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Arcsine Law as Classical Limit

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Abstract

Recently we have proved that the Arcsine Law appears as the Classical Limit of quantum harmonic oscillators, in the framework of algebraic probability thoery. In the present paper we discuss how to generalize the result by means of the notion of interacting Fock spaces, which associates algebraic probability theory and the theory of orthogonal polynomials of probability measures. As an application we show that the Classical Limit for interacting Fock spaces corresponnding to q-Gaussians and the exponential distribution are the Arcsine Law.

1 Introduction

Let us consider the time-avereged distribution of position x for a 1-dimensional classical harmonic oscillator. It is easy to see that the distribution (after standardization) has the form

$$\mu_{As}(dx) = \frac{1}{\pi}\frac{dx}{\sqrt{2-x^2}} \quad (-\sqrt{2} < x < \sqrt{2}).$$

The distribution μ_{As} is called the (normalized) **Arcsine Law**, which also plays lots of crucial roles both in pure and applied probability theory. The n-th moment $M_n := \int_{\mathbb{R}} x^n \mu_{As}(dx)$ is given by

$$M_{2m+1} = 0, \quad M_{2m} = \frac{1}{2^m}\binom{2m}{m}.$$

The moment problem for the Arcsine law is determinate, that is, the moment sequence $\{M_n\}$ characterizes μ_{As}. In [4] we have proved that the Arcsine

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Law appears as the **Classical Limit** of quantum harmonic oscillator, in the framework of algebraic probability thoery (also known as "noncommutative probability theory" or "quantum probability theory").

The purpose of this paper is to develop the result and viewpoint in [4]. Section 2 and 3 are devoted to review basic notions in algebraic probabity and the "Quantum-Classical Correspondence" for quantum harmonic oscillators proved in [4] . In section 4 we discuss how to generalize the result and viewpoint by means of "interacting Fock spaces [1]", which associates algebraic probability theory and the theory of orthogonal polynomials. In the last section we show that the Classical Limit for the interacting Fock space corresponding to Laguerre polynomials again becomes the Arcsine law (after standardization).

2 Quantum Harmonic Oscillator

Let $\mathcal{A}$ be a $*$-algebra. We call a linear map $\varphi : \mathcal{A} \to \mathbb{C}$ a state on $\mathcal{A}$ if it satisfies

$$\varphi(1) = 1, \quad \varphi(a^*a) \geq 0.$$

A pair $(\mathcal{A}, \varphi)$ of a $*$-algebra and a state on it is called an algebraic probability space. Here we adopt a notation for a state $\varphi : \mathcal{A} \to \mathbb{C}$, an element $X \in \mathcal{A}$ and a probability distribution μ on $\mathbb{R}$.

Notation 2.1. We use the notation $X \sim_\varphi \mu$ when $\varphi(X^m) = \int_{\mathbb{R}} x^m \mu(dx)$ for all $m \in \mathbb{N}$.

Remark 2.2. Existence of μ for X which satisfies $X \sim_\varphi \mu$ always holds.

Definition 2.3 (Quantum harmonic oscillator). A quantum harmonic oscillator is a triple $(\Gamma(\mathbb{C}), a, a^*)$ where $\Gamma(\mathbb{C})$ is a Hilbert space $\Gamma(\mathbb{C}) := \oplus_{n=0}^{\infty} \mathbb{C}\Phi_n$ with inner product given by $\langle \Phi_n, \Phi_m \rangle = \delta_{n,m}$, and a, a^* are operators defined as follows:

$$a\Phi_0 = 0, \quad a\Phi_n = \sqrt{n}\Phi_{n-1} (n \geq 1)$$
$$a^*\Phi_n = \sqrt{n+1}\Phi_{n+1}.$$

Let $\mathcal{A}$ be the $*$-algebra generated by a, and φ_n be the state defined as $\varphi_n(\cdot) := \langle \Phi_n, (\cdot)\Phi_n \rangle$. Then $(\mathcal{A}, \varphi_n)$ is an algebraic probability space. It is well known that

$$a + a^*$$

represents the "position" and that

$$a + a^* \sim_{\varphi_0} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx.$$

That is, in $n = 0$ case, the distribution of position is Gaussian.

