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Quasi-Subdifferential Operators and Quasi-Subdifferential Evolution Equations

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1. Introduction

Based on our previous paper [13], we introduce some useful concepts for studying variational and quasi-variational problems associated with a general, i.e., not Euler–Lagrange, partial differential operator.

Consider the following elliptic variational inequality:

$$(\mathrm{VI})\begin{cases} u \in K, \\ \displaystyle\int_\Omega \{\mathbf{a}(u, \nabla u)\cdot\nabla(u-z) + a_0(u)(u-z)\}\,dx \\ \qquad \le (f, u-z) \qquad \forall z \in K, \end{cases}$$

where $K \subset H^1(\Omega)$ is a closed convex set, $\Omega \subset \mathbb{R}^N (N \ge 1)$ is a bounded domain, $f \in L^2(\Omega)$ is a given function, $(\cdot,\cdot)$ denotes the inner product in $L^2(\Omega)$, $\mathbf{a}(r,\mathbf{p}) = \partial_{\mathbf{p}}\hat{a}(r,\mathbf{p})$, $\hat{a} \in C^1(\mathbb{R}\times\mathbb{R}^N)$, and $a_0 \in C(\mathbb{R})$ with appropriate growth conditions.

If it holds that

$$\hat{a}(r,\mathbf{p}) \text{ is convex jointly in } (r,\mathbf{p}) \in \mathbb{R}\times\mathbb{R}^N \text{ and } a_0 = \partial_r\hat{a}, \tag{1}$$

then we have

$$\begin{aligned} (\mathrm{VI}) \iff & \ (f, z-u) \le \psi(z) - \psi(u) \quad \forall z \in K, \\ \iff & \qquad \partial\psi(u) \ni f, \end{aligned}$$

where $\partial\psi$ is the subdifferential of a proper, lower-semi-continuous (l.s.c.), and convex function $\psi : L^2(\Omega) \to \mathbb{R} \cup$

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$\{+\infty\}$ defined by

$$\psi(z) := \begin{cases} \displaystyle\int_\Omega \hat{a}(z, \nabla z)dx, & \text{if } z \in K, \\ +\infty, & \text{otherwise.} \end{cases}$$

However, condition (1) is too restrictive for a general case. We have, in general:

$$\begin{aligned} \text{(VI)} \iff & \quad (f, z-u) \leq \varphi(u; z) - \varphi(u; u) \quad \forall z \in K \\ \iff & \quad \partial\varphi(u; u) \ni f, \end{aligned}$$

where $\partial\varphi$ is the subdifferential with respect to the second variable of a parameterized convex function $\varphi : L^2(\Omega) \times L^2(\Omega) \to \mathbb{R} \cup \{+\infty\}$ given by

$$\varphi(v; z) := \begin{cases} \displaystyle\int_\Omega \hat{a}(v, \nabla z)dx + \int_\Omega a_0(v) z dx, & \\ & \text{if } v \in H^1(\Omega) \text{ and } z \in K, \\ +\infty, & \text{otherwise.} \end{cases}$$

Thus, we are led to the notion of a *quasi-subdifferential operator*, which we define in the next section.

2. Quasi-subdifferential operators (QSOs)

In the following, H denotes a real Hilbert space with norm $|\cdot|_H$ and inner product $(\cdot, \cdot)$.

Definition 2.1. ([13, Definition 2.1]) A (possibly multi-valued) map $A : H \to H$ is called a *quasi-subdifferential operator* (QSO) if

$$Au = \partial\varphi(u; u) \quad \text{for } u \in D(A)$$

where $\varphi : H \times H \to \mathbb{R} \cup \{+\infty\}$ satisfies:

- $\varphi(v; \cdot) : H \to \mathbb{R} \cup \{+\infty\}$ is l.s.c. and convex $\forall v \in H$.

- $D(A) := \{v \in H | \ \varphi(v; \cdot) \not\equiv +\infty, \ v \in D(\partial\varphi(v; \cdot))\}$
$\neq \emptyset$.

We call φ the defining convex function of A, and write A^φ when this needs to be specified.

We have the following existence theorem for an equation with a quasi-subdifferential operator.

Theorem 2.2. ([13, Theorem 2.2]) *Let A be a QSO defined by φ. Let X be a reflexive Banach space with compact embedding $X \subset H$, and K be a closed convex subset of X. Assume that $D(\varphi(v; \cdot)) \subset K$ for all $v \in K$, and that there exist $C_1, C_2, C_3 > 0$, $p > q \geq 1$ satisfying the following conditions.*

(A1) *There exists $z_0 \in H$ such that for all $v \in K$*

$$\varphi(v; z_0) \leq C_1 \left(|v|_X^q + 1\right).$$

(A2) *For all $v \in K$ and $z \in X$*

$$\varphi(v; z) \geq C_2 |z|_X^p - C_3 \left(|v|_X^q + 1\right).$$

(A3) *For all $v \in K$*

$$D(\varphi(v; \cdot)) \ni z \mapsto \varphi(v; z) \text{ is strictly convex.}$$

(A4) *If $K \ni v_n \to v$ weakly in X, then $\varphi(v_n; \cdot) \to \varphi(v; \cdot)$ in the sense of Mosco.*

