Global structure of plane closed elastic curves

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1 Introduction

This is a joint work with Waichiro Matsumoto and Shoji Yotsutani (Ryukoku University).

Let $\Gamma$ be a plane closed elastic curve with length $2\pi$. We denote arc-length and curvature by $s$ and $\kappa(s)$, respectively. Let $M$ be the signed area defined by

$$M := \frac{1}{2} \int_{\Gamma} xdy - ydx,$$

where $(x, y) = (x(s), y(s)) \in \Gamma$ with $(x(0), y(0)) := (0, 0)$. Let us consider the following variational problem $(VP)$:

Find a curve $\Gamma$ (the curvature $\kappa(s)$) which minimize \(\frac{1}{2} \int_{0}^{2\pi} \kappa(s)^{2}ds\) subject to $\pi > M$ and $\omega \pi \neq M$, where $\omega$ is the winding number.

K. Watanabe ([1, 2]) considered this variational problem $(VP)$ with $\omega = 1$. He derived the Euler-Lagrange equation to $(VP)$ and showed the existence of the minimizer and investigate the profile near the disk.

The Euler-Lagrange equation to $(VP)$ is

\[
\begin{align*}
(P^\omega) \quad &
\begin{cases}
\kappa_{ss} + \frac{1}{2} \kappa^3 + \mu \kappa - \nu = 0, & s \in [0, 2\pi], \\
\kappa(0) = \kappa(2\pi), & \kappa_s(0) = \kappa_s(2\pi), \\
1 & = \frac{1}{2\pi} \int_{0}^{2\pi} \kappa(s) ds = \omega, \\
4\mu \pi^2 + \pi & = \frac{4\pi \omega \mu + \int_{0}^{2\pi} \kappa(s)^2 ds}{\int_{0}^{2\pi} \kappa(s)^2 ds} = M,
\end{cases}
\end{align*}
\]

where $\mu$ and $\nu$ are some constants. We can obtain the following proposition by using the argument of K.Watanabe [1, Lemma 3 and Lemma 4]

**Proposition 1.1** Suppose that $\kappa(s)$ is a solution of $(P^\omega)$, then the following properties hold:

(i) $\kappa(s) \in C^\infty([0, 2\pi])$.

(ii) There exists a positive integer $m$ such that $\kappa(s)$ is periodic function
with period $s = 2\pi/m$ and axially symmetric with respect to $s = \pi/m$ and $m$ denotes the number of minimum points of $\kappa(s)$ by normalizing $\kappa(0) = \max_{0\leq s\leq 2\pi} \kappa(s)$. (We call this solution "m-mode solution").

Let us normalize $\kappa(s)$ as $\kappa(0) = \max_{0\leq s\leq 2\pi} \kappa(s)$. For n-mode solution $\kappa(s)$, we may consider the following differential equation:

\[
\begin{align*}
\kappa_{ss} + \frac{1}{2}\kappa^{3} + \mu\kappa - \nu &= 0, \quad s \in \left[0, \frac{\pi}{n}\right], \\
\kappa_{s}(0) &= \kappa_{s}\left(\frac{\pi}{n}\right) = 0, \quad \kappa_{s}(s) < 0 \quad s \in \left(0, \frac{\pi}{n}\right), \\
\int_{0}^{\pi/n} \kappa(s)ds &= \frac{\omega\pi}{n}, \\
2\mu\pi^{2} + n\pi \int_{0}^{\pi/n} \kappa(s)^{2}ds \\
2\pi\omega\mu + n \int_{0}^{\pi/n} \kappa(s)^{3}ds &= M.
\end{align*}
\]

We introduce the following auxiliary problem. Let $\kappa(s)$ be unknown function, and $\mu$, $\nu$ be unknown constants. Find $(\kappa(s), \mu, \nu)$ such that

\[
\begin{align*}
\kappa_{ss} + \frac{1}{2}\kappa^{3} + \mu\kappa - \nu &= 0, \quad s \in \left[0, \frac{\pi}{n}\right], \\
\kappa_{s}(0) &= \kappa_{s}\left(\frac{\pi}{n}\right) = 0, \quad \kappa_{s}(s) < 0 \quad s \in \left(0, \frac{\pi}{n}\right).
\end{align*}
\]

First we represent all solution $(\kappa(s), \mu, \nu)$ of $(E_{n})$. Next we give the representation of the constraint (1.3) and (1.4).

We prepare notations to state our theorems.

**Definition 1.1** We define the complete elliptic integral of first, second and third kind by

\[
\begin{align*}
K(k) &:= \int_{0}^{1} \frac{d\xi}{\sqrt{(1 - \xi^{2})(1 - k^{2}\xi^{2})}}, \\
E(k) &:= \int_{0}^{1} \sqrt{\frac{1 - k^{2}\xi^{2}}{1 - \xi^{2}}}d\xi, \\
\Pi(\ell, k) &:= \int_{0}^{1} \frac{d\xi}{(1 + \ell\xi^{2})\sqrt{(1 - \xi^{2})(1 - k^{2}\xi^{2})}}.
\end{align*}
\]
Definition 1.2 Jacobi’s sn function is defined by

\[ z = \int_0^{\text{sn}(z,k)} \frac{d\xi}{\sqrt{(1-\xi^2)(1-k^2\xi^2)}} \]

and Jacobi’s cn function is defined by

\[ \text{cn}(z,k) := \sqrt{1-\text{sn}^2(z,k)} \]

for \( z \in [-K(k), K(k)] \). These elliptic functions are extended to \((-\infty, \infty)\) by using the relation \( \text{sn}(z+2K(k),k) = -\text{sn}(z,k) \) and \( \text{cn}(z+2K(k),k) = -\text{cn}(z,k) \).

