B = 7/3$.

Theorem 3.3 *Suppose that $B < C \leq 7B/3$. (3.1) has a solution $(v(x), \tau)$ if and only if d_2 satisfies*

$$\max\left\{0, \frac{B+C-2A}{C-B}\right\} \frac{a_2}{\pi^2} < d_2 < \frac{a_2}{\pi^2}.$$

Moreover, the solution is unique.

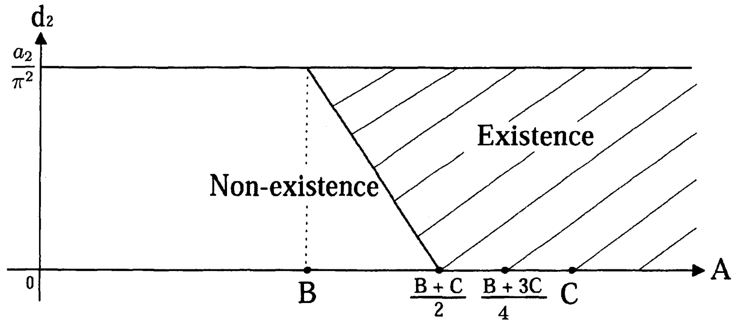

Figure 3.1: Case $B < C \leq 7B/3$

Theorem 3.4 *Suppose that $7B/3 < C$. (3.1) has the unique solution $(v(x), \tau)$ if*

$$\max\left\{0, \frac{B+C-2A}{C-B}\right\} \frac{a_2}{\pi^2} < d_2 < \frac{a_2}{\pi^2}.$$

Moreover, there exists the only one connected non-empty open set D with

$$D \subset \left\{(A, d_2) : B < A < \frac{B+C}{2},\ 0 < d_2 < \left\{\frac{B+C-2A}{C-B}\right\} \frac{a_2}{\pi^2}\right\}$$

such that (3.1) has exactly two solutions $(v(x), \tau)$ if and only if $d_2 \in D$.

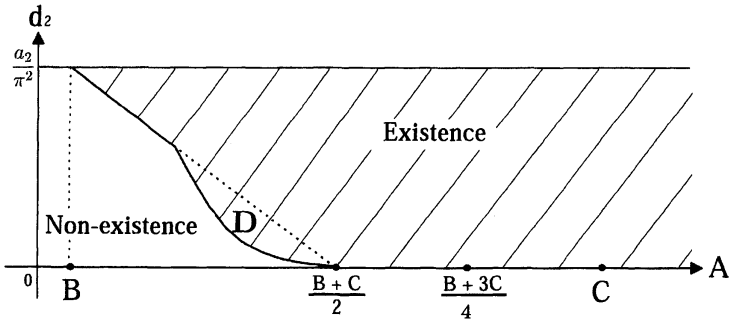

Figure 3.2: Case $7B/3 < C$

The following theorems in [3] give the shape of solutions to (3.1) as $d_2 \uparrow a_2/\pi^2$.

Theorem 3.5 (Shape of solutions as $d_2 \uparrow a_2/\pi^2$) *Suppose that $B < C$. Let $(v(x,d_2), \tau(d_2))$ be solutions of (3.1). If $A \geq B$, then*

$$v(x;d_2) \to 0, \qquad \frac{v(x;d_2) - v(0;d_2)}{v(1;d_2) - v(0;d_2)} \to \frac{1 - \cos(\pi x)}{2},$$

$$\frac{\tau(d_2)}{v(x;d_2)} \to \frac{a_2}{b_2} \cdot \frac{1}{1 - \sqrt{1 - \frac{B}{A}\cos(\pi x)}}$$

uniformly on $[0,1]$ as $d_2 \uparrow a_2/\pi^2$.

