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A Study on Uniqueness for Super-Brownian Motion in Random Environment

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Abstract

In [17], the author construct super-Brownian motion in random environment as the limit points of scaled branching random walks in random environment which are solutions of an SPDE. However, the uniqueness of the solution for such an SPDE is not still known. In the end of this paper, the author writes an idea of the proof for uniqueness which seems to do well but may fail.

We denote by $(\Omega, \mathcal{F}, P)$ a probability space. Let $\mathbb{N} = \{0,1,2,\cdots\}$, $\mathbb{N}^* = \{1,2,3,\cdots\}$, and $\mathbb{Z} = \{0,\pm 1,\pm 2,\cdots\}$. We denote by $\mathcal{M}_F(S)$ the set of finite Borel measures on S with the topology by weak convergence. Let $C_K(S)$ be the set of continuous functions with support compact. If F is a set of functions on $\mathbb{R}$, we write F_+ or F^+ for non-negative functions in F.

1 Introduction

Super-Brownian motion(SBM) is a measure valued process which was introduced by Dawson and Watanabe independently[4, 20] and is obtained as the limit of critical (or asymptotically critical) branching Brownian motions (or branching random walks). We can find many books for introduction of super-Brownian motion [5, 8] and dealing with several aspects of it [6, 7, 10, 18]. Super-Brownian motion has a lot of relations to the physics or bibliography.

There are several ways to characterize SBM, the unique solutions of martingale problem, non-linear PDE, etc. Here, we characterize it as the unique solution of the martingale problem:

Definition 1.1. *We call a measure valued process $\{X_t(\cdot) : t \in [0,\infty)\}$ super-*

Brownian motion when X_t is the unique solution of the martingale problem

$$\begin{cases} \text{For all } \phi \in \mathcal{D}(\Delta), \\ \quad Z_t(\phi) := X_t(\phi) - X_0(\phi) - \int_0^t \frac{1}{2} X_s(\Delta\phi)\, ds \\ \text{is an } \mathcal{F}_t^X\text{-continuous square-integrable martingale and} \\ \quad \langle Z(\phi)\rangle_t = \int_0^t \gamma X_s(\phi^2) ds, \end{cases}$$

where $\gamma > 0$ is a constant.

We are interested in the path property of super-Brownian motion on which many researcher wrote papers. Here is one of them, absolute continuity and singularity with respect to Lebesgue measure.

Theorem 1.2. *[9, 18, 19] Assume X is a Super-Brownian motion with $X_0 = \mu$, where $\mu \in \mathcal{M}_F(\mathbb{R}^d)$.*

(i) ($d = 1$) There exists an adapted continuous $C_K(\mathbb{R})$-valued process $\{u_t : t > 0\}$ such that $X_t(dx) = u_t(x)dx$ for all $t > 0$ P-a.s. and u satisfies the SPDE (defined on the larger probability space $(\Omega', \mathcal{F}', P')$)

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + \sqrt{\gamma u}\dot{W}, \quad u_{0+}(dx) = \mu(dx), \tag{SPDE}$$

where W is an white noise defined on the larger probability space $(\Omega', \mathcal{F}', P')$.

(ii) ($d \geq 2$) $X_t(\cdot)$ is singular with respect to Lebesgue measure almost surely.

Remark: There are some results on the detailed path properties for $d \geq 2$.

We focus on (SPDE). (SPDE) is generally expressed as

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + a(u)\dot{W}, \tag{SPDE(a)}$$

where $a(u)$ is $\mathbb{R}$-valued continuous function on $\mathbb{R}$.

There are some examples for (SPDE(a)):

(a) If $a(u) = \lambda u$, then the solution of (SPDE(a)) is the Cole-Hopf solution of KPZ equation.

(b) If $a(u) = \sqrt{u - u^2}$, then the solution of (SPDE(a)) appears as the density of stepping-stone model.

Remark: The existence of solutions for (SPDE(a)) is studied in [12] with some assumptions on $a(\cdot)$ and the initial condition μ.