1.1 Main Result

Theorem 1.1 All solutions \((\kappa(s), \mu, \nu)\) of \((E_n)\) are represented by the following (i), (ii) and (iii):

(i) \( \kappa(s) = \overline{\kappa}_n(s;k, h), \mu = \overline{\mu}_n(k, h) \) and \( \nu = \overline{\nu}_n(k, h) \) for \((k, h)\) \(\in\overline{\Sigma}\), where

\[ \overline{\Sigma} := \Sigma_{S^*} \cup \Sigma_S, \]

\( \Sigma_{S^*} := \{(k, h); -1 < k \leq 0, 2 < h < 3\}, \quad \Sigma_S := \{(k, h); 0 \leq k < 1, 0 < h \leq 3 - 2k^2\}, \]

(ii) \( \overline{\kappa}_n(s;k, h) := \begin{cases} \kappa_n^{S^*}(s;k, v(k, h)) & \text{for } (k, h) \in \Sigma_{S^*}, \\ \kappa_n^S(s;k, u(k, h)) & \text{for } (k, h) \in \Sigma_S, \end{cases} \]

(iii) \( \overline{\mu}_n(k, h) := \begin{cases} \mu_n^{S^*}(k, v(k, h)) & \text{for } (k, h) \in \Sigma_{S^*}, \\ \mu_n^S(k, u(k, h)) & \text{for } (k, h) \in \Sigma_S, \end{cases} \)

and

\( \overline{\nu}_n(k, h) := \begin{cases} \nu_n^{S^*}(k, v(k, h)) & \text{for } (k, h) \in \Sigma_{S^*}, \\ \nu_n^S(k, u(k, h)) & \text{for } (k, h) \in \Sigma_S. \end{cases} \)

Here the functions \( \kappa_n^*(s; k, v), \mu_n^*(k, v), \nu_n^*(k, v) \) and \( v(k, h) \) are defined by

\[ \kappa_n^*(s; k, v) := -\frac{\sqrt{1-v}\sqrt{(1-k^2)v+1+k^2}}{\sqrt{v+1}(2-(1+v)\text{cn}^2(n\pi K(k)(\frac{\pi}{n}-s),k))} \left( \frac{4\sqrt{2}n}{\pi}K(k) \right) \]

\[ + \frac{4-(1-k^2)(1-v)^2+4k^2(1-v)}{\sqrt{1-v^2}\sqrt{(1-k^2)v+1+k^2}} \left( \frac{n\pi}{\sqrt{2}\pi}K(k) \right), \]
\[ \mu_n^{S^*}(k, v) := \left( \frac{-3(4 - (1 - k^2)(1 - v)^2)^2}{(1 - v^2)((1 - k^2)v + 1 + k^2)^2} + 8(2 - k^2) \right) \left( \frac{n}{2\pi} K(k) \right)^2, \] (1.14)

\[ \nu_n^{S^*}(k, v) := \frac{-2\sqrt{2}(4 - (1 - k^2)(1 - v)^2)}{(1 - v^2)^{3/2}((1 - k^2)v + 1 + k^2)^{3/2}} \left( (1 + v)^2((1 - k^2)v + 1 + k^2)^2 - k^4(1 - v)^2 \right) \left( \frac{n}{2\pi} K(k) \right)^3. \] (1.15)

and

\[ v(k, h) := \frac{-2 + (2 - k^2)(2 - h) + \sqrt{(2 - k^2)^2(2 - h)^2 + 4k^4(3 - h)}}{2(1 - k^2)} \] (1.16)

and the functions \( \kappa_n(s; k, u), \mu_n(k, u), \nu_n(k, u) \) and \( u(k, h) \) are also defined by

\[ \kappa_n^{S}(s; k, u) := -\frac{(1 - k^2)(1 - ku) + k((1 - k^2)u + k)cn(\frac{2n}{\pi} K(k)(\frac{\pi}{n} - s), k)}{(1 - k^2)u + k - k(1 - ku)cn(\frac{2n}{\pi} K(k)(\frac{\pi}{n} - s), k)}. \] (1.17)

\[ \mu_n^{S}(k, u) := \left( \frac{-3(1 - ku)^2((1 - k^2)u + k)^2}{2u((1 - 2k^2)u + 2k)((1 - k^2)u^2 + 1)} + 1 - 2k^2 \right) \left( \frac{2nK(k)}{\pi} \right)^2, \] (1.18)

\[ \nu_n^{S}(k, u) := \frac{-(1 - ku)((1 - k^2)u + k)}{4u^{3/2}((1 - 2k^2)u + 2k)^{3/2}((1 - k^2)u^2 + 1)^{3/2}} \left( 4k^2((1 - k^2)u^2 + 1)^2 + (1 - k^2)u^2((1 - 2k^2)u + 2k)^2 \right) \left( \frac{2nK(k)}{\pi} \right)^3. \] (1.19)

and

\[ u(k, h) := \frac{1}{4k(1 - k^2)} \cdot \left( 2 - h + \frac{(1 - 2k^2)((2 - h)^2 + 16k^2(1 - k^2))}{8k^2(1 - k^2) + \sqrt{(1 - 2k^2)^2(2 - h)^2 + 16k^2(1 - k^2)}} \right). \] (1.20)
(ii) $\kappa(s) = \kappa_n(s; k, h), \mu = \mu_n(k, h)$ and $\nu = \nu_n(k, h)$ for $(k, h) \in \Sigma$, where

\[
\Sigma := \Sigma_R \cup \Sigma_R^*, \quad (1.21)
\]
\[
\Sigma_R^* := \{(k, h); -1 < k \leq 0, -3 < h < -2\}, \quad (1.22)
\]
\[
\Sigma_R := \{(k, h); 0 \leq k < 1, 2k^2 - 3 \leq h < 0\}. \quad (1.23)
\]

Here

\[
\kappa_n(s; k, h) := -\kappa_n\left(\frac{\pi}{n} - s, k, -h\right),
\]

\[
\mu_n(k, h) := \mu_n(k, -h), \quad \nu_n(k, h) := -\nu_n(k, -h). \quad (1.24)
\]

(iii) $\kappa(s) = \overline{\kappa}(s; k, h), \mu = \overline{\mu}(k, h)$ and $\nu = \overline{\nu}(k, h)$ for $(k, h) \in \Sigma_0$, where

\[
\Sigma_0 := \{(k, h); 0 < k < 1, h = 0\}. \quad (1.25)
\]

Here

\[
\overline{\kappa}(s; k, h) = \frac{4nkK(k)}{\pi} cn\left(\frac{2n}{\pi} K(k) s, k\right),
\]

\[
\overline{\mu}(k, h) = (1 - 2k^2) \left(\frac{2nK(k)}{\pi}\right)^2, \quad \overline{\nu}(k, h) = 0.
\]

We show the domains $\overline{\Sigma} \cup \Sigma \cup \Sigma_0$ in Figure 1.

![Diagram](image)

Figure 1: The domain of $\overline{\Sigma} \cup \Sigma \cup \Sigma_0$

**Remark 1.1** It is more useful by using the parameter $(k, u)$ and $(k, v)$ than $(k, h)$. Let us set

\[
\Sigma_v := \{(k, v); -1 < k \leq 0, -1 < v < 1/k\}, \quad (1.26)
\]

\[
\Sigma_u := \{(k, u); 0 < k < 1, 0 < u < 1/k\}. \quad (1.27)
\]

Then the following (i), (ii), (iii) and (iv) hold:

(i) Changing the parameters from $(k, h)$ to $(k, v)$ by $k = k$ and $v = v(k, h)$, $\Sigma_R^*$ becomes $\Sigma_v$. 