On the other hand, the asymptotic behavior of the distributions of position as n tends to infinity is quite nontrivial.

3 Quantum-Classical Correspondence

Then a question arises: Is it possible to see whether and in what meaning the "Quantum-Classical Correspondence" holds for harmonic oscillators? This question, which is related to fundamental problems in Quantum theory and asymptotic analysis [2], was analyzed in [4] from the viewpoint of noncommutative algebraic probability with quite a simple combinatorial argument.

The folloing is the main result in [4]:

Theorem 3.1. *Let μ_N be a probability distribution on $\mathbb{R}$ such that*

$$\frac{a+a^*}{\sqrt{2N}} \sim_{\varphi_N} \mu_N.$$

Then μ_N weakly converges to μ_{As}.

Proof. We only have to prove moment convergence because it is known that moment convergence implies weak convergence when the moment problem for the limit distribution is determinate.

First we can easily prove that

$$\varphi_N((\frac{a+a^*}{\sqrt{2N}})^{2m+1}) = \langle \Phi_N, (\frac{a+a^*}{\sqrt{2N}})^{2m+1}\Phi_N\rangle = 0$$

since $\langle \Phi_N, \Phi_M\rangle = 0$ when $N \neq M$.

To consider the moments of even degrees, we introduce the following notations:

- $\Lambda^{2m} := \{\text{maps from } \{1, 2, ..., 2m\} \text{ to } \{1, *\}\}$,
- $\Lambda^{2m}_m := \{\lambda \in \Lambda^{2m}; |\lambda^{-1}(1)| = |\lambda^{-1}(*)| = m\}$.

Note that the cardinality $|\Lambda^{2m}_m|$ equals to $\binom{2m}{m}$ because the choice of λ is equivalent to the choice of m elements which consist the subset $\lambda^{-1}(1)$ from $2m$ elements in $\{1, 2, ..., 2m\}$.

It is clear that for any $\lambda \notin \Lambda^{2m}_m$

$$\langle \Phi_N, a^{\lambda_1}a^{\lambda_2}\cdots a^{\lambda_{2m}}\Phi_N\rangle = 0$$

since $\langle \Phi_N, \Phi_M\rangle = 0$ when $N \neq M$.

On the other hand, for any $\lambda \in \Lambda_m^{2m}$ the inequality

$$N \cdots (N-m+1) \leq \langle \Phi_N, a^{\lambda_1} a^{\lambda_2} \cdots a^{\lambda_{2m}} \Phi_N \rangle \leq (N+1) \cdots (N+m)$$

holds when N is sufficiently large, because the minimum is achieved when

$$\lambda_i = \begin{cases} 1, & (1 \leq i \leq m) \\ *, & (m+1 \leq i \leq 2m) \end{cases}$$

and the maximum is achieved when

$$\lambda_i = \begin{cases} *, & (1 \leq i \leq m) \\ 1, & (m+1 \leq i \leq 2m) \end{cases}$$

by the definition of a, a^*.

Using the inequality above we have

$$\frac{1}{N^m} \langle \Phi_N, a^{\lambda_1} a^{\lambda_2} \cdots a^{\lambda_{2m}} \Phi_N \rangle \to 1 \quad (N \to \infty).$$

and then

$$\begin{aligned} \varphi_N\left(\left(\frac{a+a^*}{\sqrt{2N}}\right)^{2m}\right) &= \langle \Phi_N, \left(\frac{a+a^*}{\sqrt{2N}}\right)^{2m} \Phi_N \rangle \\ &= \tfrac{1}{2^m} \textstyle\sum_{\lambda \in \Lambda^{2m}} \tfrac{1}{N^m} \langle \Phi_N, a^{\lambda_1} a^{\lambda_2} \cdots a^{\lambda_{2m}} \Phi_N \rangle \\ &= \tfrac{1}{2^m} \textstyle\sum_{\lambda \in \Lambda_m^{2m}} \tfrac{1}{N^m} \langle \Phi_N, a^{\lambda_1} a^{\lambda_2} \cdots a^{\lambda_{2m}} \Phi_N \rangle \\ &\to \frac{1}{2^m} |\Lambda_m^{2m}| = \frac{1}{2^m} \binom{2m}{m} \quad (N \to \infty). \end{aligned}$$

□

As we have stated, the Arcsine law as the time-averaged distribution for classical harmonic oscillators emerges from the distributions for quantum harmonic oscillators. This is nothing but a noncommutative algebraic realization of Quantum-Classical Correspondence for harmonic oscillators.