2 Super-Brownian motion in random environment

In [17], the author constructs super-Brownian motion in random environment as limit points of scaled branching random walks in random environment.

2.1 branching random walks in random environment

Although there are a lot of definition of branching random walks in random environment, ours is the one introduced in [1]. Let $N \in \mathbb{N}$ be large enough. We consider the system where particles move on $\mathbb{Z}$ and the process evolves according to the following rule:

(i) There are N particles at the origin at time 0.

(ii) If a particle locates at site $x \in \mathbb{Z}$ at time n, then it moves to a uniformly chosen hearest neighbor site and split into two particles with probability $\frac{1}{2}+\frac{\beta\xi(n,x)}{2N^{1/4}}$ or dies out with probability $\frac{1}{2}-\frac{\beta\xi(n,x)}{2N^{1/4}}$, where jump and branching system are independent of each particles, $\{\xi(n,x) : (n,x) \in \mathbb{N}\times\mathbb{Z}\}$ are $\{1,-1\}$-valued i.i.d. random variables with $P(\xi(n,x)=1)=P(\xi(n,x)=-1)=\frac{1}{2}$, and $\beta>0$ is constant.

Remark: In our model, random environment is given by branching mechanics which are updated for each site and each time.

Remark: N is the scaling parameter which tends to infinity later. Also, we emphasize that the fluctuations of offspring distributions are different from the ones in [13].

We don't give the mathematically rigorous definition in this paper.

2.2 Super-Brownian motion in random environment

In this subsection, we introduce super-Brownian motion in random environment. Super-Brownian motion is obtained as the limit of scaled critical branching Brownian motions (branching random walks). When we look at our model, the mean number of offsprings from one particle is 1, so that we can regard our model as "critical" branching random walks in random environment in some sense. We will try to obtain the scaled limit process.

We denote by $B^{(N)}_{n,x}$ the number of particles at site x at time n. We define $X^{(N)}_t(dx)$ by

$$X^{(N)}_0(dx) = \delta_0(dx),$$
$$X^{(N)}_t(dx) = \frac{1}{N}\sum_{y\in\mathbb{Z}} B^{(N)}_{\lfloor tn\rfloor,y}\delta_y(N^{1/2}dx).$$

More simply, we can express the definition of $X^{(N)}_t(\cdot)$ as follows: Let $A\in\mathcal{B}(\mathbb{R})$ be a Borel set in $\mathbb{R}$. Then,

$$X^{(N)}_t(A) = \frac{\sharp\{\text{particles locates in } N^{1/2}A \text{ at time } \lfloor Nt\rfloor\}}{N}.$$

In [17], we have the following result.

Theorem 2.1. *$\{X^{(N)}_\cdot : N \in \mathbb{N}^*\}$ is C-relatively compact. Moreover, if we denote by $\{X_t(\cdot)\}$ a limit point, then $X_t(\cdot)$ is absolutely continuous with respect to Lebesgue measure for all $t > 0$ P-a.s. and its density $u(t, x)$ satisfies SPDE*

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + \sqrt{u + \frac{\beta^2 u^2}{2}}\dot{W}, \quad u_{0+}dx = \delta_0(dx). \tag{2.1}$$

Formally, $\{X_t(\cdot) : t \geq 0\}$ is a solution of the following martingale problem:

$$\begin{cases} \text{For all } \phi \in \mathcal{D}(\Delta), \\ \quad Z_t(\phi) := X_t(\phi) - \phi(0) - \int_0^t \frac{1}{2} X_s(\Delta\phi)\, ds \\ \text{is an } \mathcal{F}_t^X\text{-continuous square-integrable martingale and} \\ \quad \langle Z(\phi)\rangle_t = \int_0^t X_s(\phi^2)ds + \frac{\beta^2}{2}\int_0^t \int_{\mathbb{R}\times\mathbb{R}} \delta_{x-y}\phi(x)\phi(y)X_s(dx)X_s(dy)ds. \end{cases} \tag{2.2}$$

We shall call solutions of the above martingale problem super-Brownian motion in random environment. We remark that super-Brownian motion in random environment introduced by Mytnik is the unique solution of martingale problem (2.2) in which δ_{x-y} is replaced by $g(x, y)$, the continuous function with suitable properties.