We show the domains $\overline{\Sigma} \cup \Sigma \cup \Sigma_0$ in Figure 1.

![Diagram](image)

Figure 1: The domain of $\overline{\Sigma} \cup \Sigma \cup \Sigma_0$
(ii) Changing the parameter from \((k, h)\) to \((k, u)\) by \(k = k\) and \(u = u(k, h)\), \(\Sigma_S\) becomes \(\Sigma_u\).

(iii) Changing the parameter from \((k, h)\) to \((k, u)\) by \(k = k\) and \(u = u(k, -h)\), \(\Sigma_R\) becomes \(\Sigma_u\).

(iv) Changing the parameter from \((k, h)\) to \((k, v)\) by \(k = k\) and \(v = v(k, -h)\), \(\Sigma_R^*\) becomes \(\Sigma_v\).

We note that all changing the parameters are bijective.

We show the domains \(\Sigma_u\) and \(\Sigma_u\) in Figure 2.

![Figure 2: The domains \(\Sigma_u\) and \(\Sigma_u\)](image)

**Theorem 1.2** Let \(\kappa(s)\) be given by Theorem 1.1 and

\[
Z(k, h) := \int_0^{\pi/n} \kappa(s)ds.
\]

Then \(Z(k, h)\) is given by the following (i), (ii) and (iii):

(i) \(Z(k, h) = \overline{Z}(k, h)\) for \((k, h) \in \overline{\Sigma}\), where

\[
\overline{Z}(k, h) := \begin{cases} 
Z^{S_0}(k, v(k, h)) & \text{for } (k, h) \in \Sigma_{S_0}, \\
Z^{S}(k, u(k, h)) & \text{for } (k, h) \in \Sigma_S.
\end{cases}
\]

Here the function \(Z^{S_0}(k, v)\) is defined by

\[
Z^{S_0}(k, v) := \frac{Z^{S_0}(k, v)}{Z_0(k, v)}, \tag{1.28}
\]

\[
Z^{S_0}(k, v) := ((1 - k^2)(1 + v)(3 - v) + 4k^2v)K(k) - 4(1 - k^2)(1 - v^2)\Pi \left( \frac{1}{2}(1 - k^2)(1 - v) - 1, k \right), \tag{1.29}
\]

\[
Z_0(k, v) := \sqrt{2} \sqrt{1 - v^2} \sqrt{(1 - k^2)v + 1 + k^2} \tag{1.30}
\]
and the function $Z^S(k, u)$ is also defined by
\[ Z^S(k, u) := \frac{2((1 - k^2)u + k)Z_{\infty}(k, u)}{Z_0(k, u)}, \]  
\[ Z_{\infty}(k, u) := (2(1 - k^2)u^2 + 2 - (1 - ku)^2)K(k) - 2((1 - k^2)u^2 + 1)\Pi\left(\frac{k^2(1 - ku)^2}{u((1 - 2k^2)u + 2k)}, k\right), \]
\[ Z_0(k, u) := (1 - ku)\sqrt{u}\sqrt{(1 - 2k^2)u + 2k}\sqrt{(1 - k^2)u^2 + 1}. \]

(ii) $Z(k, h) = Z(k, h)$ for $(k, h) \in \Sigma$, where
\[ Z(k, h) := -\overline{Z}(k, -h). \]

(iii) $Z(k, h) = 0$ for $(k, h) \in \Sigma_0$.

The relation $Z(k, h)$ is represented by two forms $\overline{Z}(k, h)$ and $Z(k, h)$. The relation $Z(k, h) = \frac{\omega\pi}{n}$ is equivalent to (1.3). For example, we show the level curves of $Z(k, h) = 0$ in the case $\omega = 0$ in Figure 3.

Figure 3: The level curves of $Z(k, h) = 0$

**Theorem 1.3** Let $\kappa(s)$ be given by Theorem 1.1 and
\[ \mathcal{E}_n(k, h) := n \int_0^{\pi/n} \kappa(s)^2 ds. \]
Then $\mathcal{E}_n(k, h)$ is given by the following (i), (ii) and (iii):

(i) $\mathcal{E}_n(k, h) = \mathcal{E}_n(k, h)$ for $(k, h) \in \bar{\Sigma}$, where

\[
\bar{\mathcal{E}}_n(k, h) := \begin{cases} 
\mathcal{E}_n^{s*}(k, v(k, h)) & \text{for } (k, h) \in \Sigma_{S^{*}}, \\
\mathcal{E}_n^{s}(k, u(k, h)) & \text{for } (k, h) \in \Sigma_{S}.
\end{cases}
\]  

Here the function $\mathcal{E}_n^{s*}(k, v)$ is defined by

\[
\mathcal{E}_n^{s*}(k, v) := \frac{2n^{2}}{\pi}K(k) \cdot \left( \frac{(4-(1-k^{2})(1-v)^{2})^{2}}{(1-v^{2})((1-k^{2})v+k^{2}+1)}+8k^{2}-16 \right)K(k)+16E(k)
\]

and the function $\mathcal{E}_n^{s}(k, u)$ is also defined by

\[
\mathcal{E}_n^{s}(k, u) := \frac{4n^{2}}{\pi}K(k) \cdot \left( \frac{(1-ku)^{2}((1-k^{2})u+k)^{2}K(k)}{u((1-2k^{2})u+2k)((1-k^{2})u^{2}+1)}-4((1-k^{2})K(k)-E(k)) \right).
\]

(ii) $\mathcal{E}_n(k, h) = \mathcal{E}_n(k, h)$ for $(k, h) \in \Sigma$, where

$\mathcal{E}_n(k, h) := \mathcal{E}_n(k, -h)$.