In other words, **the Arcsine Law appears as the classical limit of quantum harmonic oscillators**. The "time averaged" nature is deeply related to the notion of Bohr's "complementarity" for energy and time. Starting from energy eigenstates, one cannot obtain the classical harmonic oscillator itself but time averaged distribution of it.

4 Generalization: Interacting Fock space

In this section we discuss a generalization of Theorem 3.1.

Definition 4.1 (Jacobi sequence). A sequence $\{\omega_n\}$ is called a Jacobi sequence if it satisfies one of the conditions below:

- (finite type) There exist a number m such that $\omega_n > 0$ for $n < m$ and $\omega_n = 0$ for $n \geq m$;
- (infinite type) $\omega_n > 0$ for all n.

Definition 4.2 (Interacting Fock space). Let $\{\omega_n\}$ be a Jacobi sequence. An interacting Fock space $\Gamma_{\{\omega_n\}}$ is a triple $(\Gamma(\mathbb{C}), a, a^*)$ where $\Gamma(\mathbb{C})$ is a Hilbert space $\Gamma(\mathbb{C}) := \oplus_{n=0}^{\infty}\mathbb{C}\Phi_n$ with inner product given by $\langle \Phi_n, \Phi_m \rangle = \delta_{n,m}$, and a, a^* are operators defined as follows:

$$a\Phi_0 = 0, \quad a\Phi_n = \sqrt{\omega_n}\Phi_{n-1}(n \geq 1)$$

$$a^*\Phi_n = \sqrt{\omega_{n+1}}\Phi_{n+1}.$$

As before, let $\mathcal{A}$ be the $*$-algebra generated by a, and φ_n be the state defined as $\varphi_n(\cdot) := \langle \Phi_n, (\cdot)\Phi_n \rangle$. Then $(\mathcal{A}, \varphi_n)$ is an algebraic probability space.

The following result proved in [5] is a generalization of Theorem 3.1:

Theorem 4.3. *Let $\Gamma_{\{\omega_n\}} := (\Gamma(\mathbb{C}), a, a^*)$ be an interacting Fock space satisfying the condition*

$$\lim_{n\to\infty} \frac{\omega_{n+1}}{\omega_n} = 1$$

and μ_N be a probability distribution on $\mathbb{R}$ such that

$$\frac{a + a^*}{\sqrt{2\omega_N}} \sim_{\varphi_N} \mu_N.$$

Then μ_N weakly converges to μ_{As}.

The theorem above has an interpretation in terms of orthogonal polynomials. To see this we explain the relation between interacting Fock spaces, probability measures and orthogonal polynomials.

Let μ be a probability measure on $\mathbb{R}$ having finite moments. (For the rest of the present paper, we always assume that all the moments are finite.) Then the space of polynomial functions is contained in the Hilbert space $L^2(\mathbb{R}, \mu)$. A Gram-Schmidt procedure provides orthogonal polynomials which only depend on the moment sequence.

Let $\{p_n(x)\}_{n=0,1,\cdots}$ be the monic orthogonal polynomials of μ such that the degree of p_n equals to n. Then there exist sequences $\{\alpha_n\}_{n=0,1,\cdots}$ and Jacobi sequence $\{\omega_n\}_{n=1,2,\cdots}$ such that

$$xp_n(x) = p_{n+1}(x) + \alpha_{n+1}p_n(x) + \omega_n p_{n-1}(x) \quad (p_{-1}(x) \equiv 0).$$

$\alpha_n \equiv 0$ if μ is symmetric, i.e., $\mu(-dx) = \mu(dx)$.