Then, for each $f \in H$, there exists $u \in K$ satisfying

$$Au \ni f.$$

The idea of the proof of this theorem is as follows. For each $v \in K$, assumptions (A2) and (A3) mean that there exists a unique $z_v \in K$ minimizing $\varphi(v; z) - (f, z)$ $(z \in H)$. By (A1) and (A2), the map $v \mapsto z_v$, if restricted to an appropriate compact and convex set $\tilde{K} \subset K$, maps to itself. By (A4), this map is continuous with respect to the topology of H. Therefore, from Schauder's fixed point theorem, it follows that there is a fixed point u that is a solution of the desired equation. We refer to [13] for the detail.

We note that, under different assumptions, we can use another type of fixed point theorem to obtain an existence theorem of a different type. In the next section, we introduce a concept based on such an argument.

This theorem can be applied to (VI) as well as to the following quasi-variational inequality (cf. [13, Section 3]):

$$\text{(QVI)} \begin{cases} u \in K(u), \\ \displaystyle\int_\Omega \{\mathbf{a}(u, \nabla u) \cdot \nabla(u - z) + a_0(u)(u - z)\}\, dx \\ \qquad \le (f, u - z) \qquad \forall z \in K(u) \end{cases}$$

Here, $K(v) \subset H^1(\Omega)$ is a closed convex set depending on v. We have

$$\text{(QVI)} \Longleftrightarrow \quad Au \ni f,$$

where A is a QSO defined by

$$\varphi(v; z) := \begin{cases} \displaystyle\int_\Omega \hat{a}(v, \nabla z) dx + \int_\Omega a_0(v) z dx, \\ \qquad\qquad \text{if } v \in H^1(\Omega) \text{ and } z \in K(v), \\ +\infty, \qquad \text{otherwise.} \end{cases}$$

For a pseudo-monotone operator approach to (VI) and (QVI), we refer to Kenmochi et al. [10, 5]. For an earlier study of elliptic quasi-variational inequalities, see Joly and Mosco [4].

3. Quasi-variational principles

A variational principle is expressed using a proper, l.s.c., and convex function ψ and its subdifferential as follows:

$$\partial\psi(u) \ni 0 \iff \psi(u) = \min_z \psi(z).$$

Here, the equation (or inclusion) $\partial\psi(u) \ni 0$ represents a variational inequality or a differential equation with a boundary condition according to the constraint posed by the function ψ. This principle has played an important role in mathematical physics and related fields. However, there is a simple limitation to the principle, since it can only be applied to problems associated with Euler–Lagrange-type differential operators. Problems associated with non-Euler–Lagrange-type differential operators, e.g., the Navier–Stokes equations, the diffusion equation with a convection term and so on, are not derived directly from the variational principle.

Let us consider the following idea:

$$\partial\varphi(u;u) \ni 0 \iff \begin{cases} u \text{ is a fixed point of } v \mapsto z_v : \\ \varphi(v; z_v) = \min_z \varphi(v; z). \end{cases} \tag{2}$$

Here, we have a function $\varphi : H \times H \to \mathbb{R} \cup \{+\infty\}$ such that $\varphi(v; \cdot) : H \to \mathbb{R} \cup \{+\infty\}$ is l.s.c. and convex for each $v \in H$ and proper for some $v \in H$. In (2), $\partial\varphi$ denotes the subdifferential with respect to the second variable. Hence, we have

$$\partial\varphi(u;u) \ni 0 \iff A^\varphi \ni 0,$$

where A^φ is the QSO defined by φ. We call the idea in (2) a *quasi-variational principle* (QVP). Thus, QVP is closely related to QSOs. A similar concept to this (2) was used by Joly and Mosco [4] to study quasi-variational inequalities, that is, variational inequalities with constraints depending on the unknown functions. However, the idea can be applied to various problems with non-Euler–Lagrange-type differential operators. In fact, the proof of Theorem 2.2 is based on QVP and can be applied to variational and quasi-variational inequalities with non-Euler–Lagrange-type differential operators.

In addition to this, QVP plays an essential role in a standard proof of the existence theorem for the stationary Navier–Stokes equations. These are stated below in a slightly abstract form.