(iii) $\mathcal{E}_n(k, h) = \mathcal{E}_n(k, h)$ for $(k, h) \in \Sigma_0$, where

\[
\mathcal{E}_n(k, h) = \frac{-16n^{2}}{\pi}K(k) \cdot ((1-k^{2})K(k)-E(k)).
\]

**Theorem 1.4** Let $(\kappa(s), \mu, \nu)$ be given by Theorem 1.1 with (1.3) and $h \neq 0$ and

\[
M_n(k, h) := \frac{2\mu\pi^{2}+n\pi\int_{0}^{\pi/n} \kappa(s)^{2}ds}{2\pi\omega\mu+n\int_{0}^{\pi/n} \kappa(s)^{3}ds}.
\]

Then $M_n(k, h)$ is given by the following (i) and (ii):

(i) $M_n(k, h) = M_n(k, h)$ for $(k, h) \in \bar{\Sigma}$, where

\[
M_n(k, h) := \begin{cases} 
M_n^{s*}(k, v(k, h)) & \text{for } (k, h) \in \Sigma_{S^{*}}, \\
M_n^{s}(k, u(k, h)) & \text{for } (k, h) \in \Sigma_{S}.
\end{cases}
\]
Here the function $M^S_n(k, v)$ is defined by

$$M^S_n(k, v):=\frac{\sqrt{2}\pi^2}{n} \cdot \frac{\sqrt{1-v^2}\sqrt{(1-k^2)v+k^2+1}}{((1-k)v+1+k)} \cdot \varphi_1(k, v) \cdot \frac{1}{K(k)^2},$$

\[(1.39)\]

$$\varphi_1(k, v) := (- (1-k^2)(1-v)^2 + 4)^2 K(k) - 8(1-v^2)((1-k^2)v+k^2+1)E(k),$$

\[(1.40)\]

$$\varphi_2(k, v) := ((1+k)v+1-k)((1-k^2)(v+1)^2+4k^2).$$

\[(1.41)\]

and the function $M^S_n(k, u)$ is also defined by

$$M^S_n(k, u):=\frac{-\pi^2}{2n} \cdot \frac{\sqrt{u}\sqrt{(1-k^2)u^2+1}\sqrt{(1-2k^2)u+2k}}{(1-ku)((1-k^2)u+k)} \cdot \varphi_3(k, u) \cdot \frac{1}{K(k)^2},$$

\[(1.42)\]

$$\varphi_3(k, u) := -((1-ku)^2((1-k^2)u+k)^2$$

$$+u((1-2k^2)u+2k)((1-k^2)u^2+1) K(k)$$

$$+2u((1-2k^2)u+2k)((1-k^2)u^2+1)E(k),$$

\[(1.43)\]

$$\varphi_4(k, u) := k^2(1-ku)^2((1-k^2)u^2+1)$$

$$+u((1-2k^2)u+2k)((1-k^2)u+k)^2.$$

\[(1.44)\]

(ii) $M_n(k, h) = M_n(k, h)$ for $(k, h) \in \Sigma$, where

$$M_n(k, h) = -\overline{M}_n(k, -h).$$

For given $M$, the solutions of transcendental equations

$$Z(k, h) = \frac{\omega \pi}{n}, \quad M_n(k, h) = M$$

\[(1.45)\]

give the solution of $(P_n^\omega)$ by Theorem 1.1.

For example, let us determine the solution $\kappa(s)$ of $(P_n^0)$. Figure 4 shows 1-mode solution which is obtained by solving (1.45) with $\omega = 0$ and $n = 1$. Figure 5 shows the curve which is corresponding to Figure 4. The thick-line is corresponding to $0 \leq s \leq \pi$. 

We note that the other curves are not closed except for $k = k_0$ with $h = 0$ in Theorem 1.1, where $k_0$ is the unique solution of $2E(k) - K(k) = 0$ ($0 < k < 1$).
Figure 6 shows the energy curves of stationary solutions for $\omega = 0$ which are obtained from the equation (1.45) and Theorem 1.3.

Investigating the global structure, we obtain the following theorems.

**Theorem 1.5** Let $\omega = 0$ and $n \geq 1$. Then, there exists a unique $n$-mode solution $\kappa(s) = \kappa_n(s; M)$ of $(P_n^0)$ for $-\frac{\pi}{n} < M < \frac{\pi}{n}$. Further there exists no solution for $M \leq -\frac{\pi}{n}$, $\frac{\pi}{n} \leq M$.

**Theorem 1.6** Let $\omega = 0$ and $n \geq 1$. Then, there exists a unique minimizer $\kappa(s) = \kappa(s; M)$ for $-\pi < M < \pi$ with the normalizing condition $\kappa(0) := \max_{0 \leq s \leq 2\pi} \kappa(s)$. This minimizer is 1-mode solution.

**Theorem 1.7** Let $\omega = 0$ and $n \geq 1$. Then, the $n$-mode solution $\kappa(s) = \kappa_n(s; M)$ of $(P_n^0)$ with property $\kappa(0) := \max_{0 \leq s \leq \pi/n} \kappa(s)$ for $0 \leq s \leq \pi/n$ satisfies the following relations:

(i) \[ \lim_{M \uparrow \frac{\pi}{n}} \kappa_n(s; M) = \begin{cases} n & \text{for } s \in \left[0, \frac{\pi}{n}\right), \quad \text{uniformly in } \left[0, \frac{\pi}{n}\right) \\ -\infty & \text{for } s = \frac{\pi}{n} \end{cases} \]

(ii) \[ \lim_{M \downarrow -\frac{\pi}{n}} \kappa_n(s; M) = \begin{cases} \infty & s \in \left(0, \frac{\pi}{n}\right], \quad \text{uniformly in } \left(0, \frac{\pi}{n}\right] \\ -n & s = 0 \end{cases} \]
In this paper, we show the proof of Theorem 1.1. We need long calculation to obtain Theorem 1.2 $\sim 1.7$. The complete proofs of them will appear in a forthcoming papers.


2 Proof of Theorem 1.1

We rewrite $(E_n)$ as first order differential equation to find the solution $\kappa(s)$.

Let us set

$$\kappa(0) := \alpha, \quad \kappa \left( \frac{L}{2n} \right) := \beta (\alpha > 0, \ \alpha > \beta).$$

Multiplying $2\kappa_s$ to $(E_n)$, we have

$$\frac{d}{ds} \left( \frac{d\kappa}{ds} \right)^2 = \frac{d}{ds} \left( -\frac{1}{4} \kappa^4 - \mu \kappa^2 + 2\nu \kappa \right).$$

Integrating above equation on $[0, s]$, we obtain

$$\frac{d\kappa}{ds} = -\sqrt{\tilde{g}(\kappa)},$$

(2.1)

where

$$\tilde{g}(\kappa) = -\frac{1}{4} \kappa^4 - \mu \kappa^2 + 2\nu \kappa + \frac{1}{4} \alpha^4 + \mu \alpha^2 - 2\nu \alpha.$$  

(2.2)

By the Neumann boundary condition of $(E_n)$, we can rewrite $\tilde{g}(\kappa)$ as

$$\tilde{g}(\kappa) = \frac{1}{4} (\alpha - \kappa)(\kappa - \beta) \left( (\kappa + \frac{\alpha + \beta}{2})^2 + 4\delta \right),$$