It is known that there exist an isometry $U : \Gamma_{\{\omega_n\}} \to L^2(\mathbb{R}, \mu)$ through which we obtain

$$a + a^* + a^o \sim_{\varphi_N} |P_N(x)|^2\mu(dx)$$

where a^o is an operator defined by $a^o\Phi_n := \alpha_{n+1}\Phi_n$ and P_n denotes the normalized orthogonal polynomial of degree n [3]. Then Theorem 4.3 implies the following:

Theorem 4.4. *Let μ be a symmmetric measure such that the corresponting Jacobi sequence $\{\omega_n\}$ satisfies*

$$\lim_{n\to\infty} \frac{\omega_{n+1}}{\omega_n} = 1$$

Then the measure μ_n defined as $\mu_n(dx) := |P_n(\sqrt{2\omega_n}x)|^2\mu(\sqrt{2\omega_n}dx)$ weakly converge to μ_{As}.

Since "q-Gaussians" ($0 \le q \le 1$, $q = 1$ is Gaussian and $q = 0$ is Wigner Semicircle Law), corresponding to $\omega_n = [n]_q := 1+q+q^2+\cdots+q^{n-1}$, satisfy the condition above, μ_{As} is turned out to be the Classical Limit of these measures.

In the next section we discuss the Classical Limit for the case of exponential distribution as an example of asymmetric measure.

5 Example: Exponential-Laguerre case

Let μ be the exponential distribution, i.e., $\mu(dx) := e^{-x}dx$ $(x > 0)$.Then

$$xl_n(x) = l_{n+1}(x) + (2n+1)l_n(x) + n^2 l_{n-1}(x) \quad (l_{-1}(x) \equiv 0),$$

holds, where l_n denotes the Laguerre polynomial of n-th degree modified to be monic. Let us consider the interacting Fock space $\Gamma_{\{\omega_n\}}$ for $\omega_n = n^2$. As we have discussed,

$$a + a^* + a^o \sim_{\varphi_N} |L_N(x)|^2 e^{-x}dx \quad (x > 0).$$

where L_n denotes the usual (normalized) Laguerre polinomial of n-th order.

Then we can calculate the "Limit moment" of $\mu_n(dx) := |L_n(nx)|^2 ne^{-nx}dx$ $(x > 0)$ in the spirit of the proof of Theorem 3.1 and Theorem 4.3 [5].

Proposition 5.1.

$$\lim_{N\to\infty} \varphi_N\left(\left(\frac{a+a^*+a^o}{N}\right)^m\right) = \sum_l 2^{m-2l}\binom{m}{m-2l}\binom{2l}{l}.$$

The right hand side of the proposition above is simplified as follows.

Lemma 5.2.

$$\sum_l 2^{m-2l}\binom{m}{m-2l}\binom{2l}{l} = \binom{2m}{m}.$$

Proof. Consider two sets of maps

$$L := \{f : \mathbf{m} \to \mathbf{4}; |f^{-1}(0)| = |f^{-1}(1)|\}$$

$$R := \{\tilde{f} : \mathbf{2} \times \mathbf{m} \to \mathbf{2}; |\tilde{f}^{-1}(0)| = |\tilde{f}^{-1}(1)|\},$$

where $\mathbf{m} := \{0, 1, 2, \ldots, m-1\}$. Since we can construct an isomorphism between L and R, $|L| = |R|$. This is what to be proved. (This proof is obtained in discussion with Hiroki Sako). □

It is easy to show that

$$\binom{2m}{m} = \int_0^4 x^m \frac{1}{\pi}\frac{dx}{\sqrt{4-(x-2)^2}},$$

and hence we obtain the following theorem[5].

Theorem 5.3. *Let L_n be the normalized Laguerre polynomial of n-th degree. Then $\mu_n(dx) := |L_n(nx)|^2 ne^{-nx}dx$ $(x > 0)$ weakly converge to*

$$\frac{1}{\pi}\frac{dx}{\sqrt{4-(x-2)^2}} \quad (0 < x < 4).$$

That is, the **Classical Limit of "Laguerre oscillator" is also the Arcsine Law** (just translated and dilated).

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