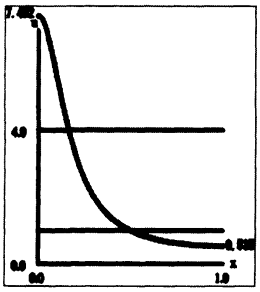

Figure 3.3: u as $d_2 \uparrow a_2/\pi^2$

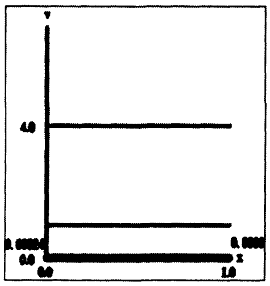

Figure 3.4: v as $d_2 \uparrow a_2/\pi^2$

The following theorems in [3] give the shape of solutions to (3.1) as $d_2 \downarrow 0$. A new number $(B+3C)/4$ appears. The shape is drastically change at $A = (B+3C)/4$

Theorem 3.6 (Shape of solutions as $d_2 \to 0$ for $A < \frac{B+3C}{4}$) *Suppose that $B \neq C$. Let $(v(x,d_2), \tau(d_2))$ be solutions of (3.1). If $A < \frac{B+3C}{4}$ and $B < C$, then*

$$v(0;d_2) \to 2 \cdot \frac{a_2}{c_2} \cdot \frac{\frac{B+3C}{4} - A}{C - B}, \qquad v(x;d_2) \to \frac{a_2}{c_2} \cdot \frac{A-B}{C-B} \quad \text{for } x > 0,$$

$$\frac{\tau(d_2)}{v(0;d_2)} \to \frac{a_2}{2c_2} \cdot \frac{C-A}{C-B} \cdot \frac{A-B}{\frac{B+3C}{4} - A}, \qquad \frac{\tau(d_2)}{v(x;d_2)} \to \frac{a_2}{b_2} \cdot \frac{C-A}{C-B} \quad \text{for } x > 0,$$

as $d_2 \downarrow 0$.

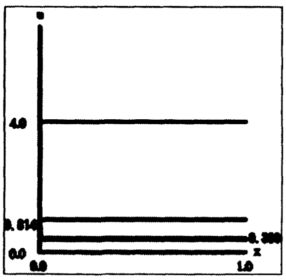

Figure 3.5: u for $A \leq (B+3C)/4$

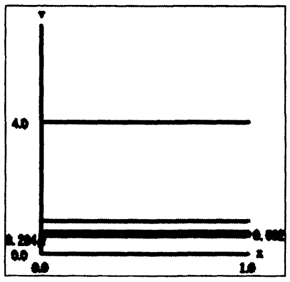

Figure 3.6: v for $A \leq (B+3C)/4$

Theorem 3.7 (Shape of solutions as $d_2 \to 0$ for $A \geq \frac{B+3C}{4}$) *Suppose that $B \neq C$. Let $(v(x,d_2), \tau(d_2))$ be solutions of (3.1). If $B < C$ and $A \geq \frac{B+3C}{4}$, then*

$$v(0;d_2) \to 0, \qquad v(x;d_2) \to \frac{3a_2}{4c_2} \quad \text{for } x > 0,$$
$$\frac{\tau(d_2)}{v(0;d_2)} \to \infty, \qquad \frac{\tau(d_2)}{v(x;d_2)} \to \frac{a_2}{4c_2} \quad \text{for } x > 0, \ \text{as } \ d_2 \to 0.$$

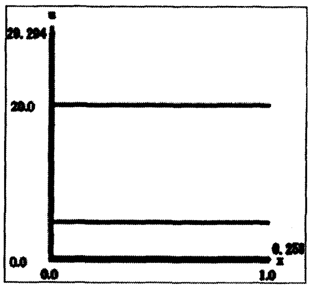

Figure 3.7: u for $(B+3C)/4 < A$

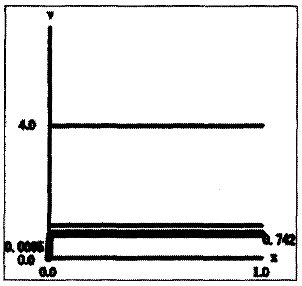

Figure 3.8: v for $(B+3C)/4 < A$

4 Stability in one-dimensional problem

Let us consider the stability of stationary solutions, and multi-dimensional solutions with their stability.