(2.3)

where $\delta$ is some constant. Comparing the coefficients of (2.2) with that of (2.3), we obtain

$$\mu = \frac{-1}{8} (3(\alpha + \beta)^2 - \frac{1}{2} (3\alpha + \beta)(\alpha + 3\beta) - 8\delta),$$

$$\nu = \frac{1}{32} (\alpha + \beta)((\alpha - \beta)^2 + 16\delta).$$

Let us set

$$A := \frac{3\alpha + \beta}{4}, \quad B := \frac{\alpha + 3\beta}{4}. $$
Then $\mu$ and $\nu$ are represented by
\[
\begin{align*}
\mu &= -\frac{1}{8}(3(A + B)^2 - 8(AB + \delta)), \\
\nu &= \frac{1}{8}(A + B)((A - B)^2 + \delta). 
\end{align*}
\] (2.4)

Further, let us set
\[
\hat{\kappa} := \frac{1}{2}\left(\kappa + \frac{A + B}{2}\right). 
\] (2.5)

Then (2.1) is represented by
\[
\begin{align*}
\frac{d\hat{\kappa}}{ds} &= -\sqrt{\hat{g}(\hat{\kappa})}, \\
\hat{\kappa}(0) &= A, \quad \hat{\kappa}\left(\frac{L}{2n}\right) = B, 
\end{align*}
\] (2.6)

where
\[
\hat{g}(\hat{\kappa}) = (A - \hat{\kappa})(\hat{\kappa} - B)(\hat{\kappa}^2 + \delta). 
\] (2.7)

We need to consider the following five cases in (2.6):

(i) $A + B < 0$, $\delta \leq 0$,
(ii) $A + B < 0$, $\delta > 0$,
(iii) $A + B > 0$, $\delta > 0$,
(iv) $A + B > 0$, $\delta \leq 0$,
(v) $A + B = 0$.

After the proof of Theorem 1.1, we obtain the following five equivalent relations:

(i) $A + B < 0$, $\delta \leq 0 \iff \Sigma_{S^{*}}$,
(ii) $A + B < 0$, $\delta > 0 \iff \Sigma_{S}$,
(iii) $A + B > 0$, $\delta > 0 \iff \Sigma_{R}$,
(iv) $A + B > 0$, $\delta \leq 0 \iff \Sigma_{R^{*}}$,
(v) $A + B = 0$, $\delta \geq 0 \iff \Sigma_{0}$,

where $\Sigma_{S^{*}}$, $\Sigma_{S}$, $\Sigma_{R^{*}}$, $\Sigma_{R}$ and $\Sigma_{0}$ are given by (1.8), (1.9), (1.22), (1.23) and (1.25), respectively.

We note that there exists no solution for $A + B = 0$, $\delta < 0$. We also note that if $(\kappa(s), \mu, \nu)$ is a solution of $(E_n)$, then $(-\kappa(\pi/n - s), \mu, -\nu)$ is also the solution of $(E_n)$ by (2.1) and (2.3). Hence if we have the solutions of $(E_n)$ in the case of (i) and (ii), then we also obtain the solutions of $(E_n)$ in the case (iv) and (iii), respectively. Thus we treat the case (i) and (ii).
2.1 Representation of solutions for $A + B < 0, \delta \leq 0$

**Lemma 2.1** Suppose that $A + B < 0$ and $\delta \leq 0$ in (2.6). Then the solution $\kappa(s)$ of $(E_n)$ is represented by

$$
\kappa(s) = \kappa_n^*(s; A, B, \eta) := 2\hat{\kappa}_n^*(s; A, B, \eta) - \frac{A + B}{2} + 2\eta,
$$

where $\eta := \sqrt{-\delta}$ and

$$
\hat{\kappa}_n^*(s; A, B, \eta) := \frac{(A - \eta)(B - \eta)}{B - \eta + (A - B)cn^2 \left( \frac{2\pi}{n}K(k) \left( \frac{\pi}{n} - s \right), k \right)}.
$$

Moreover it holds that

$$
\sqrt{(A - \eta)(B + \eta)} = \frac{2n}{\pi}K(k)
$$

with

$$
k = -\sqrt{\frac{2\eta(A - B)}{(A - \eta)(B + \eta)}}.
$$

**Proof.** Under the condition that $A + B < 0$ and $\delta = -\eta^2 \leq 0$ ($\eta \geq 0$), we have

$$
B < A \leq -\eta \leq 0 \leq \eta,
$$

since $A > B$ and (2.7) is positive on the interval $(B, A)$. Now we show $A \neq -\sqrt{-\delta}$. Assume that $A = -\sqrt{-\delta} < 0$. Then, substituting $\hat{\kappa} = A - 1/\xi$ into (2.6), we have

$$
\frac{L}{2n} = \int_{B}^{A} \frac{d\hat{\kappa}}{(A - \hat{\kappa})\sqrt{-(\hat{\kappa} - B)(\hat{\kappa} + A)}}
$$

$$
= \frac{1}{\sqrt{-2A(A - B)}} \int_{1/(A - B)}^{\infty} \frac{d\xi}{\sqrt{\left( \frac{\xi - 1}{A - B} \right) - \frac{1}{2A}}}
$$

$$
= \frac{1}{\sqrt{-2A(A - B)}} \left[ 2\log \left| \sqrt{\frac{\xi - 1}{A - B} + \frac{1}{2A}} \right| \right]_{1/(A - B)}^{\infty}
$$

$$
= \infty.
$$

This is a contradiction. In the same way, we obtain

$$
\frac{L}{2n} = \int_{B}^{0} \frac{d\hat{\kappa}}{-\hat{\kappa}\sqrt{-(\hat{\kappa} - B)}} = \infty.
$$
in the case $A = -\sqrt{\eta} = 0$. This is also contradiction. Thus it holds that

$$B < A < -\eta \leq 0 \leq \eta.$$  \hfill (2.12)

Let us set

$$\tilde{\kappa}(s) = \frac{1}{\kappa(s)} + \eta.$$  \hfill (2.13)

Then (2.6) becomes

$$\frac{d\tilde{\kappa}}{ds} = \sqrt{(A-\eta)(B-\eta)} \sqrt{\left(\tilde{\kappa} - \frac{1}{A-\eta}\right) \left(\frac{1}{B-\eta} - \tilde{\kappa}\right)} (2\eta\tilde{\kappa} + 1).$$

Further we introduce change of variable $\tilde{\kappa}$ to $\varphi$ by

$$\tilde{\kappa}(s) = \frac{1}{B-\eta} - \left(\frac{1}{B-\eta} - \frac{1}{A-\eta}\right) \sin^2 \varphi(s) \text{ for } \varphi(s) \in [0, \pi/2].$$  \hfill (2.14)

We obtain

$$\frac{d\varphi}{ds} = \frac{-1}{2} \sqrt{(A-\eta)(B+\eta)} \sqrt{1-k^2 \sin^2 \varphi}.$$  

Integrating the above equation on $[0, s]$, we have

$$\frac{\sqrt{(A-\eta)(B+\eta)}}{2} s = K(k) - \int_0^{\varphi(s)} \frac{d\varphi}{\sqrt{1-k^2 \sin^2 \varphi}},$$  \hfill (2.15)

since $\varphi(0) = \pi/2$. At $s = \pi/n$, we obtain (2.10) by $\varphi(\pi/n) = 0$.