Now, we don't have any proof of the uniqueness of the solutions of (2.1). In the next section, we show give some strategy for proof which may end in failure.

3 A strategy for proof of uniqueness

Although there are several definition of the uniqueness for SPDE, we consider the uniqueness in law for our model. The readers can refer some papers on the uniqueness (in law or pathwise) of the solutions of (SPDE(a)) [11, 14, 15, 16]. In most cases, Hölder continuity of $a(\cdot)$ influences on the uniqueness. Actually, the uniqueness in law holds when $a(u) = u^\gamma$, $\gamma \in [\frac{1}{2}, 1]$. In our case, the Hölder contiuity of $a(\cdot)$ is $\frac{1}{2}$ so that we can conjecture the uniqueness in law does hold.

We suppose that X_t is a solution of (SPDE(a)) with $a(u) = \sqrt{u + \beta^2 u^2}$, that is

$$\frac{\partial X_t}{\partial t} = \frac{1}{2}\Delta X_t + \sqrt{X + \beta^2 X^2}\dot{W}, \text{ and } X_{0+}(dx) = \phi(x)dx,$$

where ϕ is rapidly decreasing function in x, that is

$$\phi \in C^+_{\text{rap}}(\mathbb{R}) = \left\{ g \in C^+(\mathbb{R}) : |g|_p \equiv \sup_{x\in\mathbb{R}} e^{p|x|}|g(x)| < \infty, \forall p > 0 \right\}.$$

Then, it is known that there exists $X_t(x) \in C^+_{\text{rap}}(\mathbb{R})$ such that $X_t(dx) = X_t(x)dx$ almost surely.

The main idea to prove the uniqueness in law is to prove the existence of the "dual" process $\{Y_t : t \geq 0\}$, which is $\mathcal{M}_F(\mathbb{R})$-valued process and satisfies the equation

$$E\left[\exp\left(-\langle Y_t, X_0\rangle\right)\right] = E\left[\exp\left(-\langle \nu, X_t\rangle\right)\right]$$

for each $\nu \in \mathcal{M}_F(\mathbb{R})$, where $\langle \mu, \phi \rangle = \int_{\mathbb{R}} \phi(x)\mu(dx)$ for $\phi \in \mathcal{D}(\Delta)$ and $\mu \in \mathcal{M}_F(\mathbb{R})$. However, the problem of the existence of Y can be reduced the existence of an approximating sequence, $\{Y_t^{(n)} : t \geq 0\}_{n \in \mathbb{N}^*}$.

Strategy 3.1. *We construct an approximating sequence, $\{Y^{(n)}\}_{n \geq 1}$ such that*

$$\lim_{n \to \infty} E\left[\exp\left(-\langle Y_t^{(n)}, X_0 \rangle\right)\right] = E\left[\exp\left(-\langle \nu, X_t \rangle\right)\right]. \tag{3.1}$$

for each $t \geq 0$ and each solution X is independent of $Y^{(n)}$.

In the rest of this report, we show a sequence which seems to satisfies (3.1). Let $\{\tau_k^{(n)} : k \in \mathbb{N}^*\}$ be the i.i.d. exponential random variables with parameter n, that is $P(\tau_k^{(n)} > t) = \exp(-nt)$ and let $\{\mu_k^{(n)} : k \in \mathbb{N}^*\}$ be Poisson random measures on $\mathbb{R}$ with intensity $\beta^{-2} n dx$ where $\tau_\cdot^{(\cdot)}$ and $\mu_\cdot^{(\cdot)}$ are independent of X. We identify $\mu_k^{(n)}$ as random points $\{x_k^{(n)}(i) : i \in \mathbb{N}^*\} \subset \mathbb{R}$ by

$$\mu_k^{(n)} = \sum_{i \in \mathbb{N}^*} \delta_{x_k^{(n)}(i)}.$$

Let $T_k^{(n)} = \sum_{j=1}^k \tau_j^{(n)}$.