Theorem 3.1. (*abstract Navier–Stokes equations*) *Let* $V \subset H \subset V^*$ *be a Hilbert triplet with compact embeddings,* $\langle \cdot, \cdot, \rangle$ *be the duality pairing, and* $F : V \to V^*$ *be the duality map. Let* $B : V \to V^*$ *be a compact map satisfying* $\langle B(z), z\rangle = 0$ *for all* $z \in V$. *Let* $A : H \to H$ *be a QSO defined by*

$$\varphi(v; z) := \begin{cases} \frac{1}{2}|z|_V^2 + \langle B(v), z\rangle, & \text{if } v, z \in V, \\ +\infty, & \text{otherwise.} \end{cases}$$

Then, for each $f \in H$, *there exists a* $u \in H$ *such that*

$$Au = f.$$

This theorem can be proved as follows. For each $v \in V$, there exists a unique $z_v \in V$ such that

$$\Phi_{\lambda,f}(v; z_v) = \min_z \Phi_{\lambda,f}(v; z),$$

where, for $\lambda \in [0, 1]$, we define

$$\Phi_{\lambda,f}(v; z) := \begin{cases} \frac{1}{2}|z|_V^2 + \lambda\big(\langle B(v), z\rangle - (f, z)\big), & \text{if } v, z \in V, \\ +\infty, & \text{otherwise.} \end{cases}$$

That is, we have

$$z_v + \lambda F^{-1}\big(B(v) - f\big) = 0.$$

By Leray–Schauder's fixed point theorem, we can show that there exists a fixed point u of the map $v \mapsto z_v$ that is a desired solution to the equation.

4. Quasi-subdifferential evolution equations

In this section, we study *quasi-subdifferential evolution equations* (QSEs), which are evolution equations related to QSOs. We consider two types of QSE. The first is given as follows:

$$\text{(QSE1)} \qquad u'(t) + A(t)u(t) \ni 0 \quad \text{a.e. } t \in (0,T).$$

Here, $A(t)$ $(0 \le t \le T)$ is a QSO defined by $\varphi^t : H \times H \to \mathbb{R} \cup \{+\infty\}$. Consider the following conditions:

(Φ1) $\varphi^t(v;z) \ge G(|z|_X)$ $\forall (v,z) \in H \times H$, where X is a Banach space with compact embedding $X \subset H$ and $\lim_{r\to+\infty} G(r) = +\infty$.

(Φ2) There are two functions $\alpha \in W^{1,2}(0,T)$ and $\beta \in W^{1,1}(0,T)$ such that, for all $v, w \in H, 0 \le s \le t \le T$ and $z \in D(\varphi^s(v;\cdot))$, there exists $\tilde{z} \in D(\varphi^t(v;\cdot))$ satisfying the following inequalities:

$$|\tilde{z} - z|_H \le |\alpha(t) - \alpha(s)| \left(\varphi^s(v;z)\right)^{1/2},$$

$$\varphi^t(w;\tilde{z}) - \varphi^s(v;z)$$

$$\le |\beta(t) - \beta(s)|\varphi^s(v;z) + |w - v|_H \left(\varphi^s(v;z)\right)^{1/2}.$$

Put $K(t) := \{z \in H |\ \varphi^t(z;z) < +\infty\}$.

Theorem 4.1. ([13, Theorem 4.1]) *Assume* (Φ1) *and* (Φ2). *Then, for each $u_0 \in K(0)$, there exists a solution $u \in W^{1,2}(0,T;H)$ of* (QSE1) *satisfying $u(0) = u_0$.*

The idea of this theorem has been developed by Kenmochi, Kubo, Yamazaki, Shirakawa and Fukao [12, 16, 20, 17, 18, 2, 15, 3] and is based on the theory of time-dependent subdifferential evolution equations (TSEs). In fact, by assumption (Φ2), for each $v \in W^{1,2}(0,T;H)$ the function

$$t \mapsto \Phi(t) := \varphi^t(v(t);\cdot)$$

satisfies the condition of the standard theory of TSEs developed by Kenmochi [8, 9] and Yamada [19]. Hence, there exists a unique solution of the problem:

$$\begin{cases} u'(t) + \varphi^t(v(t); u(t)) \ni 0 & \text{a.e. } t \in (0,T), \\ u(0) = u_0. \end{cases}$$

Using assumption (Φ1) and the energy inequality derived by TSE theory, we can show that there is a fixed point of the map $v \mapsto u$ that gives a desired solution of (QSE1).

The second type of QSE is given as follows:

$$\text{(QSE2)} \qquad \mathcal{L}_{u_0} u + \mathcal{A} u \ni 0 \quad \text{in } \mathcal{H}.$$

Here, $\mathcal{H} := L^2(0,T;H)$, $\mathcal{A} : \mathcal{H} \to \mathcal{H}$ is a QSO, $\mathcal{L}_{u_0} u := u'$, and $D(\mathcal{L}_{u_0}) := \{w \in W^{1,2}(0,T;H) |\ w(0) = u_0\}$.

This type of problem arises in hysteresis models, nonlocal obstacle problems, and so on (cf. [11, 1, 14, 6]). In particular, Kano, Murase and Kenmochi [7] studied this type of abstract problem by employing the theory of TSEs.

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