Substituting (2.10) and $\xi = \sin \varphi$ into (2.15), we have

$$\sin(\varphi(s)) = \text{sn} \left(\frac{n}{\pi} K(k) \left(\frac{\pi}{n} - s\right), k\right),$$

which implies that

$$\tilde{\kappa}(s) = \frac{1}{A-\eta} + \left(\frac{1}{B-\eta} - \frac{1}{A-\eta}\right) \text{cn}^2 \left(\frac{n}{\pi} K(k) \left(\frac{\pi}{n} - s\right), k\right).$$  \hfill (2.16)

On the other hand, it follows from (2.5) and (2.13) that we have

$$\kappa(s) = 2 \tilde{\kappa}(s) - \frac{A + B}{2} = \frac{2}{\tilde{\kappa}(s)} - \frac{A + B}{2} + 2\eta.$$

Thus, substituting (2.16) into above relation, the lemma holds.  \hfill \Box
2.2 Representation of solutions for $A + B < 0, \delta > 0$

**Lemma 2.2** Suppose that $A + B < 0, \delta > 0$ in (2.6). Then the solution of $\kappa(s)$ of $(E_n)$ is represented by

$$\kappa(s) = \kappa_n^S(s; A, B, \delta) := 2\hat{\kappa}_n^S(s; A, B, \delta) - \frac{A + B}{2},$$  \hspace{1cm} (2.17)

where

$$\hat{\kappa}_n^S(s; A, B, \delta) := \frac{AB_\delta + A_\delta B - (AB_\delta - A_\delta B)cn \left( \frac{2n}{\pi} K(k) \left( \frac{\pi}{n} - s \right), k \right)}{A_\delta + B_\delta + (A_\delta - B_\delta)cn \left( \frac{2n}{\pi} K(k) \left( \frac{\pi}{n} - s \right), k \right)},$$  \hspace{1cm} (2.18)

$$A_\delta := \sqrt{A^2 + \delta}, \quad B_\delta := \sqrt{B^2 + \delta}.$$  \hspace{1cm} (2.19)

Moreover it holds that

$$\sqrt{A_\delta B_\delta} = \frac{2n}{\pi} K(k),$$  \hspace{1cm} (2.20)

with

$$k = \sqrt{\frac{1}{2} \left( 1 - \frac{AB + \delta}{A_\delta B_\delta} \right)}.$$  \hspace{1cm} (2.21)

**Proof.** Let us set

$$\hat{\kappa}(s) = \frac{1}{\tilde{\kappa}(s)} + B.$$  \hspace{1cm} (2.22)

Then (2.6) becomes

$$d\tilde{\kappa} = B_\delta \sqrt{A - B} \sqrt{\left( \tilde{\kappa} - \frac{1}{A - B} \right) \left( \tilde{\kappa}^2 + \frac{2B}{B_\delta^2} \tilde{\kappa} + \frac{1}{B_\delta^2} \right)}.$$  \hspace{1cm} (2.23)

Further we introduce change of variable $\tilde{\kappa}$ to $\varphi$ by

$$\tilde{\kappa}(s) = \frac{1}{A - B} \left( 1 + \frac{A_\delta}{B_\delta} \tan^2 \varphi(s) \right),$$  \hspace{1cm} (2.24)

where $\varphi(s) \in [0, \pi]$. We get

$$\frac{A_\delta}{(A - B)B_\delta} \tan \frac{\varphi}{2} \left( 1 + \tan^2 \frac{\varphi}{2} \right) \frac{d\varphi}{ds}$$

$$= \sqrt{A_\delta B_\delta} \cdot \frac{A_\delta}{(A - B)B_\delta} \tan \frac{\varphi}{2} \sqrt{1 + \tan^4 \frac{\varphi}{2} + 2AB + \delta} \cdot \frac{A_\delta}{A_\delta B_\delta} \cdot \tan \frac{\varphi}{2}$$

$$= \sqrt{A_\delta B_\delta} \cdot \frac{A_\delta}{(A - B)B_\delta} \cdot \tan \frac{\varphi}{2} \sqrt{\left( 1 + \tan^2 \frac{\varphi}{2} \right)^2 - 4k^2 \tan^2 \frac{\varphi}{2}}$$

$$= \sqrt{A_\delta B_\delta} \cdot \frac{A_\delta}{(A - B)B_\delta} \cdot \tan \frac{\varphi}{2} \left( 1 + \tan^2 \frac{\varphi}{2} \right) \sqrt{1 - k^2 \sin^2 \varphi}.$$  \hspace{1cm} (2.25)
Thus we obtain
\[ \frac{d\varphi}{ds} = \sqrt{A_\delta B_\delta} \sqrt{1 - k^2 \sin^2 \varphi}. \]

Integrating the above equation on \([0, s]\), we have
\[ \sqrt{A_\delta B_\delta} s = \int_{0}^{\varphi(s)} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}} \]
(2.24)

by \( \varphi(0) = 0 \). At \( s = \pi/n \), we have (2.20) by \( \varphi(\pi/n) = \pi \).

Substituting (2.20) and \( \xi = \sin \varphi \) into (2.24), we obtain
\[ \sin \varphi(s) = \text{sn} \left( \frac{2n}{\pi} K(k)s, k \right), \]
which implies that
\[ \cos(\varphi(s)) = \text{cn} \left( \frac{2n}{\pi} K(k)s, k \right). \]

Thus we have
\[ \tan^2 \frac{\varphi(s)}{2} = \frac{1 - \cos \varphi(s)}{1 + \cos \varphi(s)} = \frac{1 - \text{cn} \left( \frac{2n}{\pi} K(k)s, k \right)}{1 + \text{cn} \left( \frac{2n}{\pi} K(k)s, k \right)}. \]

Substituting above relation into (2.23), \( \tilde{\kappa}(s) \) becomes
\[ \tilde{\kappa}(s) = \frac{A_\delta + B_\delta - (A_\delta - B_\delta) \text{cn} \left( \frac{2n}{\pi} K(k)s, k \right)}{(A - B)B_\delta \left( 1 + \text{cn} \left( \frac{2n}{\pi} K(k)s, k \right) \right)}. \]
(2.25)