Now, we are ready to define $Y^{(n)}$. Suppose $Y_0^{(n)} = \nu$. Then, $Y_t^{(n)}$ is given by

$$Y_t^{(n)} = \begin{cases} S_t(Y_0^{(n)}) - \int_0^t \frac{1}{2} S_{t-s}\left(Y_s^{(n)^2}\right) ds, & t \in [0, T_1^{(n)}), \\ \sum_{i \in \mathbb{N}^*} \frac{\beta^2}{n} Y_{T_1^{(n)}-}^{(n)}(x_1^{(n)}(i)) \delta_{x_1^{(n)}(i)}, & t = T_1^{(n)}, \\ S_{t-T_1^{(n)}}\left(Y_{T_1^{(n)}}^{(n)}\right) - \int_0^{t-T_1^{(n)}} \frac{1}{2} S_{t-s-T_1^{(n)}}\left(Y_{s+T_1^{(n)}}^{(n)^2}\right) ds, & t \in [T_1^{(n)}, T_2^{(n)}), \\ \vdots & \\ \sum_{i \in \mathbb{N}^*} \frac{\beta^2}{n} Y_{T_k^{(n)}-}^{(n)}(x_k^{(n)}(i)) \delta_{x_k^{(n)}(i)}, & t = T_k^{(n)}, \\ S_{t-T_k^{(n)}}\left(Y_{T_k^{(n)}}^{(n)}\right) - \int_0^{t-T_k^{(n)}} \frac{1}{2} S_{t-s-T_k^{(n)}}\left(Y_{s+T_k^{(n)}}^{(n)^2}\right) ds & t \in [T_k^{(n)}, T_{k+1}^{(n)}), \end{cases}$$

for any $k \in \mathbb{N}$, where $S_t\mu(x) = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(x-y)^2}{2t}\right) \mu(dy)$ for $\mu \in \mathcal{M}_F(\mathbb{R})$.

We need give some remarks on the definition of $Y_t^{(n)}$.

Remark : The integration equation

$$Y_t(x) = S_t\nu(x) - \int_0^t \frac{1}{2} S_{t-s}\left(Y_s^2\right) ds, \quad \text{for } \nu \in \mathcal{M}_F(\mathbb{R}) \tag{3.2}$$

has the unique solution and $Y_t(x)$ is continuous in $x \in \mathbb{R}$ for each $t > 0$ [2]. Then, the definition of $Y_{T_k}^{(n)}$ is well-defined. Also, the integral equation (3.2) is equivalent to the partial differential equation,

$$\frac{\partial Y}{\partial t} = \frac{1}{2}\Delta Y_t - \frac{1}{2} Y_t^2, \quad t > 0 \text{ and } Y_{0+}(x)dx = \nu(dx). \tag{3.3}$$

Moreover, $Y_t(\cdot) \in L^p(\mathbb{R})$ for any $1 \le p < 3$, but in general $Y_t(\cdot) \notin L^3(\mathbb{R})$ as the function in x.

Remark : At time $t = T_k^{(n)}$, the continuous function $Y_{t-}^{(n)}$ is approximated by using Poisson random measure.

Hereafter, we abbreviate superscript (n) for τ, μ, etc.

Strategy 3.2. *We will look at the difference between* $E[\exp(-\langle Y_t^{(n)}, X_0\rangle)]$ *and* $E[\exp(-\langle Y_0^{(n)}, X_t\rangle)]$.