Time dependent limiting equation is as follow. Unknown functions are $\tau(t)$, $v(x,t)$, and

$$\begin{cases} \dfrac{d}{dt}\left(\displaystyle\int_\Omega \frac{\tau}{v}\,dx\right) = \displaystyle\int_\Omega \frac{\tau}{v}\left(a_1 - b_1\frac{\tau}{v} - c_1 v\right) dx, & \\ \dfrac{\partial v}{\partial t} = d_2 \Delta v + v(a_2 - c_2 v) - b_2 \tau & \text{in } \Omega, \\ \dfrac{\partial v}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases}$$

We suspect from a lot of numerical computation that the equation is a nice approximation of the original time dependent problem with sufficiently large $r := \rho_{12}/d_1$. For instance, for $r = 700,000$, it is not easy to distinguish each other.

The following Figure 4.1 shows numerical results for

$$d_1 = 1, \quad d_2 = *, \quad r = 700,000$$
$$a_2 = *, \quad b2 = 1, \quad c2 = 2.$$
$$a_2 = 1, \quad b2 = 1, \quad c2 = 1.$$

We note that $C < 7B/3$, $(B+C)/2 = 1.5$ and $(B+3C)/4 = 1.75$, since $B = 1$ and $C = 2$.

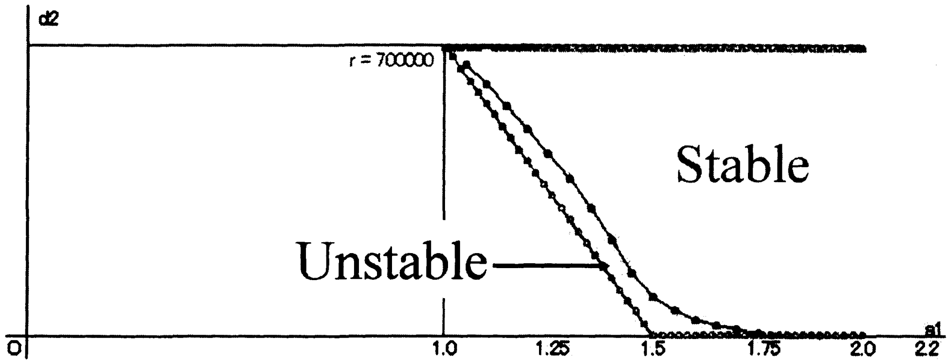

Figure 4.1: Stability and instability

Wu[8] gave a proof of instability for

d_2 sufficiently small with $(B+C)/2 < A < (B+4C)/4$

in one-dimensional case. Recently, she have also given a proof of stability for

$d_2(< a_2/\pi^2)$ sufficiently close to a_2/π^2 with $(B+C)/2 < A < (B+4C)/4$

in one-dimensional case.

5 Multi-dimensional problem

We have done various numerical computations for the case Ω is rectangles in 2-dimensional space. It seems that the structure of stable stationary solutions is essentially very similar to 1-dimensional case, though there are much varieties of shape of solutions in 2-dimensional case than in one-dimensional case.