On the other hand, we obtain
\[ \kappa(s) = 2 \tilde{\kappa}(s) - \frac{A + B}{2} = 2 \left( \frac{1}{\tilde{\kappa}(s)} + B \right) - \frac{A + B}{2} \]
by (2.5) and (2.22). Thus, substituting (2.25) into above relation, we obtain (2.17) since
\[ \text{cn} \left( \frac{2n}{\pi} K(k)s, k \right) = -\text{cn} \left( \frac{2n}{\pi} K(k) \left( \frac{\pi}{n} - s \right), k \right). \]

\[ \square \]
2.3 Change of parameters for $A + B < 0, \delta \leq 0$

Let us consider the case $A + B < 0, \delta \leq 0$. It is difficult for us to investigate the global structure by using the parameters $A, B$ and $\eta := \sqrt{-\delta}$. $A$ and $B$ belong to semi-infinite interval and $\eta$ is constrained by (2.20) and (2.21). Thus we change the parameter.

Let us see $(k, \tilde{h})$ be known and $A, B$ and $\eta$ be the solutions of the system of

\[
\begin{cases}
    k^2 = \frac{2\eta(A - B)}{(A - \eta)(B + \eta)}, \\
    \sqrt{(A - \eta)(B + \eta)} = \frac{2n}{\pi} K(k), \\
    A = (1 - \tilde{h})B (0 < \tilde{h} < 2).
\end{cases}
\]

Then we obtain the following lemma:

**Lemma 2.3** Suppose that $A + B < 0$ and $\delta \leq 0$. Then $A, B$ and $\eta$ are represented by

\[
A = -\frac{(1 - k^2)v + 1}{\sqrt{2}\sqrt{1 + \sqrt{(1 - k^2)v + k^2 + 1}}} \cdot \left( \frac{2n}{\pi} K(k) \right),
\]

\[
B = -\frac{(2 - k^2)v + k^2 + 2}{\sqrt{2}\sqrt{1 + \sqrt{(1 - k^2)v + k^2 + 1}}} \cdot \left( \frac{2n}{\pi} K(k) \right),
\]

\[
\eta = \frac{k^2\sqrt{1 - v}}{\sqrt{2}\sqrt{1 + v}\sqrt{(1 - k^2)v + k^2 + 1}} \cdot \left( \frac{2n}{\pi} K(k) \right)
\]

and $v = v(k, h)$ for $(k, h) \in \Sigma_{S\ast}$, where $\Sigma_{S\ast}$ and $v(k, h)$ are defined by (1.8) and (1.16), respectively.

**Proof.** It follows from (2.26),(2.27) and (2.28) that we obtain

\[
\eta = -\frac{k^2}{2hB} \left( \frac{2n}{\pi} K(k) \right)^2.
\]

Substituting (2.28) and (2.30) into (2.27), we have

\[
(1 - \tilde{h})B^4 - \frac{1}{2}(2 - k^2) \left( \frac{2n}{\pi} K(k) \right)^2 B^2 - \frac{k^4}{4h^2} \left( \frac{2n}{\pi} K(k) \right)^4 = 0.
\]

If $\tilde{h} \geq 1$, then the left hand side of above equation is negative. Thus, we may consider $0 < \tilde{h} < 1$. Solving the above equation with respect to $B$, we obtain

\[
A = -(1 - \tilde{h})\xi \left( \frac{2n}{\pi} K(k) \right), B = -\xi \left( \frac{2n}{\pi} K(k) \right),
\]

\[
\eta = \frac{k^2}{2h\xi} \left( \frac{2n}{\pi} K(k) \right)
\]

(2.31)
since $B < 0$, where

$$\xi := \frac{\sqrt{(2 - k^2)\tilde{h} + \sqrt{(2 - k^2)^2\tilde{h}^2 + 4k^4(1 - \tilde{h})}}}{2\sqrt{\tilde{h}\sqrt{1 - \tilde{h}}}}$$

for $(k, \tilde{h}) \in \{(k, \tilde{h}); 0 < k < 1, 0 < \tilde{h} < 1\}$.

To simplify the representation, we set

$$v = \frac{-2 + (2 - k^2)\tilde{h} + \sqrt{(2 - k^2)^2\tilde{h}^2 + 4k^4(1 - \tilde{h})}}{2(1 - k^2)},$$

which implies that

$$2(1 - k^2)v + 2 - (2 - k^2)\tilde{h} = \sqrt{(2 - k^2)^2\tilde{h}^2 + 4k^4(1 - \tilde{h})}. \quad (2.32)$$

Solving (2.32) with respect to $\tilde{h}$ yields

$$\tilde{h} = \frac{(v + 1)((1 - k^2)v + k^2 + 1)}{(2 - k^2)v + k^2 + 2}.$$

Hence we have

$$(2 - k^2)\tilde{h} + \sqrt{(2 - k^2)^2\tilde{h}^2 + 4k^4(1 - \tilde{h})} = 2(1 - k^2)v + 2,$$

$$1 - \tilde{h} = \frac{(1 - v)((1 - k^2)v + 1)}{(2 - k^2)v + k^2 + 2}$$

by (2.32). Thus $\xi$ becomes

$$\xi = \frac{(2 - k^2)v + k^2 + 2}{\sqrt{2\sqrt{1 - v^2}\sqrt{(1 - k^2)v + k^2 + 1}}}.$$

Substituting $h = \tilde{h} + 2$ and above relation into (2.31), the lemma holds. □

### 2.4 Change of parameters for $A + B < 0, \delta > 0$

Let us consider the case $A + B < 0, \delta > 0$. It is also difficult for us to investigate the global structure by using the parameters $A, B$ and $\delta$. Thus we change the parameters.
Let \((k, \tilde{h})\) be known and \(A, B\) and \(\delta\) be the solutions of the system of

\[
\begin{align*}
\left\{\begin{array}{ll}
k^2 &= \frac{1}{2} \left(1 - \frac{\sqrt{(A^2 + \delta)(B^2 + \delta)}}{AB + \delta}\right), \\
\sqrt{(A^2 + \delta)(B^2 + \delta)} &= \frac{2n}{\pi} K(k), \\
A &= (1 - \tilde{h})B \ (0 < \tilde{h} < 2).
\end{array}\right.
\end{align*}
\]  

Then we obtain the following lemma:

**Lemma 2.4** Suppose that \(A + B < 0, \ \delta > 0\). Then \(A, B\) and \(\delta\) are represented by

\[
\begin{align*}
A &= -\frac{\sqrt{u}(1 - 2k^2 - 2k(1 - k^2)u)}{\sqrt{(1 - k^2)u^2 + 1}\sqrt{(1 - 2k^2)u + 2k}} \cdot \left(\frac{2n}{\pi} K(k)\right), \\
B &= -\frac{\sqrt{(1 - 2k^2)u + 2k}}{\sqrt{u}\sqrt{(1 - k^2)u^2 + 1}} \cdot \left(\frac{2n}{\pi} K(k)\right), \\
\delta &= \frac{(1 - k^2)u((1 - 2k^2)u + 2k)}{(1 - k^2)u^2 + 1} \left(\frac{2n}{\pi} K(k)\right)^2,
\end{align*}
\]

and \(u = u(k, h)\) for \((k, h) \in \Sigma_S\), where \(u(k, h)\) and \(\Sigma_S\) are defined by (1.9) and (1.20), respectively.