We have by using Ito's formula and (3.3) that

$$\begin{aligned}
&E_X\left[\exp(-\langle Y_{T_k-}^{(n)}, X_{T-T_k}\rangle)\right] \\
&= E_X\left[\exp(-\langle Y_{T_{k-1}}^{(n)}, X_{T-T_{k-1}}\rangle)\right] \\
&\quad + \frac{1}{2}E_X\left[\int_{T_{k-1}+}^{T_k} \exp(-\langle Y_{s-}^{(n)}, X_{T-s}\rangle)\left\{\langle Y_{s-}^{(n)^2}, X_{T-s}\rangle - \langle Y_{s-}^{(n)^2}, X_{T-s} + \beta^2 X_{T-s}^2\rangle\right\} ds\right] \\
&= E_X\left[\exp(-\langle Y_{T_{k-1}}^{(n)}, X_{T-T_{k-1}}\rangle)\right] \\
&\quad + \frac{1}{2}E_X\left[\int_{T_{k-1}+}^{T_k} \exp(-\langle Y_{s-}^{(n)}, X_{T-s}\rangle)\left\{-\langle Y_{s-}^{(n)^2}, \beta^2 X_{T-s}^2\rangle\right\} ds\right],
\end{aligned}$$

for each $0 < T_k \le T$. The definition of $Y^{(n)}$ with this implies that

$$\begin{aligned}
&E_X\left[\exp(-\langle Y_T^{(n)}, X_0\rangle)\right] \\
&= E_X[\exp(-\langle Y_0^{(n)}, X_T\rangle)] \\
&\quad + \frac{1}{2}E_X\left[\int_0^T \exp(-\langle Y_s^{(n)}, X_{T-s}\rangle)\left\{-\beta^2\langle Y_s^{(n)^2}, X_{T-s}^2\rangle\right\} ds\right] \\
&+ \int_0^T \int_{\mathcal{M}(\mathbb{R})} E_X\left[\left\{\exp\left(-\left\langle \mu, \frac{\beta^2}{n} Y_{s-}^{(n)} X_{T-s}\right\rangle\right) - \exp(-\langle Y_{s-}^{(n)}, X_{T-s}\rangle)\right\}\right] N(d\mu, ds),
\end{aligned}$$

where $N(d\mu, ds)$ is the corresponding counting measure of point process $\{(\mu_k, \tau_k) : k \in \mathbb{N}^*\}$.

Taking expectation of both sides, we have that

$$\begin{aligned}
&E\left[\exp(-\langle Y_T^{(n)}, X_0\rangle)\right] \\
&= E\left[\exp(-\langle Y_0^{(n)}, X_T\rangle)\right] \\
&\quad + \frac{1}{2}E\left[\int_0^T \exp(-\langle Y_s^{(n)}, X_{T-s}\rangle)\left\{-\beta^2\langle Y_s^{(n)^2}, X_{T-s}^2\rangle\right\} ds\right] \\
&+ E\left[\int_0^T \exp\left(-\langle Y_s^{(n)}, X_{T-s}\rangle\right)\right.
\end{aligned}$$

$$\times \left\{ \exp\left(\frac{n}{\beta^2} \int_{\mathbb{R}} \left(\exp\left(-\frac{\beta^2 Y_{s-}^{(n)}(x) X_{T-s}(x)}{n} \right) - 1 + \frac{\beta^2 Y_{s-}^{(n)}(x) X_{T-s}(x)}{n} \right) dx \right) - 1 \right\} nds \Bigg].$$

When we focus on the terms in brackets $\{\}$, we have that

$$\underbrace{\exp\left(\frac{n}{\beta^2} \int_{\mathbb{R}} \underbrace{\left(\exp\left(-\frac{\beta^2 Y^{(n)} X}{n} \right) - 1 - \frac{\beta^2 Y^{(n)} X}{n} \right)}_{\approx \frac{\beta^4 Y^{(n)2} X^2}{2n^2}} dx \right) - 1 - \frac{\beta^2}{2n} \int_{\mathbb{R}} Y^{(n)^2} X^2 dx,}_{\approx 0}$$

a.s. Thus, if we can show the expectation of integral of this term with respect to nds converges to 0, then we obtain the result on the uniqueness in law. It is known that if X_0 has rapidly decreasing density, then $\sup_{t \le T} \sup_{x \in \mathbb{R}} E_X[X_t(x)^p] < \infty$. So if we find some nice estimate of $P(\langle Y^{(n)^2}, 1 \rangle \ge n\varepsilon)$ as $n \to \infty$, then the problem will be solved. However, one of the difficulty of it comes from the fact $Y^{(n)} \notin L^3(\mathbb{R})$ almost surely.

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