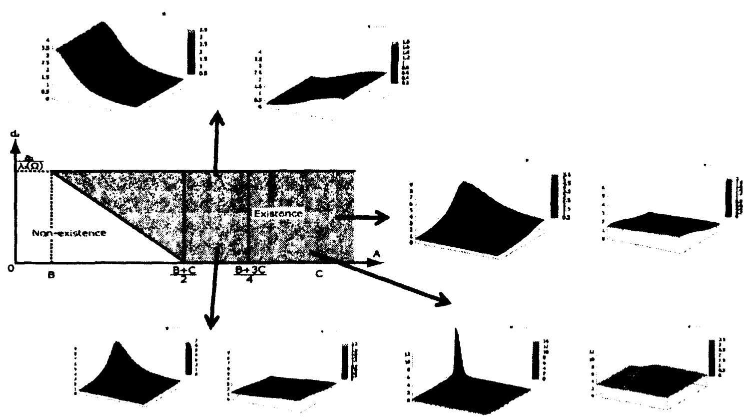

Figure 5.1: 2D global

Now, we will state some mathematical results. We prepare notations. Let

$$\lambda_0 = 0 \ < \lambda_1 \leq \lambda_2 \leq \cdots$$

$$\varphi_0 = const., \quad \varphi_1, \quad \varphi_2, \quad \cdots.$$

be eigen values and corresponding eigen functions of $-\Delta$ in $\Omega \subset R^N$ with Neumann boundary.

Theorem 5.1 *Suppose that $N \leq 3$ and λ_1 be a simple eigen values with an eigen function φ_1. Then, there exists exactly two positive non-constant solutions (v_-, τ_-) and (v_+, τ_+) of (2.1) for d_2 sufficiently close to a_2/λ_1 with $d_2 < a_2/\lambda_1$*

Moreover,

$$\tau \to 0,$$
$$\frac{\tau_\pm(d_2)}{v_\pm(x; d_2)} \to \frac{a_2}{b_2} \cdot \frac{1}{1 + \mu_\pm \varphi_1(x)}$$

as $d_2 \uparrow a_2/\lambda_1$, where μ_-, μ_+ $(\mu_- < 0 < \mu_+)$ are solutions of

$$\frac{\int_\Omega (1 + \mu\ \varphi_1(x))^{-2}\, dx}{\int_\Omega (1 + \mu\ \varphi_1(x))^{-1}\, dx} = \frac{A}{B}.$$

Remark. *The set $\{(v_-, \tau_-), (v_+, \tau_+)\}$ is uniquely determined though there is a freedom to pick up φ_1.* The condition $N \leq 3$ comes from Harnack's inequality in our proof.

Remark. *For $N = 1$, $\Omega = (0,1)$, it is easy to see that*

$$\lambda_1 = \pi^2, \quad \varphi_1(x) = \cos \pi x, \quad \frac{1}{1 - \mu^2} = \frac{A}{B}, \quad \mu_\pm = \pm\sqrt{1 - \frac{B}{A}}.$$

Remark. *For $N = 2$, $\Omega = (0,1) \times (0, \ell)$ with $0 < \ell < 1$, it is easy to see that*

$$\lambda_1 = \pi^2, \quad \varphi_1(x, y) = \cos \pi x, \quad \frac{1}{1 - \mu^2} = \frac{A}{B}, \quad \mu_\pm = \pm\sqrt{1 - \frac{B}{A}}.$$

Theorem 5.2 *Suppose that $N \leq 3$ and λ_1 be a simple eigen values. Then, (v_-, τ_-) and (v_+, τ_+) defined by Theorem 5.1 are asymptotically stable for d_2 sufficiently close to a_2/λ_1 with $d_2 < a_2/\lambda_1$.*

The following general lemma plays crucial role to prove Theorems 5.1 and 5.2.

Lemma 5.3 *Suppose that $N \geq 1$ and φ_1 be eigen values corresponding to λ_1. Let $g(\mu)$ be defined by*

$$g(\mu) := \frac{\int_\Omega (1+\mu\ \varphi_1(x))^{-2}\,dx}{\int_\Omega (1+\mu\ \varphi_1(x))^{-1}\,dx}$$

for $\mu \in (-1/max_{\bar\Omega}\ \varphi_1, -1/min_{\bar\Omega}\ \varphi_1)$. *Then*

$$\frac{dg(\mu)}{d\mu} = \begin{cases} + & for\ \mu > 0, \\ 0 & for\ \mu = 0, \\ - & for\ \mu < 0. \end{cases}$$

Moreover, for $N \leq 4$,

$$\begin{cases} g(\mu) \to \infty & as \quad \mu \uparrow \mu_+, \\ g(\mu) \to \infty & as \quad \mu \downarrow \mu_-. \end{cases}$$

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