**Proof.** It follows from (2.34) and (2.33) that we obtain

\[
\delta = (1 - 2k^2) \left(\frac{2n}{\pi} K(k)\right)^2 - (1 - \tilde{h})B^2.
\]  

Substituting (2.35) and (2.37) into (2.34), we have

\[
-\tilde{h}^2(1 - \tilde{h})B^4 + (1 - 2k^2)(1 - \tilde{h})^2 \left(\frac{2n}{\pi} K(k)\right)^2 B^2 - 4k^2(1 - k^2) \left(\frac{2n}{\pi} K(k)\right)^4 = 0.
\]

Solving above equation with respect to \(B\), we obtain the following two solutions (i) and (ii):

\[
\begin{align*}
(i) \quad B &= \frac{-2\sqrt{2k\sqrt{1 - k^2}}}{\sqrt{\tilde{h}\sqrt{(1 - 2k^2)\tilde{h}} + \sqrt{\tilde{h}^2 - 4k^2(1 - k^2)(2 - \tilde{h})^2}}} \cdot \left(\frac{2n}{\pi} K(k)\right) \\
&\quad \text{for } (k, \tilde{h}) \in \{(k, \tilde{h}); \ 0 < k \leq 1/\sqrt{2}, \ H(k) < \tilde{h} < 2\} \\
&\quad \cup \{(k, \tilde{h}); \ 1/\sqrt{2} < k \leq 1, \ 1 < \tilde{h} < 2\}, \\
(ii) \quad B &= \frac{-2\sqrt{2k\sqrt{1 - k^2}}}{\sqrt{\tilde{h}\sqrt{(1 - 2k^2)\tilde{h}} - \sqrt{\tilde{h}^2 - 4k^2(1 - k^2)(2 - \tilde{h})^2}}} \cdot \left(\frac{2n}{\pi} K(k)\right) \\
&\quad \text{for } (k, \tilde{h}) \in \{(k, \tilde{h}); \ 0 < k \leq 1/\sqrt{2}, \ H(k) < \tilde{h} < 1\},
\end{align*}
\]
since $B < 0$, where
\[ H(k) := \frac{4k\sqrt{1-k^2}}{1+2k\sqrt{1-k^2}}. \]

Further changing the parameter $(k, \tilde{h})$ to $(k, h)$ by
\[ h = \begin{cases} 2 - \sqrt{\tilde{h}^2 - 4k^2(1-k^2)(2-\tilde{h})^2} & \text{for case (i)}, \\ 2 + \sqrt{\tilde{h}^2 - 4k^2(1-k^2)(2-\tilde{h})^2} & \text{for case (ii)}, \end{cases} \]

$\tilde{h}$ becomes
\[ \tilde{h} = \frac{(2-h)^2 + 16k^2(1-k^2)}{8k^2(1-k^2) + \sqrt{(1-2k^2)^2(2-h)^2 + 16k^2(1-k^2)}} \quad (2.38) \]

for $(k, h) \in \Sigma_S$, where $\Sigma_S$ is defined by (1.9).

To simplify the representation, we set
\[ u = \frac{1}{4k(1-k^2)} \cdot \left( 2 - h + \frac{(1-2k^2)((2-h)^2 + 16k^2(1-k^2))}{8k^2(1-k^2) + \sqrt{(1-2k^2)^2(2-h)^2 + 16k^2(1-k^2)}} \right) \]

\[ = \frac{1}{4k(1-k^2)} \cdot \left( 2 - h + \frac{-8k^2(1-k^2) + \sqrt{(1-2k^2)^2(2-h)^2 + 16k^2(1-k^2)}}{1-2k^2} \right), \quad (2.39) \]

which implies that
\[(1-2k^2)(4k(1-k^2)u-2+h)+8k^2(1-k^2) = \sqrt{(1-2k^2)^2(2-h)^2 + 16k^2(1-k^2)}.\]

Solving the above equation with respect to $h$ yields
\[ h = \frac{2(1-ku)((1-k^2)(1-2k^2)u+k(3-2k^2))}{(1-2k^2)u+2k} \]

Substituting the above relation into (2.38) gives
\[ \tilde{h} = \frac{4k(1-k^2)u - (2-h)}{(1-2k^2)} \]
\[ = \frac{2k((1-k^2)u^2 + 1)}{(1-2k^2)u + 2k} \]
by (2.39). Hence we have

$$1 - \tilde{h} = \frac{u(1 - 2k^2 - 2k(1 - k^2)u)}{(1 - 2k^2)u + 2k}.$$ 

Further we obtain

$$(1 - 2k^2)\tilde{h} + \sqrt{\tilde{h}^2 - 4k^2(1 - k^2)(2 - \tilde{h})^2} = 4k(1 - k^2)u$$

by (2.38) and (2.39) in the case (i). We also obtain

$$(1 - 2k^2)\tilde{h} - \sqrt{\tilde{h}^2 - 4k^2(1 - k^2)(2 - \tilde{h})^2} = 4k(1 - k^2)u.$$ 

in the case (ii). Using above relations, the lemma holds. \(\square\)

**Proof of Theorem 1.1.** Substituting (2.29) and (2.36) into (2.4), (2.8) and (2.17), we obtain (i) of Theorem 1.1.

We obtain (ii) of Theorem 1.1 since if \((\kappa(s), \mu, \nu)\) is a solution of \((E_n)\), then \((-\kappa(\pi/n - s), \mu, -\nu)\) is also the solution of \((E_n)\) by (2.1) and (2.3).

It follows from Lemma 2.4 that \(A + B = 0, \delta \geq 0\) is equivalent to \(\Sigma_0\). Thus we obtain (iii) of Theorem 1.1 since \(u(k, 0) = 1/k\). \(\square\)

**References**


